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Quantum Models of Cognition and Decision

2nd December 2013

Outline

- Motivation Example
- Classical Probability vs Quantum Probability
- Violations of Classical Probability
  - Order of Effects
  - Conjunction Errors
  - The Sure Thing Principle
  - The Double Slit Experiment
- Implications and Research Questions

Two Probability Theories

CLASSICAL THEORY

Andrei Kolmogorov (1933)

QUANTUM THEORY

John von Neumann (1932)

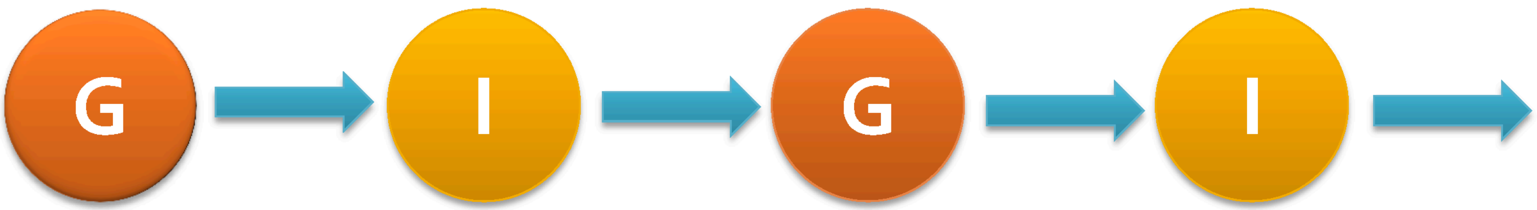
Motivation Example

- Suppose you are a juror trying to judge whether a defendant is Guilty or Innocent

- What beliefs do you experience?

Definite States?

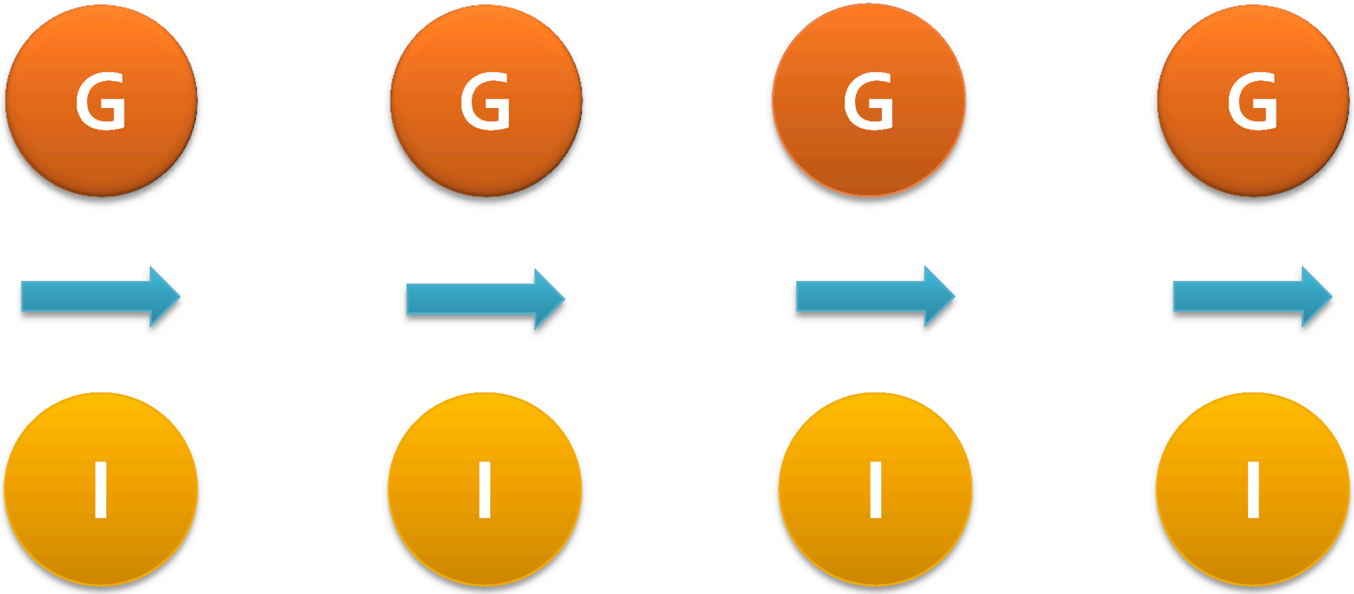
- **Classical Information Processing**



- Single path trajectory. Jump between states.
- At each moment favors Guilty and another moment favors Innocent

Indefinite States?

- **Quantum Information Processing**

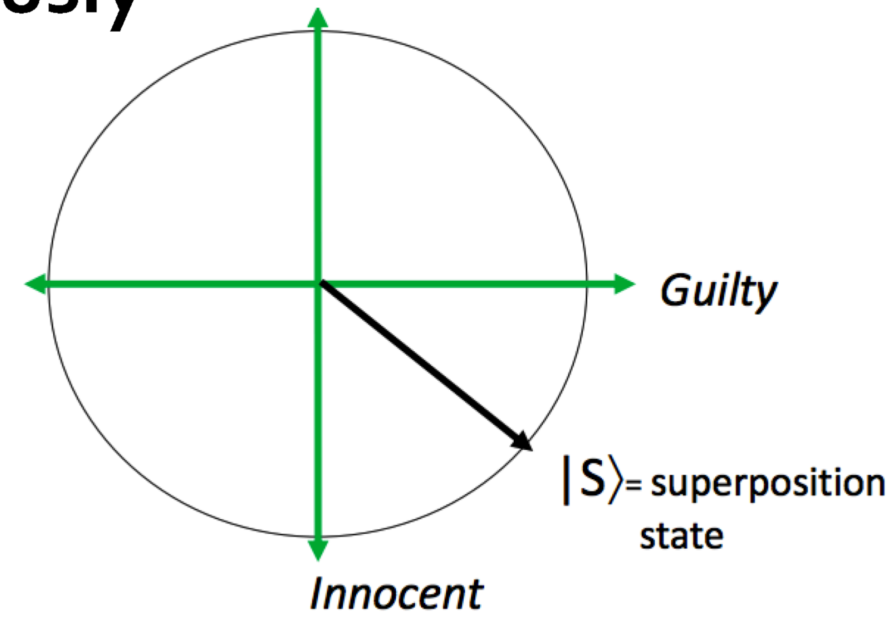


- Beliefs **don't jump** between each other. They are in a **superposition!**

Indefinite States?

- **Quantum Information Processing**

- We experience a feeling of **ambiguity, confusion** or **uncertainty** about all states **simultaneously**



Question Order Effects Example

A Gallup Poll question in 1997 (N = 1002, split sample)
(Politician's names differ from the original work)

Q1. Do you generally think that **Passos Coelho** is honest and trustworthy?

Q2. How about **Ramalho Eanes**?

Q1. Do you generally think that **Ramalho Eanes** is honest and trustworthy?

Q2. How about **Passos Coelho?**

Moore, D.W. (2002). Measuring new types of question order effects. *Public Opinion Quaterly*, **66**, 80-91

Question Order Effects Example

Q1. Do you generally think that **Passos Coelho** is honest and trustworthy? **(50%)**

Q2. How about **Ramalho Eanes**? **(60%)**

Q1. Do you generally think that **Ramalho Eanes** is honest and trustworthy? **(68%)**

Q2. How about **Passos Coelho?** **(57%)**

Moore, D.W. (2002). Measuring new types of question order effects. *Public Opinion Quaterly*, **66**, 80-91

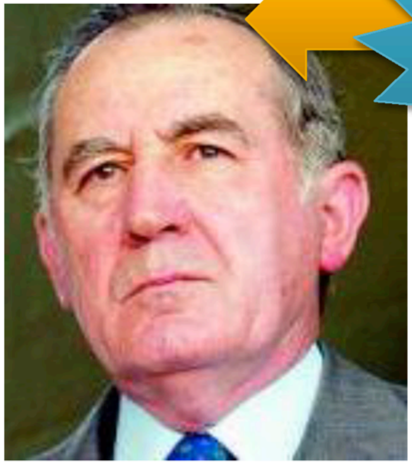
Question Order Effects Example

Q1. Do you generally think that **Passos Coelho** is honest an trustworthy? **(50%)**

18%

Q1. Do you generally think that **Ramalho Eanes** is honest and stworthy? **(68%)**

Q2. How about **Ramalh**
(60%)



3 %

Q2. How about **Passos Coelho?**
(57%)

Moore, D.W. (2002). Measuring new types of question order effects. *Public Opinion Quaterly*, **66**, 80-91

Question Order Effects Example

Q1. Do you generally think that **Passos Coelho** is honest and trustworthy? **(50%)**

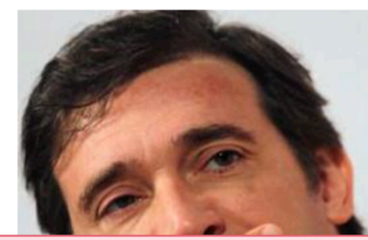
Q2. How about **Ramalho** …?
(60%)

18%

3 %

Q1. Do you generally think that **Ramalho Eanes** is honest and trustworthy? **(68%)**

Q2. How about **Passos Coelho?**
(57%)

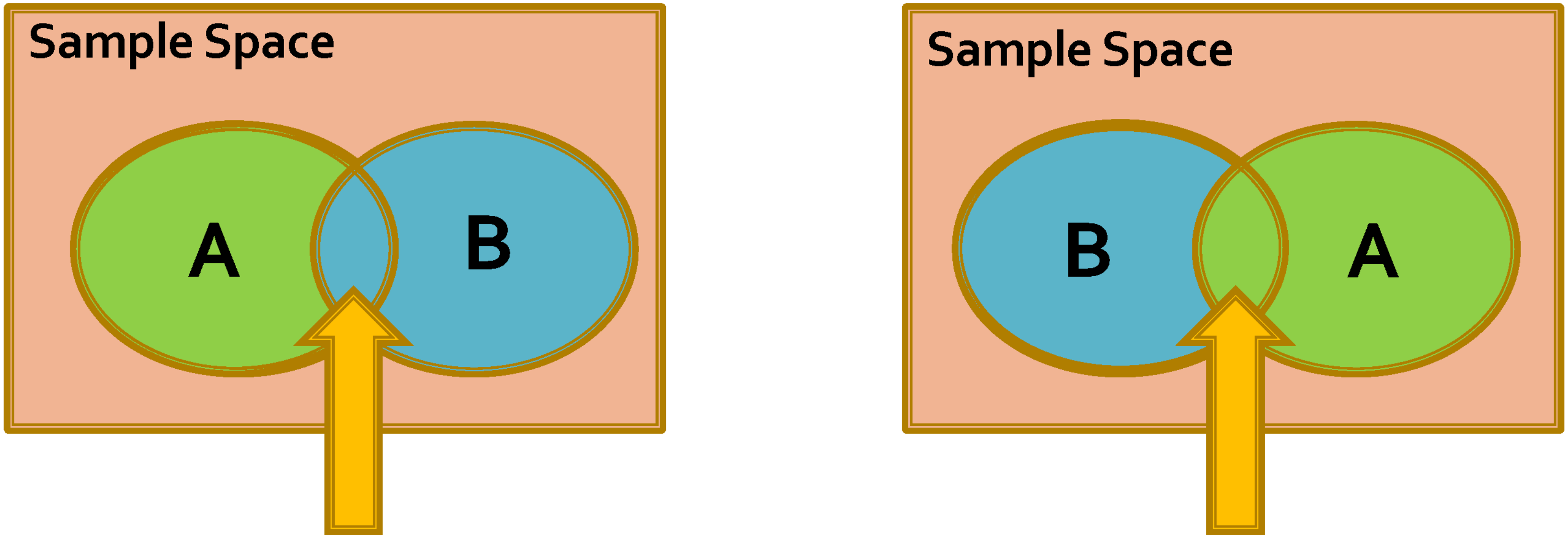


Assimilation Effect!

Moore, D … *inion*
Quaterly …

Violations on Question Order Effects

- Classical Probability cannot explain order of effects, because events are represented as **sets** and are **commutative!**



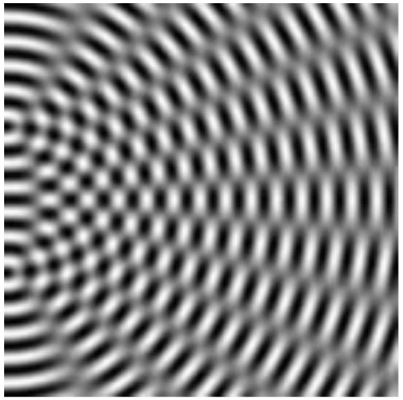
$P(A \cap B) = P(B \cap A)$

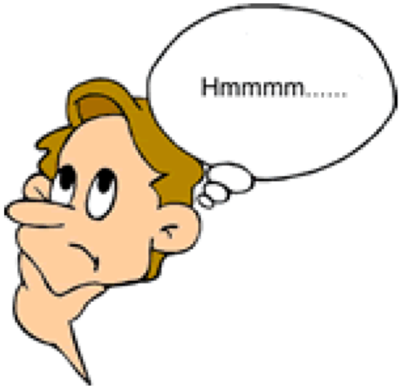
Judgments can Disturb Each Other!

- Order of effects are responsible for introducing uncertainty into a person’s judgments.

Judgment 1

Judgment 2





Differences Between Classic and Quantum Probability

- Events
- System State
- State Revision
- Compatible Events
- Incompatible Events (quantum only)

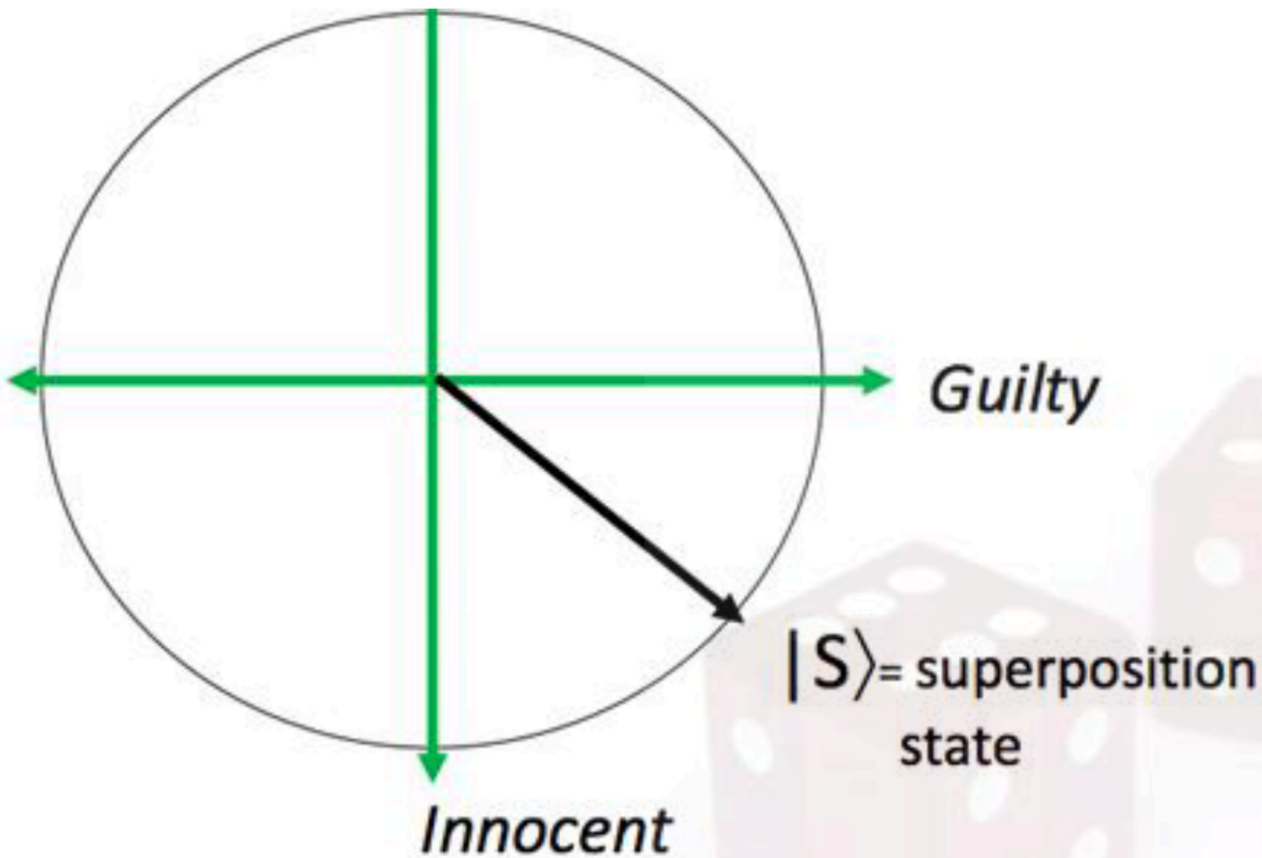
Events – Classical Theory

- **Sample space** (Ω). Contains a finite number of points N, Ω = { Guilty, Innocent }.
- Events are mutually exclusive and sample space is exhaustive.
- Combining events obey to logic of set theory (conjunction and disjunction operations) and to the distributive axiom of set theory.

Events – Quantum Theory

- **Hilbert Space** (H). Contains a (in)finite number of basis vectors , $V = \{|Guilty\rangle, |Innocent\rangle\}$. Allows complex numbers!
- Basis vectors are orthonormal (i.e, mutual exclusive)
- Events are defined by subspaces. Combining events obey the logic of subspaces. Does NOT OBEY the DISTRIBUTIVE AXIOM!

Events – Quantum Theory



$$|S\rangle = \frac{1}{\sqrt{2}}|Guilty\rangle + \frac{1}{\sqrt{2}}|Innocent\rangle$$

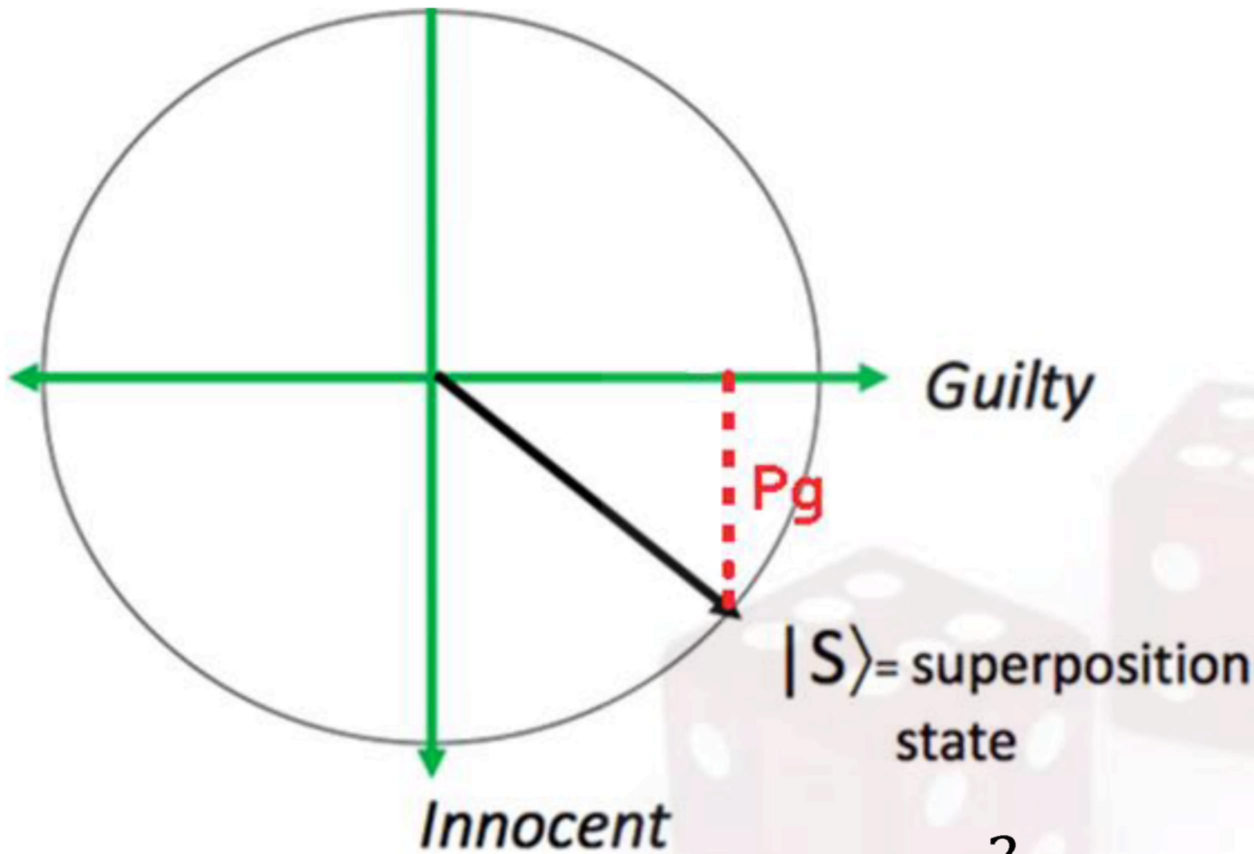
System State – Classical Theory

- State is a probability function, denoted by *Pr(.)*
- Function directly maps elementary events into probabilities. $\Pr(Guilty) = \frac{1}{2}$
- Empty set receives probability zero.
- Sample space receives probability one.

System State – Quantum Theory

- State is a unit-length vector in the N-dimensional vector space, defined by $|S\rangle$, used to map events into probabilities
- The state is projected onto the subspaces corresponding to an event, and the squared length of this projection equals the event probability.

System State – Quantum Theory



$$\Pr(Guilty) = \|Pg\|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

State Revision – Classical Theory

- An event is observed and want to determine other probabilities after observing this fact.
- Uses conditional probability function.

$$\Pr(Innocent|Guilty) = \frac{\Pr(Innocent \cap Guilty)}{\Pr(Guilty)}$$

$$\Pr(Innocent|Guilty) = 0$$

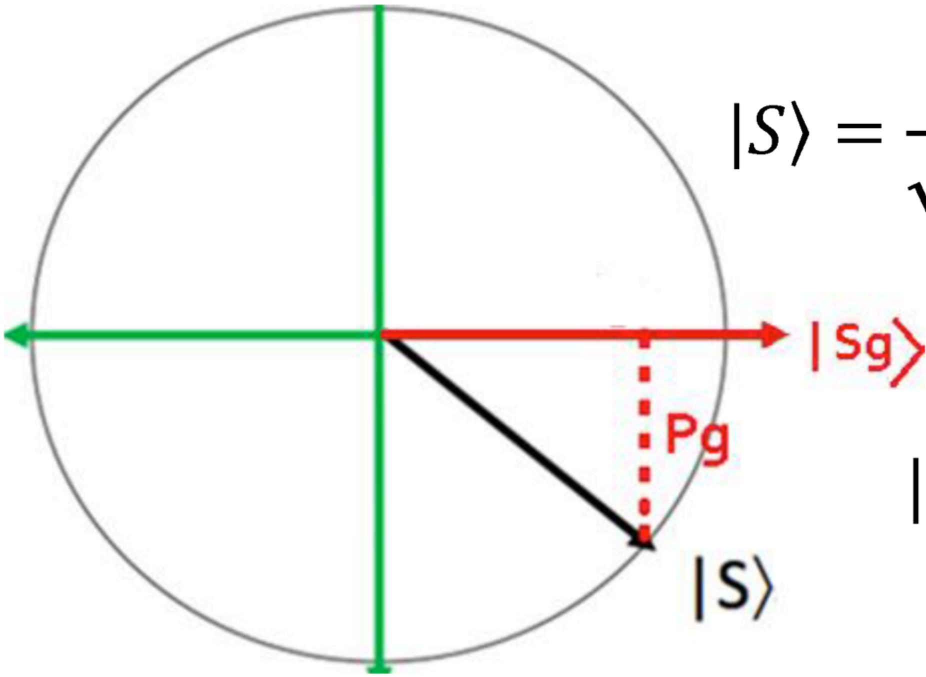
State Revision – Quantum Theory

- Changes the original state vector by projecting the original state onto the subspace representing the observed event.

- The lenght of the projection is used as normalization factor

$$|S_g\rangle = \frac{P_g|S\rangle}{\|P_g|S\rangle\|}$$

State Revision – Quantum Theory



$$|S\rangle = \frac{1}{\sqrt{2}}|Guilty\rangle + \frac{1}{\sqrt{2}}|Innocent\rangle$$

$$|S_g\rangle = \frac{(1/\sqrt{2})|Guilty\rangle}{\sqrt{(1/\sqrt{2})^2}}$$

$$|S_g\rangle = 1|Guilty\rangle + 0|Innocent\rangle$$

$$\Pr(Innocent) = 0^2 = 0$$

Compatibility

- Can all events be described within a single sample space?

- Classical theory: YES! Unicity assumption!

- Quantum theory: Only if they share a common basis!

Compatibility – Classical Theory

- There is only one sample space. All events are contained in this single sample space.
- A single probability function is sufficient to assign probabilities to all events. **Principle of Unicity**.
- Conjunction and disjunction operations are well defined

Compatibility – Quantum Theory

- There is only one Hilbert Space where all events are contained in.
- For a single fixed basis, the intersection and union of two events spanned by a **common set of basis vectors** is always well defined.
- A probability function assigns probabilities to all events defined with respect to the basis.

Incompatibility– Quantum Theory

- Event A is spanned by $V = \{|V_i\rangle, i = 1, \ldots, N\}$, such that $V_A \subset V$
- Event B by W$= \{|W_i\rangle, i = 1, \ldots, N\}$, such that $W_B \subset W$
- Then the intersection and union of these events is **not defined**.
- Probabilities are assigned to sequences of events using **Luders rule**. Distributive axiom does not hold!

Incompatibility– Example

- Luders Rule: Compute the probability of the sequence of events *A* followed by *B*.

$\Pr(A) = \|P_A|S\rangle\|^2$ the revised state is $|S\rangle = \frac{P_A|S\rangle}{\|P_A|S\rangle\|}$

- The probability of *B* has to be conditioned on the first event , that is $\Pr(B|S_A) = \Pr(A) \,.\, \Pr(B|A)$

$$\Pr(A) \,.\, \Pr(B|A) = \|P_A|S\rangle\|^2 . \|P_B|S_A\rangle\|^2$$

Incompatibility– Example

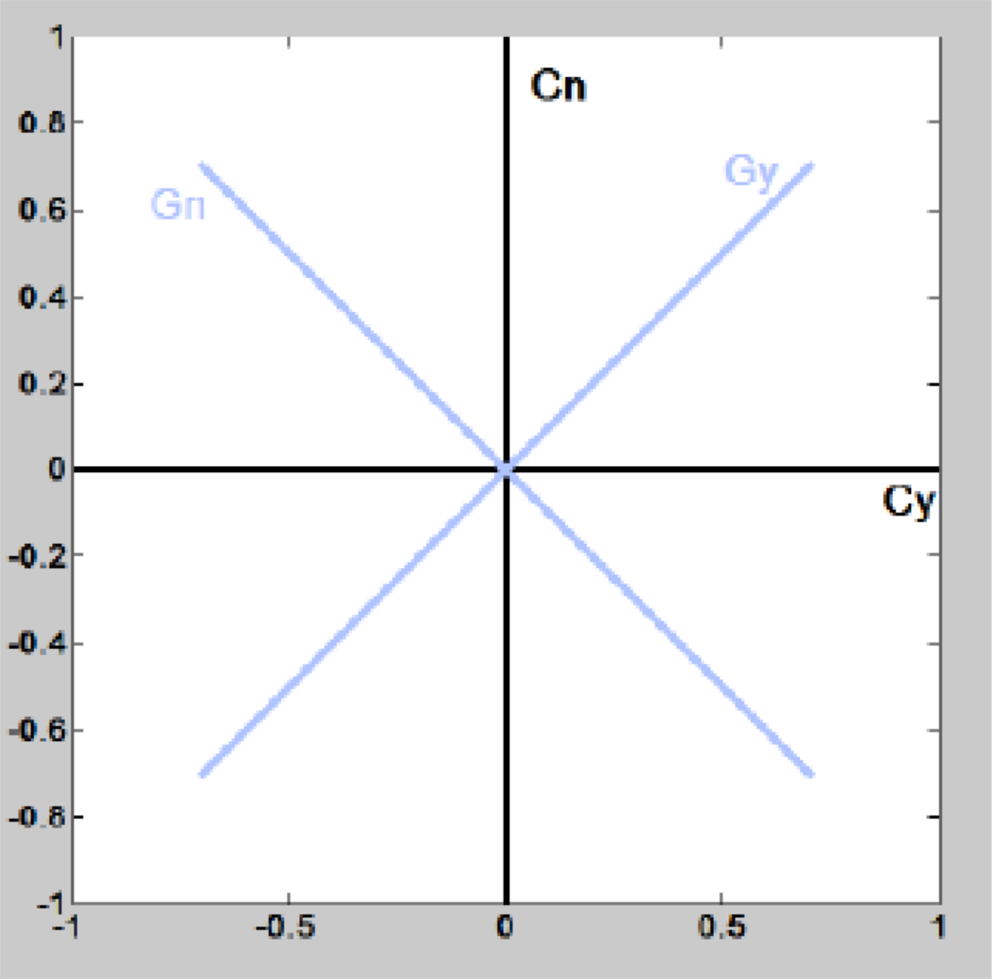
$$\Pr(A)\,.\Pr(B|A) = \|P_A|S\rangle\|^2.\,\|P_B|S_A\rangle\|^2$$

$$= \|P_A|S\rangle\|^2.\left\|P_B\frac{P_A|S\rangle}{\|P_A|S\rangle\|}\right\|^2$$

$$= \|P_A|S\rangle\|^2.\frac{1}{\|P_A|S\rangle\|^2}\|P_BP_A|S\rangle\|^2$$

$$= \|P_BP_A|S\rangle\|^2 \neq \|P_AP_B|S\rangle\|^2$$

Incompatibility– Quantum Theory



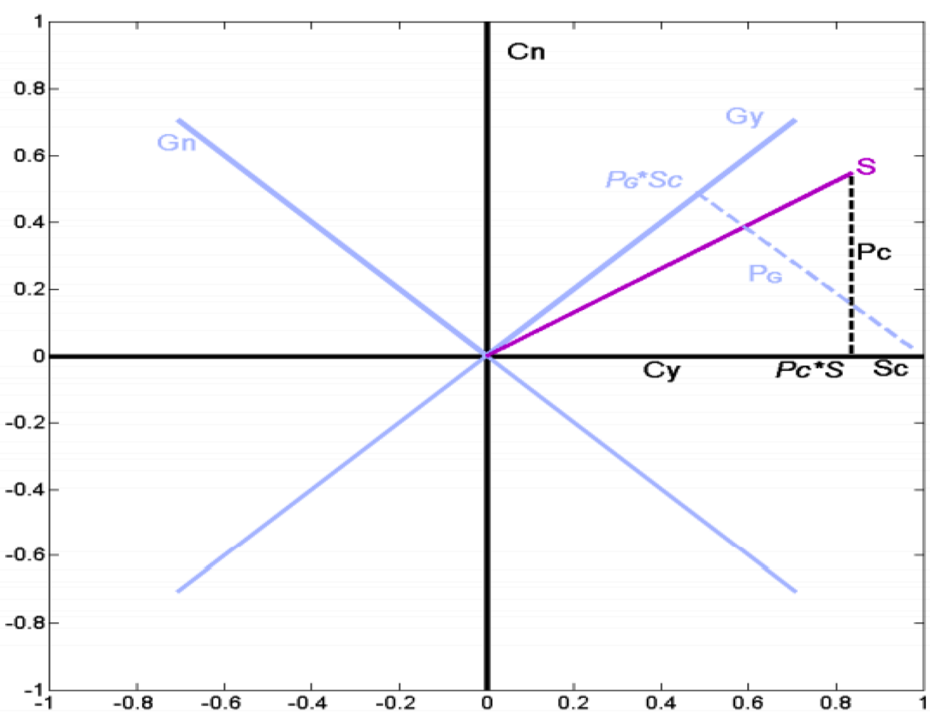
Cx Axis: Passos Coelho

Gx Axis: General Ramalho Eanes

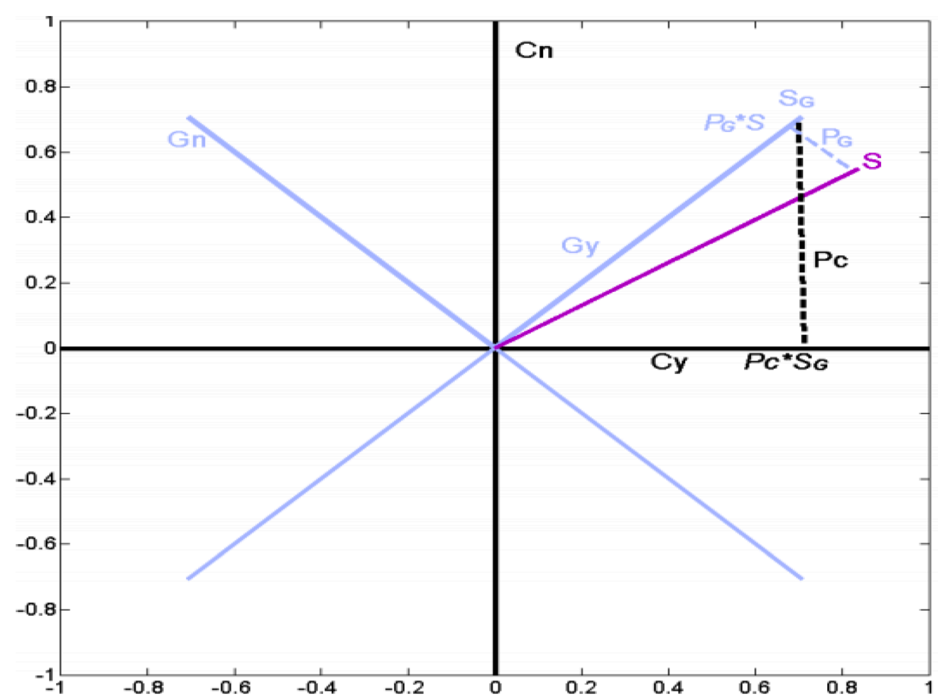
Question Order Effects - Quantum

- Using a quantum model, the probability of responses differ when asked first vs. when asked second

- **Coelho-Eanes**



- **Eanes-Coelho**



Question Order Effects - Quantum

- Passos Coelho is a honest person

$$|S\rangle = 0.8367|P\rangle + 0.5477\bar{P}\rangle$$

- General Eanes is a honest person

$$|S\rangle = 0.9789|G\rangle - 0.2043\bar{G}\rangle$$

- Analysis of first question – Passos Coelho

$$\Pr(Cy) = \|P_C|S\rangle\|^2 = |0.8367|^2 = 0.70$$
$$\Pr(Cn) = \|P_C|S\rangle\|^2 = |0.5477|^2 = 0.30$$

Question Order Effects - Quantum

- Passos Coelho is a honest person

$$|S\rangle = 0.8367|P\rangle + 0.5477\bar{P}\rangle$$

- General Eanes is a honest person

$$|S\rangle = 0.9789|G\rangle - 0.2043\bar{G}\rangle$$

- Analysis of first question – General Eanes

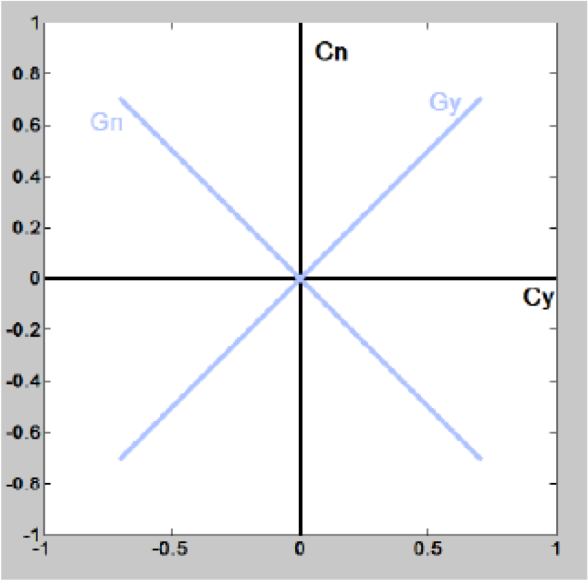
$$\Pr(Gy) = \|P_G|S\rangle\|^2 = |0.9789|^2 = 0.9582$$
$$\Pr(Gn) = \|P_G|S\rangle\|^2 = |-0.2043|^2 = 0.0417$$

Question Order Effects - Quantum

- Analysis of the second question
  - The probability of saying “yes” to Passos Coelho is the probability of saying “yes” to General Eanes and then “yes” to Passos Coelho plus the probability of saying “no” to General Eanes and then “yes ” to Passos Coelho

$$\Pr(Cy) = (0.96).(0.50) + (0.04).(0.50) = 0.50$$

$$\Pr(Gy) = (0.70).(0.50) + (0.30).(0.50) = 0.50$$



Question Order Effects - Quantum

- According to this simplified two dimensional model:
- Large difference between the agreement rates for two politicians in non-comparative context: 70% for Passos Coelho and 96% for General Eanes
- There is no difference in the comparative context: 50% for both.

Question Order Effects - Quantum

- According to this simplified two dimensional model:
- Large difference between the agreement rates for two politicians in non-comparative context: 70% for Passos Coelho and 96% for General Eanes
- There is no difference in the comparative context:

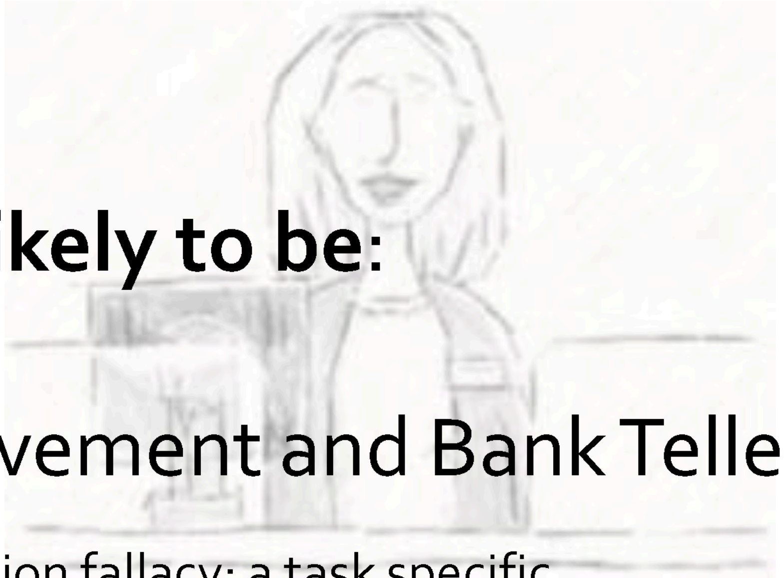
Explains Assimilation Effect!

Violations Classical Probability

- ~~Effects on question order~~
- **Human Probability Judgment Errors**
- The Sure Thing Principle
- The Double Slit Experiment

Conjunction and Disjunction Errors

" Linda is 31 years old, single, outspoken and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations."



Choose what Linda is **more likely to be**:

(a) Bank Teller;

(b) Active in the Feminist Movement and Bank Teller

Morier, D.M. & Borgida, E. (1984). The conjunction fallacy: a task specific phenomena? *Personality and Social Psychology Bulletin*, **10**, 243-252

Conjunction Errors

- 90% of people answered option (b) over option (a).
- People judge Linda to be: "Active in the **feminist** movement and a **bank teller**" over being a "Bank Teller"

$$\Pr(Feminist \cap Bank\ Teller) \geq \Pr(Bank\ Teller)$$

Morier, D.M. & Borgida, E. (1984). The conjunction fallacy: a task specific phenomena? *Personality and Social Psychology Bulletin*, **10**, 243-252

Conjunction Errors

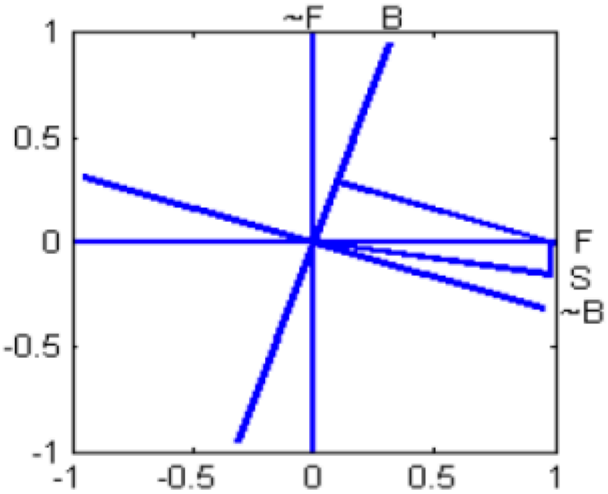
- According to mathematics and logic, we were expecting to find:

$$\Pr(Bank\ Teller) \geq \Pr(Feminist\ \cap Bank\ Teller)$$

- Even if we considered:
  $\Pr(Bank\ Teller) = 0.03 \quad \Pr(Feminist) = 0.95$

$$\begin{aligned}\Pr(Feminist\ \cap Bank\ Teller) &= 0.03 \times 0.95 \\ &= 0.0285 \leq \Pr(Bank\ Teller)\end{aligned}$$

Quantum Model for the Conjunction Error Effects



$$\Pr(B) = \|P_B|S\rangle\|^2 = \|P_B I|S\rangle\|^2$$
$$= \|P_B(P_F + P_{\bar{F}})|S\rangle\|^2$$
$$= \|P_B P_F|S\rangle + P_B P_{\bar{F}}|S\rangle\|^2$$

$$= \|P_B P_F|S\rangle\|^2 + \|P_B P_{\bar{F}}\|^2 + Int_B$$
$$Int_B = 2.Re[\langle S|P_F P_B P_{\bar{F}}|S\rangle]Cos\theta$$

$$Pr(F)\Pr(B|F) = \|P_B P_F|S\rangle\|^2$$

Violations Classical Probability

- ~~Effects on question order~~
- ~~Human Probability Judgment Errors~~
- **The Sure Thing Principle**
- The Double Slit Experiment

Violations of the Sure Thing Principle

The Sure Thing Principle:

" If under state of the world X, people prefer action A over action B and in state of the world ~X prefer action A over B, then if the state of the world in unknown, a person should always prefer action A over B" (Savage, 1954)

Savage, L.J. (1954). The Foundations of Statistics. John Wiley & Sons.

The Two Stage Gambling Game

- At each stage, the decision was wether or not to play a gamble that has an equal chance of winning $2.00 or loosing $1.00.

- Three conditions for participants:
  - Informed they won the first gamble
  - Informed they lost the first gamble
  - Did not know the outcome of the first gamble

Tversky, A. & Shafir, E. (1992). The disjunction effect in choice under uncertainty. *Psichological Science*, **3**, 305-309

The Two Stage Gambling Game

- Results:
  - If participants **knew they won** the first gamble, **(68%)** chose to **play again.**
  - If participants **knew they lost** the first gamble, **(59%)** chose to **play again.**

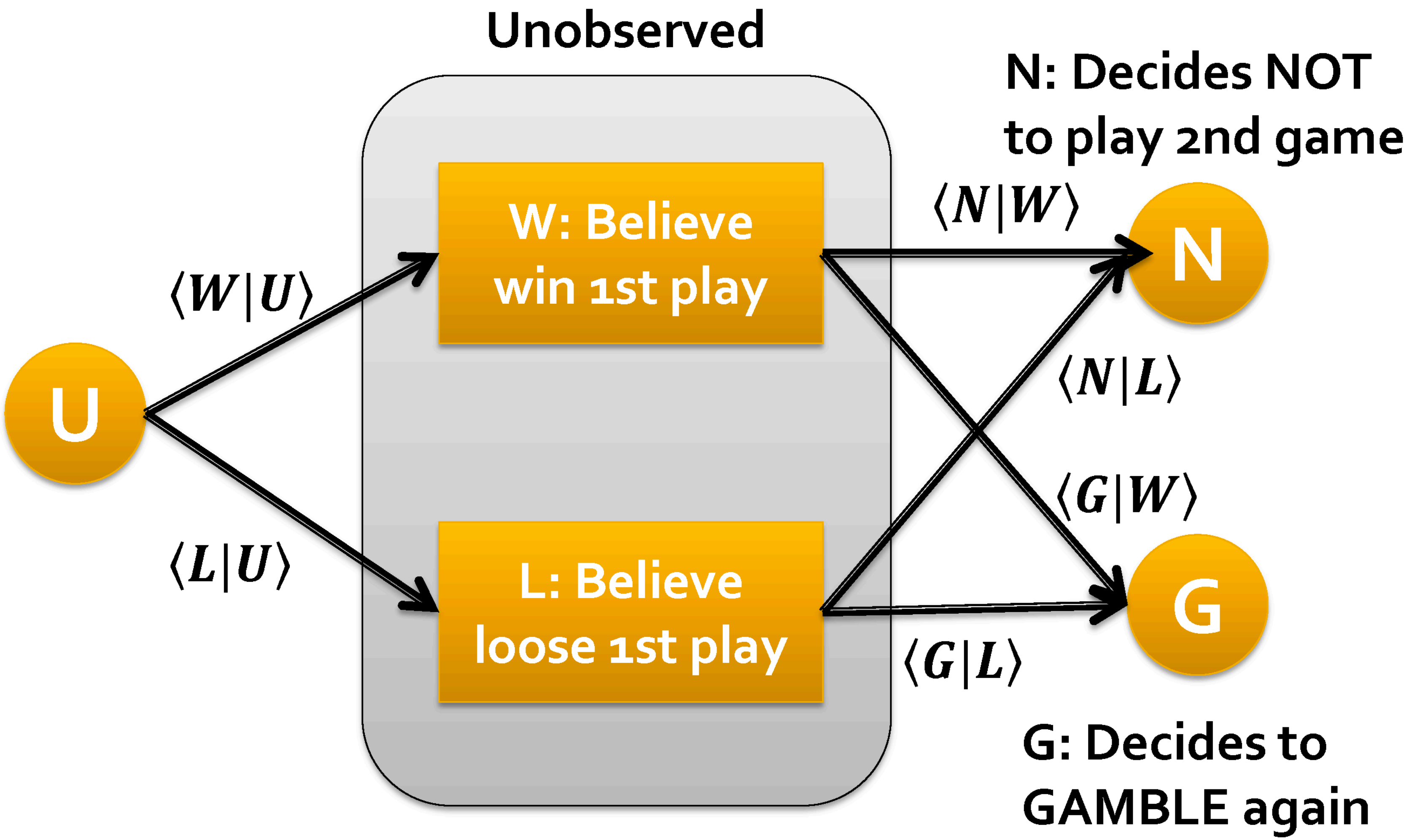
Tversky, A. & Shafir, E. (1992). The disjunction effect in choice under uncertainty. *Psichological Science*, **3**, 305-309

The Two Stage Gambling Game

- Results:
  - If participants **knew they won** the first gamble, **(68%)** chose to **play again.**
  - If participants **knew they lost** the first gamble, **(59%)** chose to **play again.**
  - If participants **did not know the outcome** of the first gamble, **(64%)** chose **not to play.**

Tversky, A. & Shafir, E. (1992). The disjunction effect in choice under uncertainty. *Psichological Science*, **3**, 305-309

The Two Stage Gambling Game



The Two Stage Gambling Game

- **Classical theory - Law of total probability**

$$\Pr(G|U) = Pr(W|U).Pr(G|W) + \Pr(L|U).\Pr(G|L)$$

- From this law, one would expect:
  $\Pr(G|W) = 0.69 > \Pr(G|U) > \Pr(G|L) = 0.59$

- Tversky & Shafir (1992) found that
  $\Pr(G|U) = 0.36 < \Pr(G|L) = 0.59 < \Pr(G|W) = 0.69$

The Two Stage Gambling Game

- **Classical theory - Law of total probability**

$$\mathrm{Pr}(G|U) = Pr(W|U).\,Pr(G|W) + \mathrm{Pr}(L|U)\,.\,\mathrm{Pr}(G|L)$$

- From this law, one would expect:

$$\mathrm{Pr}(G|W) = 0.69 > \mathrm{Pr}(G|U) > \mathrm{Pr}(G|L) = 0.59$$

Violates Law of Total Probability!

- Tve

Pr ... $W)$

$= 0.69$

The Two Stage Gambling Game

- **Quantum theory - Law of total amplitude**

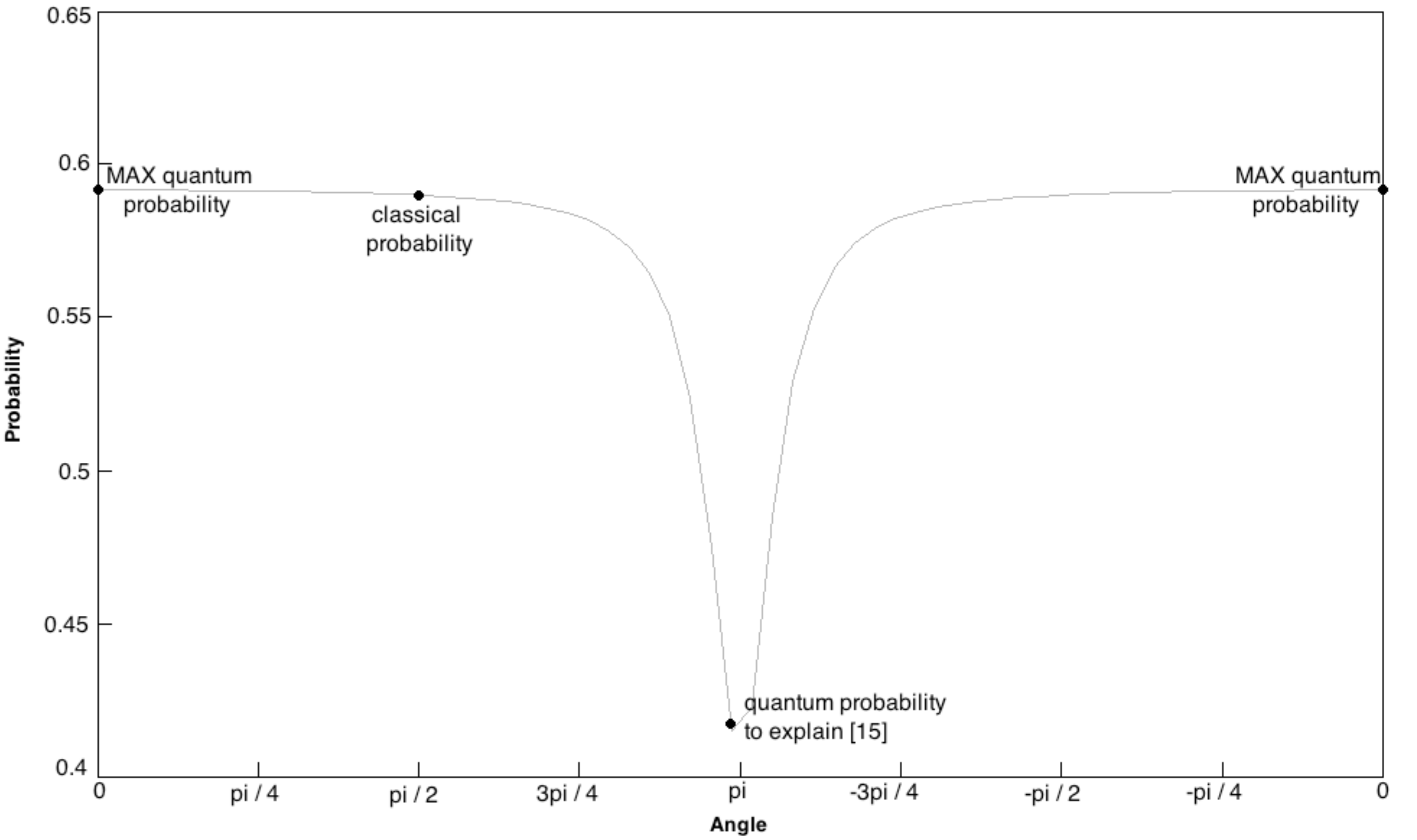
$$\Pr(\langle G|U\rangle) = |\langle W|U\rangle\langle G|W\rangle + \langle L|U\rangle\langle G|L\rangle|^2$$

$$= |\langle W|U\rangle\langle G|W\rangle|^2 + |\langle L|U\rangle\langle G|L\rangle|^2 +$$
$$+2. Re[\langle W|U\rangle\langle G|W\rangle\langle L|U\rangle\langle G|L\rangle. Cos\ \theta]$$

- To account for Tversky and Shafir results, θ must be chosen such that

$$2. Re[\langle W|U\rangle\langle G|W\rangle\langle L|U\rangle\langle G|L\rangle. Cos\ \theta] < 0$$

The Two Stage Gambling Game



Violations Classical Probability

- ~~Effects on question order~~
- ~~Human Probability Judgment Errors~~
- ~~The Sure Thing Principle~~
- **The Double Slit Experiment**

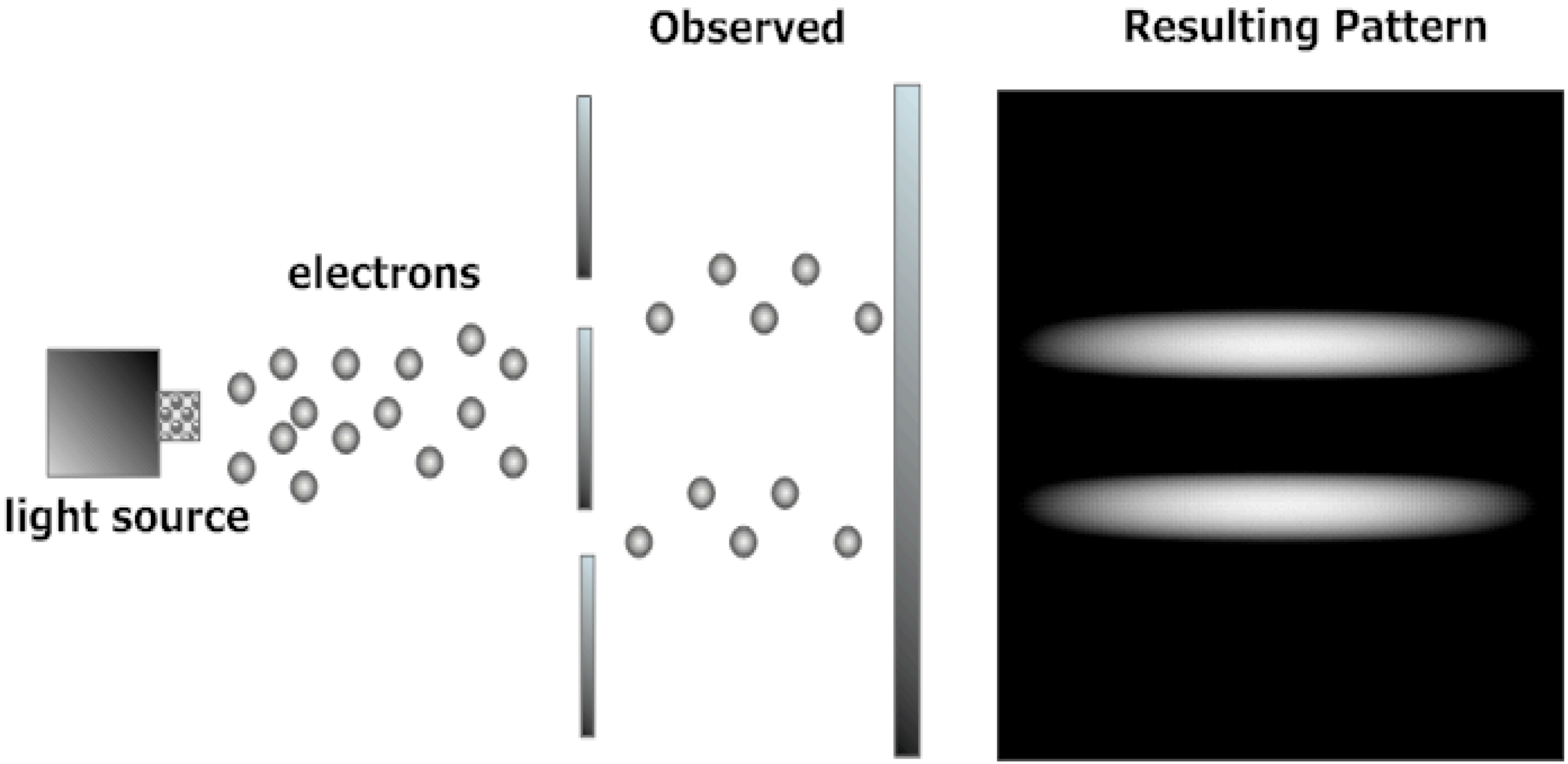
The Double Slit Experiment

- A single electron is dispersed from a light source.
- The electron is split into one of two channels (C_1 or C_2) from which it can reach one of the two detectors (D_1 or D_2).

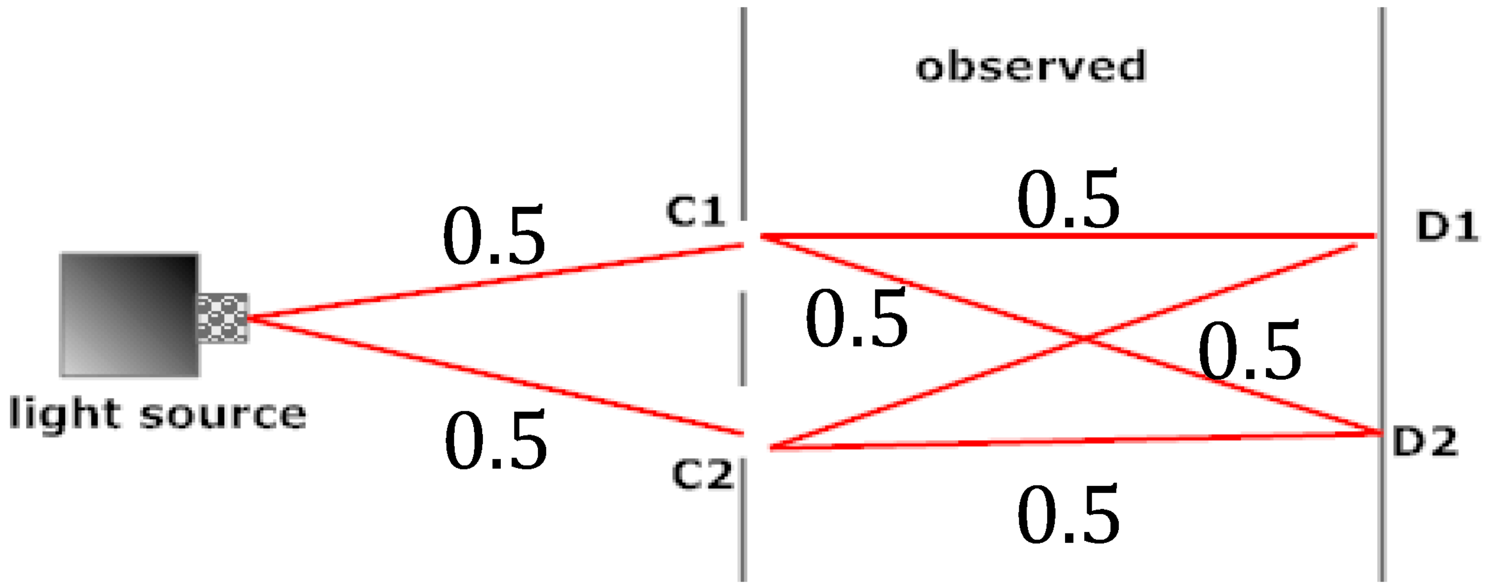
The Double Slit Experiment

- Two conditions are examined:
  - The channel through which the electron passes is **observed**.
  - The channel through which the electron passes in **not observed**.

The Double Slit Experiment



The Double Slit Experiment (C)

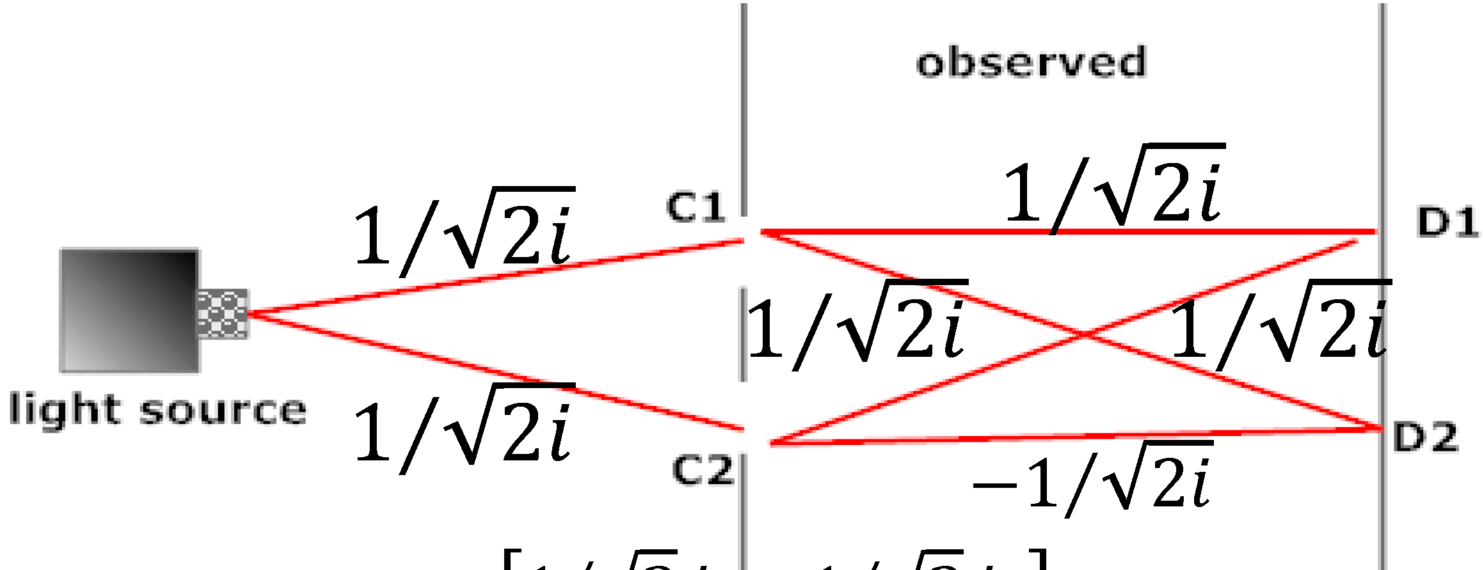


$$Pr(c1) = [0.5 \quad 0] \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} = [0.25 \quad 0.25]$$

$$\Pr(c2) = [0 \quad 0.5] \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} = [0.25 \quad 0.25]$$

$$\Pr(c1 \; or \; c2) = \Pr(c1) + \Pr(c2) = [0.5 \quad 0.5]$$

The Double Slit Experiment (Q)

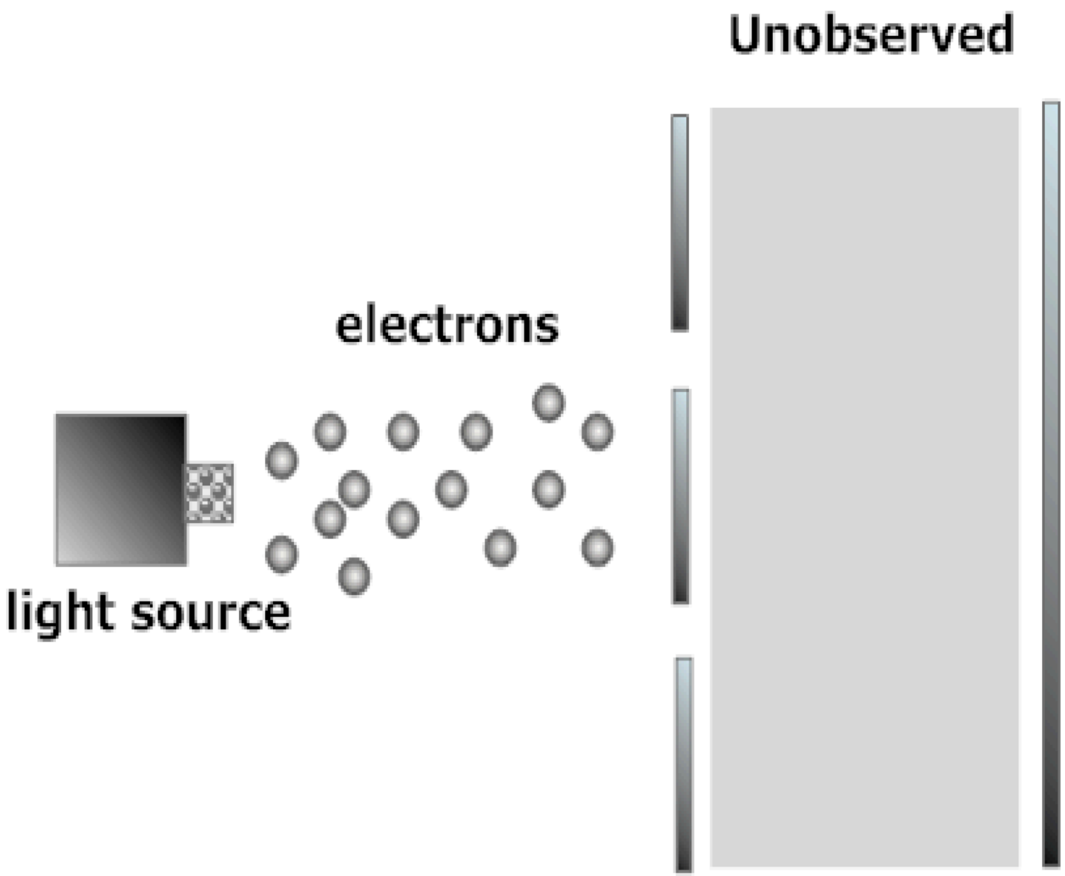


$$c1 = \begin{bmatrix} 1/\sqrt{2}\,i & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2}\,i & 1/\sqrt{2}\,i \\ 1/\sqrt{2}\,i & -1/\sqrt{2}\,i \end{bmatrix} = \begin{bmatrix} -0.5 & -0.5 \end{bmatrix}$$

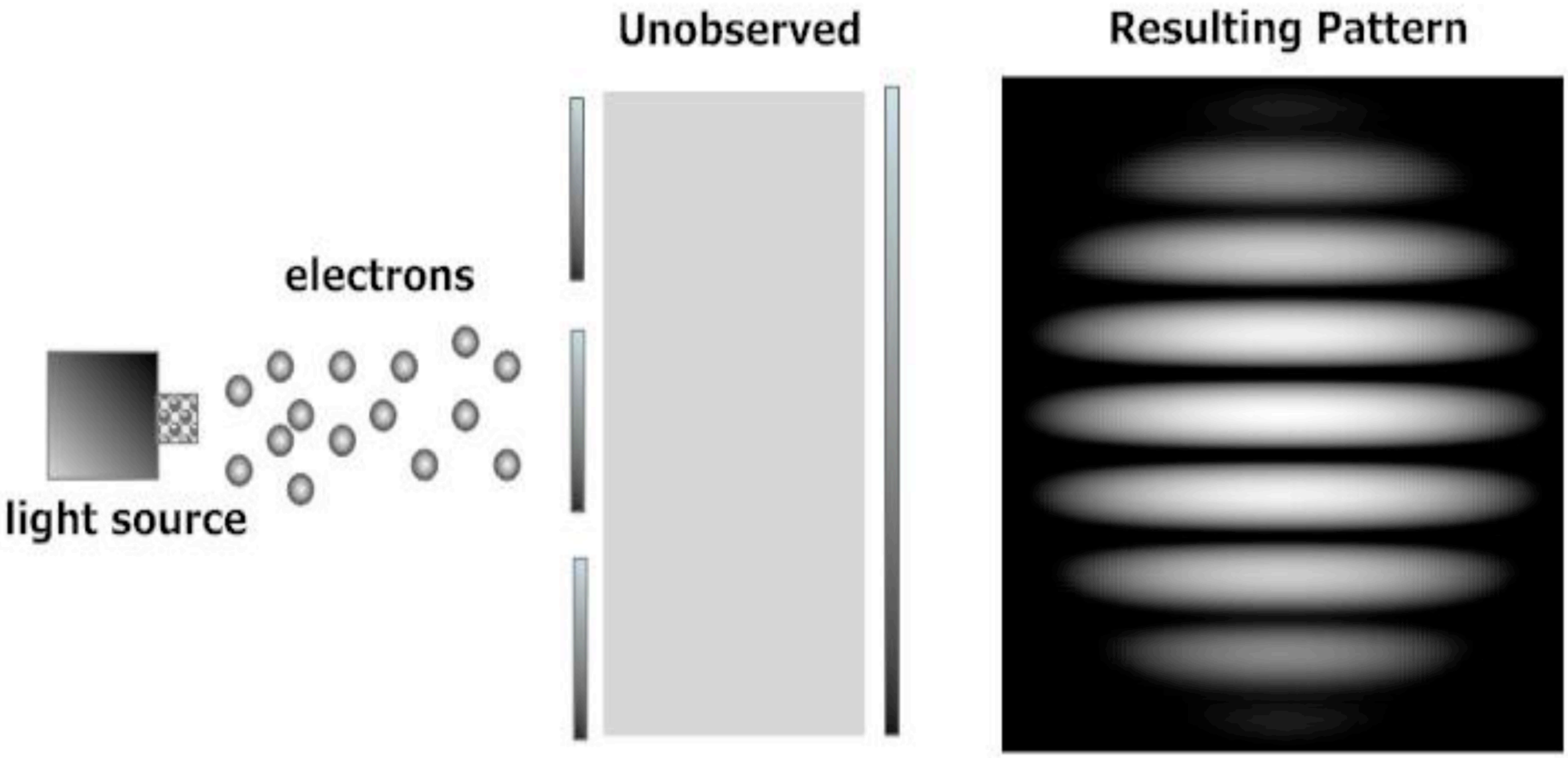
$$c2 = \begin{bmatrix} 0 & 1/\sqrt{2}\,i \end{bmatrix} \begin{bmatrix} 1/\sqrt{2}\,i & 1/\sqrt{2}\,i \\ 1/\sqrt{2}\,i & -1/\sqrt{2}\,i \end{bmatrix} = \begin{bmatrix} -0.5 & -0.5 \end{bmatrix}$$

$$\Pr(c1\ or\ c2) = c1^2 + c2^2 = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$$

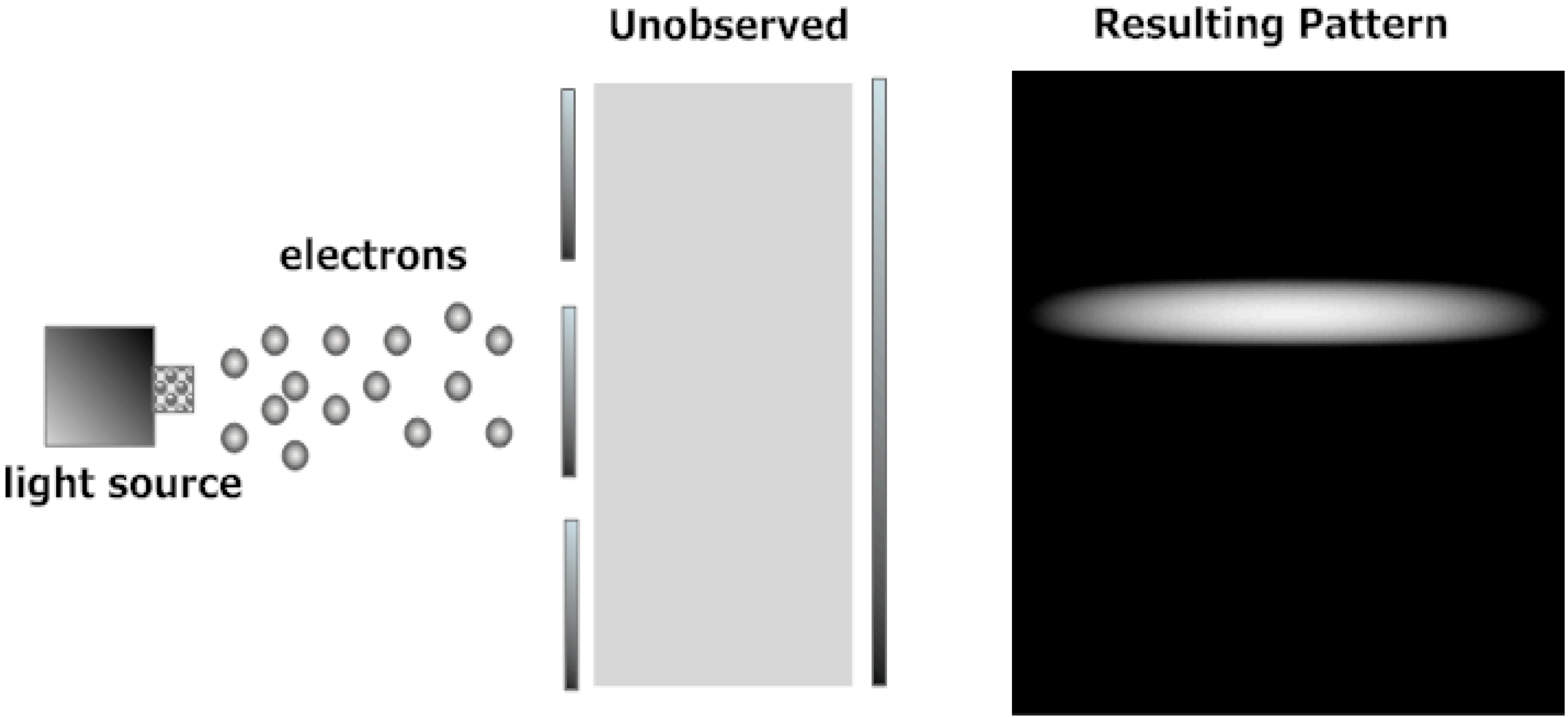
The Double Slit Experiment



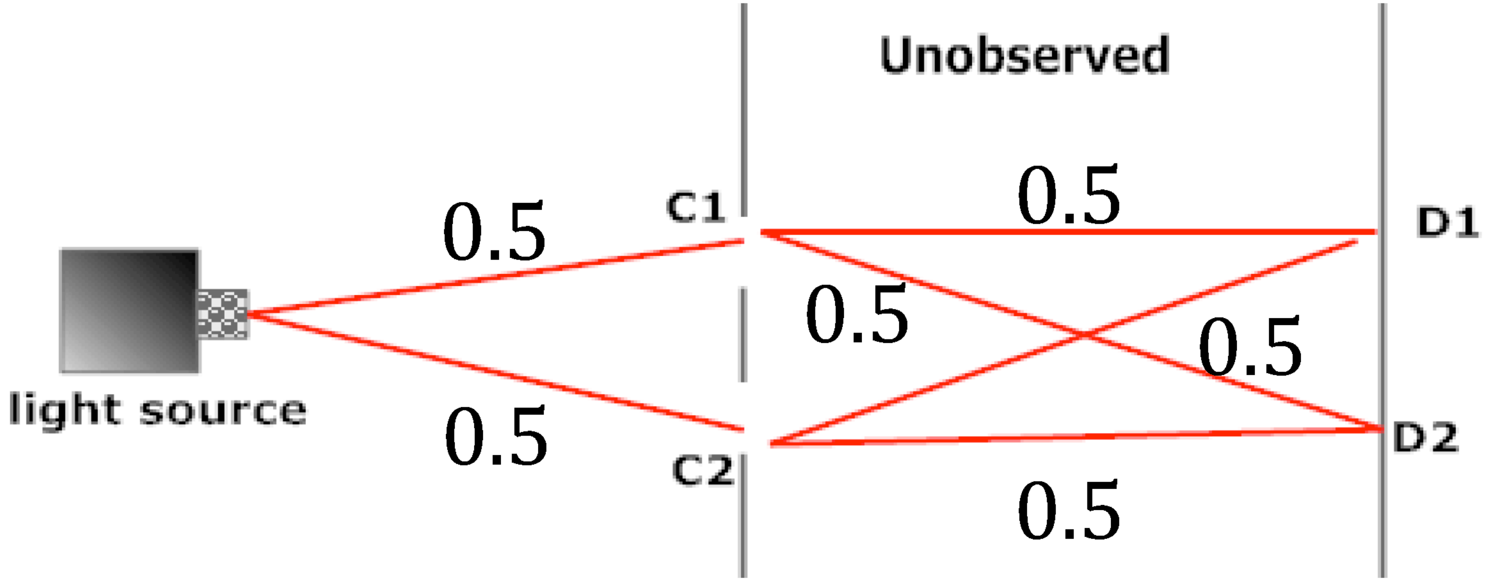
The Double Slit Experiment



The Double Slit Experiment



The Double Slit Experiment (C)



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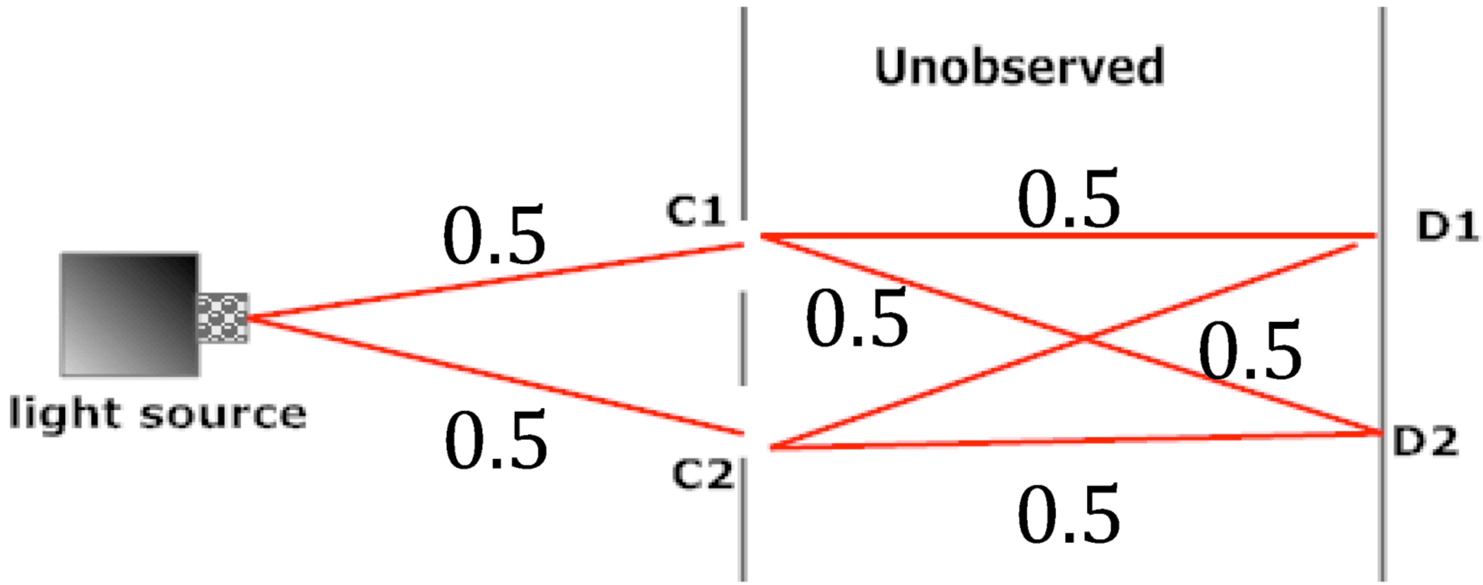
$$\Pr(c2) = [0 \quad 0.5] \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} = [0.25 \quad 0.25]$$

$$\Pr(c1 \; or \; c2) = \Pr(c1) + \Pr(c2) = [0.5 \quad 0.5]$$

The Double Slit Experiment (C)

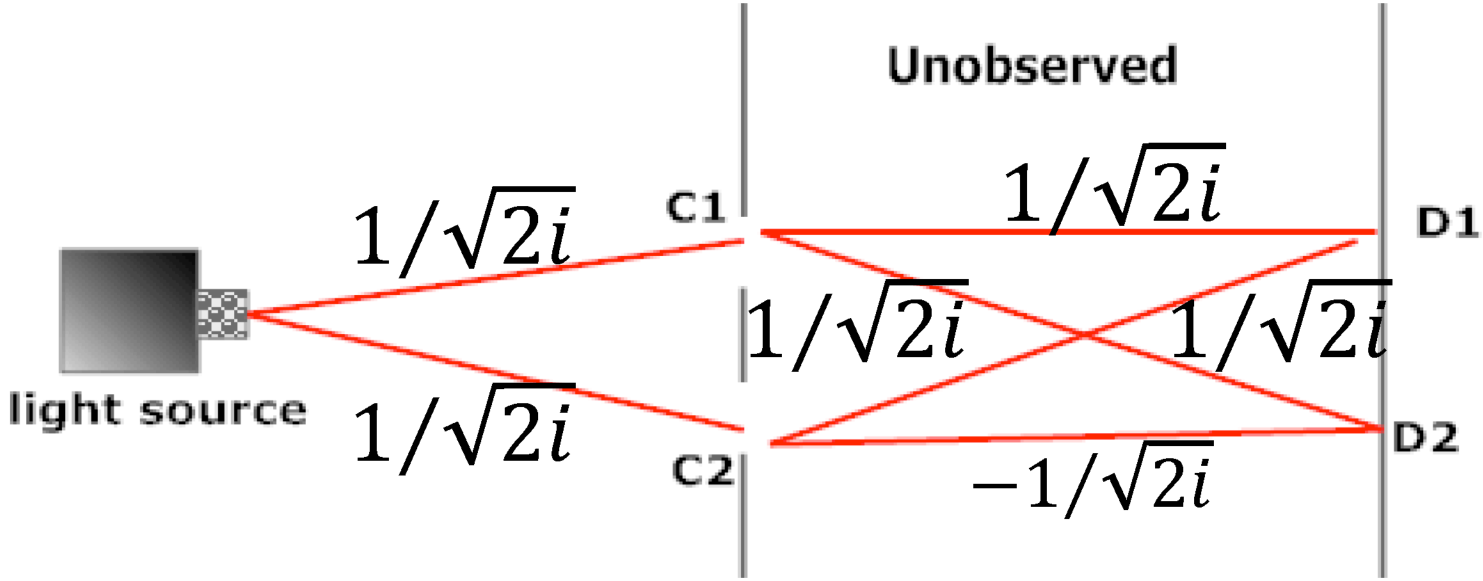


$Pr(c1) = [0.5 \quad 0] [0.5 \quad 0.5] = [0.25 \quad 0.25]$

Cannot Explain the Interference Pattern!

$\Pr(c1 \; or \; c2) = \Pr(c1) + \Pr(c2) = [0.5 \quad 0.5]$

The Double Slit Experiment (Q)



$$c12 = \begin{bmatrix} 1/\sqrt{2}\,i & 1/\sqrt{2}\,i \end{bmatrix} \begin{bmatrix} 1/\sqrt{2}\,i & 1/\sqrt{2}\,i \\ 1/\sqrt{2}\,i & -1/\sqrt{2}\,i \end{bmatrix} = \begin{bmatrix} -1 & 0 \end{bmatrix}$$

$$\Pr(c12) = c12^2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Quantum Rejection of Single Path

- If we do not **observe** the system
- Then, we cannot assume that one of only two possible paths are taken
- In quantum theory, the state is **superposed** between the two possible paths!

Classical Law of Total Probability

Suppose that events A_1, …, A_N form a set of mutually disjoint events, such that their union is al in the sample space for any other event B.

Then the classical law of total probability can be formulated in the following way:

$$Pr(B) = \sum_{i=1}^{N} Pr(A_i) Pr(B|A_i) \qquad \text{where:} \qquad \sum_{i=1}^{N} A_i = 1$$

Quantum Law of Total Probability

One can convert a classical probability into quantum probabilities using Born's Rule:

$$Pr(A) = \mid e^{i\theta_A} \psi_A \mid^2$$

Then the classical law of total probability can be formulated in the following way (using Born's rule):

$$Pr(B) = \left| \sum_{x=1}^{N} e^{i\theta_x} \psi_{A_x} \psi_{B|A_x} \right|^2 \quad \text{where:} \quad \sum_{x=1}^{N} \left| e^{i\theta_x} \psi_{A_x} \right|^2 = 1$$

Deriving the Interference Term

For simplicity, let's assume that N = 2:

$$Pr(B) = \left| \sum_{x=1}^{N} e^{i\theta_x} \psi_{A_x} \psi_{B|A_x} \right|^2$$

$$Pr(B) = \left| e^{i\theta_1} \psi_{A_1} \psi_{B|A_1} + e^{i\theta_2} \psi_{A_2} \psi_{B|A_2} \right|^2$$

$$Pr(B) = \left(e^{i\theta_1} \psi_{A_1} \psi_{B|A_1} + e^{i\theta_2} \psi_{A_2} \psi_{B|A_2} \right) \left(e^{-i\theta_1} \psi_{A_1} \psi_{B|A_1} + e^{-i\theta_2} \psi_{A_2} \psi_{B|A_2} \right)$$

Deriving the Interference Term

$$Pr(B) = \left(e^{i\theta_1}\psi_{A_1}\Psi_{B|A_1} + e^{i\theta_2}\psi_{A_2}\Psi_{B|A_2}\right)\left(e^{-i\theta_1}\psi_{A_1}\Psi_{B|A_1} + e^{-i\theta_2}\psi_{A_2}\Psi_{B|A_2}\right)$$

$$Pr(B) = e^{i\theta_1}\psi_{A_1}\Psi_{B|A_1}e^{-i\theta_1}\psi_{A_1}\Psi_{B|A_1} + e^{i\theta_1}\psi_{A_1}\Psi_{B|A_1}e^{-i\theta_2}\psi_{A_2}\Psi_{B|A_2} +$$

$$+e^{i\theta_2}\psi_{A_2}\Psi_{B|A_2}e^{-i\theta_1}\psi_{A_1}\Psi_{B|A_1} + e^{i\theta_2}\psi_{A_2}\Psi_{B|A_2}e^{-i\theta_2}\psi_{A_2}\Psi_{B|A_2}$$

$$Pr(B) = \left|\psi_{A_1}\Psi_{B|A_1}\right|^2 + \left|\psi_{A_2}\Psi_{B|A_2}\right|^2 +$$

$$e^{i\theta_1}\psi_{A_1}\Psi_{B|A_1}e^{-i\theta_2}\psi_{A_2}\Psi_{B|A_2} + e^{i\theta_2}\psi_{A_2}\Psi_{B|A_2}e^{-i\theta_1}\psi_{A_1}\Psi_{B|A_1}$$

Deriving the Interference Term

$$Pr(B) = \left(e^{i\theta_1}\psi_{A_1}\psi_{B|A_1} + e^{i\theta_2}\psi_{A_2}\psi_{B|A_2}\right)\left(e^{-i\theta_1}\psi_{A_1}\psi_{B|A_1} + e^{-i\theta_2}\psi_{A_2}\psi_{B|A_2}\right)$$

$$Pr(B) = e^{i\theta_1}\psi_{A_1}\psi_{B|A_1}e^{-i\theta_1}\psi_{A_1}\psi_{B|A_1} + e^{i\theta_1}\psi_{A_1}\psi_{B|A_1}e^{-i\theta_2}\psi_{A_2}\psi_{B|A_2} + \\ + e^{i\theta_2}\psi_{A_2}\psi_{B|A_2}e^{-i\theta_1}\psi_{A_1}\psi_{B|A_1} + e^{i\theta_2}\psi_{A_2}\psi_{B|A_2}e^{-i\theta_2}\psi_{A_2}\psi_{B|A_2}$$

$$Pr(B) = \left|\psi_{A_1}\psi_{B|A_1}\right|^2 + \left|\psi_{A_2}\psi_{B|A_2}\right|^2 + \\ e^{i\theta_1}\psi_{A_1}\psi_{B|A_1}e^{-i\theta_2}\psi_{A_2}\psi_{B|A_2} + e^{i\theta_2}\psi_{A_2}\psi_{B|A_2}e^{-i\theta_1}\psi_{A_1}\psi_{B|A_1}$$

Classical Probability

Quantum Interference Effect

Deriving the Interference Term

Interference = $\psi_{A_1}\psi_{B|A_1}\psi_{A_2}\psi_{B|A_2}\left(e^{i(\theta_1-\theta_2)}+e^{i(\theta_2-\theta_1)}\right)$

Knowing that

$$\cos(\theta_1-\theta_2)=\frac{e^{i\theta_1-i\theta_2}+e^{i\theta_2-i\theta_1}}{2}$$

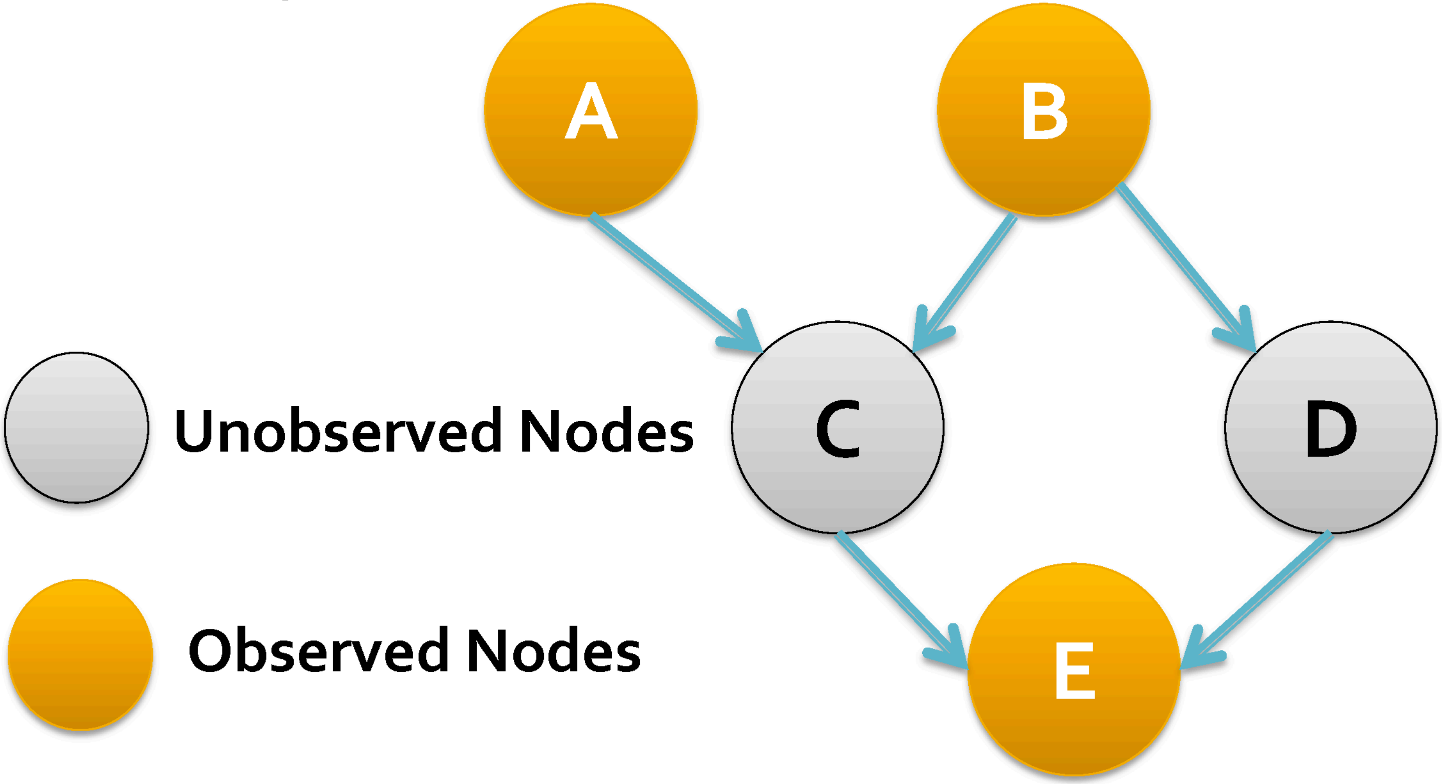
Then,

Interference = $2\,\psi_{A_1}\psi_{B|A_1}\psi_{A_2}\psi_{B|A_2}\cos(\theta_1-\theta_2)$

$$Pr(B)=\left|\psi_{A_1}\psi_{B|A_1}\right|^2+\left|\psi_{A_2}\psi_{B|A_2}\right|^2+2\,\psi_{A_1}\psi_{B|A_1}\psi_{A_2}\psi_{B|A_2}\cos(\theta_1-\theta_2)$$

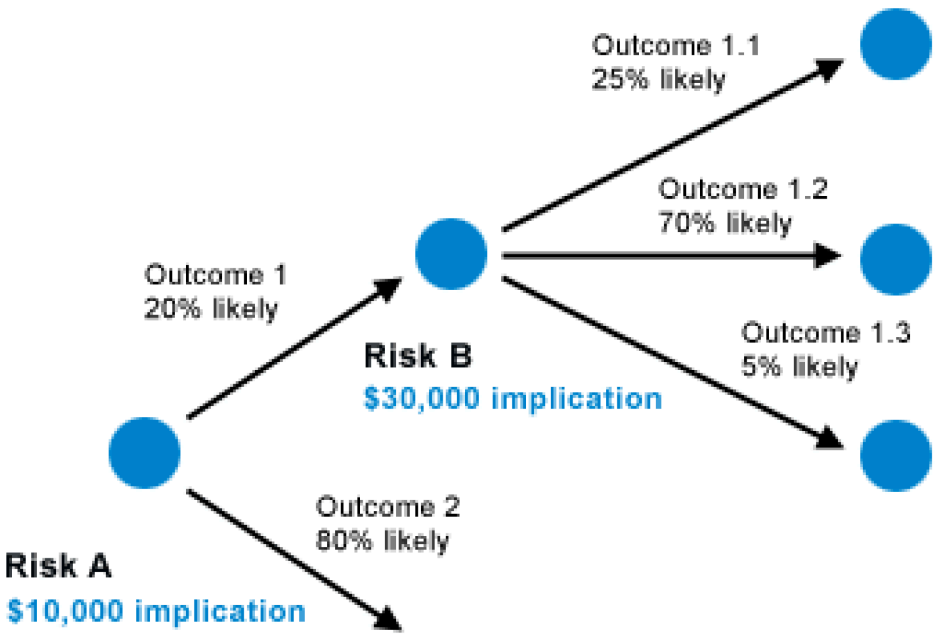
Research Questions

- What are the implications of quantum probabilities in Computer Science models?
  - Bayesian Networks, Markov Networks



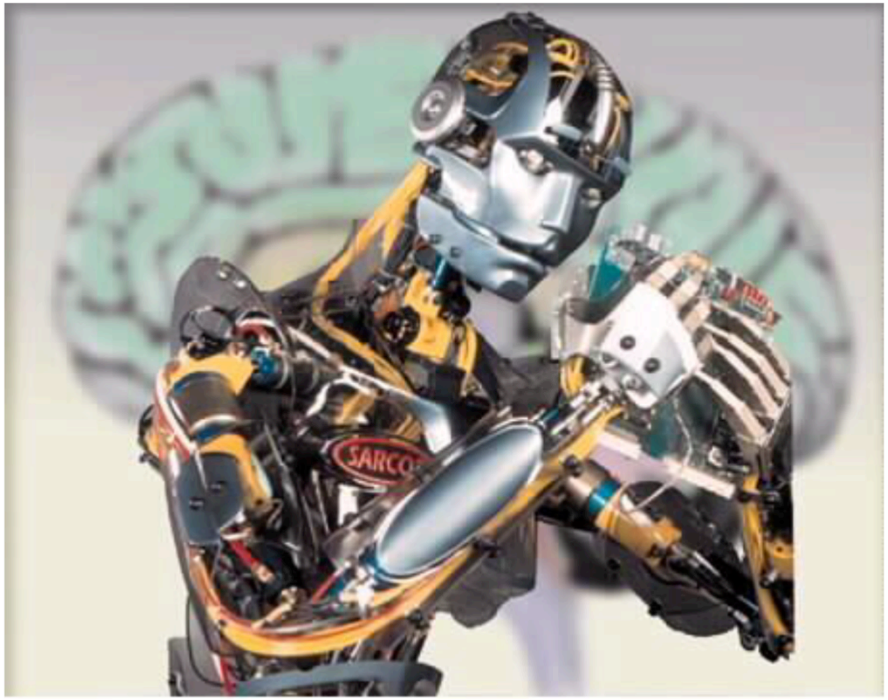
Research Questions

- What are the implications of quantum probabilities in Decision Making?
  - Decision trees, utility functions, risk management…



Research Questions

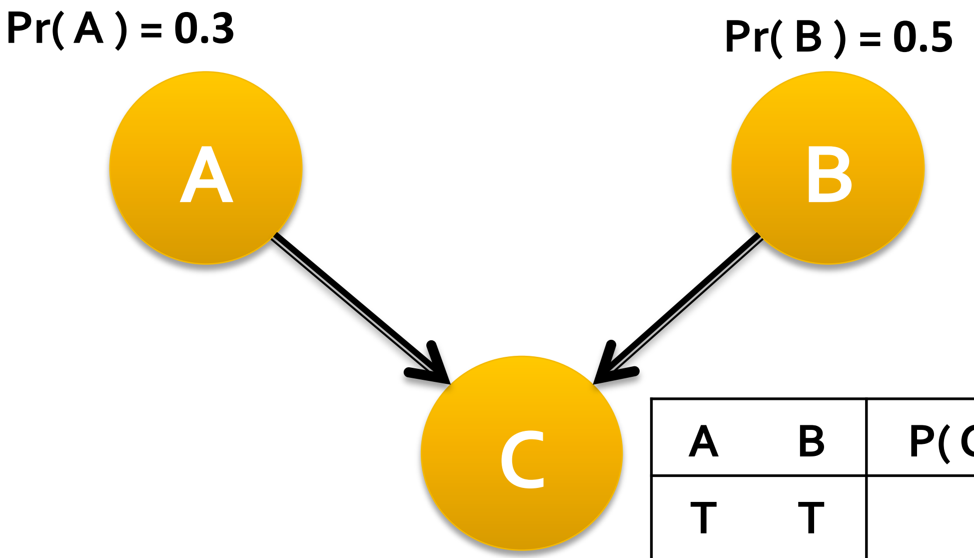
- What are the implications of quantum theory in machine learning?
  - A couple of works in the literature state that it is possible!



Bayesian Networks - Classical

- Directed acyclic graph;
- Each node represents a **random variable**
- Each edge represents a direct causal influence from the source node (**cause**) to the target node (**effect**)

Bayesian Networks - Classical



A	B	P(C | A B)
T	T	0.8
T	F	0.7
F	T	0.4
F	F	0.02

Bayesian Networks - Classical

- What is the probability of node **C** given that node **A** was observed to occur?

$$Pr(C = t \mid A = t, B) = \ ?$$

Bayesian Networks - Classical

- What is the probability of node **C** given that node **A was observed to occur**?
  - We need to compute the **full joint distribution**!

A	B	C	Pr(A, B, C)
T	T	T	0.3 X 0.5 X 0.8 = 0.12
T	T	F	0.3 X 0.5 X 0.2 = 0.03
T	F	T	0.3 X 0.5 X 0.7 = 0.105
T	F	F	0.3 X 0.5 x 0.3 = 0.045
F	T	T	
F	T	F	
F	F	T	
F	F	F	

Bayesian Networks - Classical

- What is the probability of node **C** given that node **A was observed to occur**?
  - We need to compute the full joint distribution!

A	B	C	Pr(A, B, C)
T	T	T	0.3 X 0.5 X 0.8 = 0.12
T	T	F	0.3 X 0.5 X 0.2 = 0.03
T	F	T	0.3 X 0.5 X 0.7 = 0.105
T	F	F	0.3 X 0.5 x 0.3 = 0.045
F	T	T	
F	T	F	
F	F	T	
F	F	F	

We don't need to compute the entries where A is False!

Bayesian Networks - Classical

- What is the probability of node **C** given that node **A was observed to occur**?
  - We need to compute the full joint distribution!

A	B	C	Pr(A, B, C)	Pr(A, B, C)
T	T	T	0.3 X 0.5 X 0.8 = 0.12	0.4
T	T	F	0.3 X 0.5 X 0.2 = 0.03	0.1
T	F	T	0.3 X 0.5 X 0.7 = 0.105	0.35
T	F	F	0.3 X 0.5 x 0.3 = 0.045	0.15
Sum			0.3	1

Bayesian Networks - Classical

- What is the probability of node **C** given that node **A was observed to occur**?
  - Just sum the entries where **C = T**

A	B	C	Pr(A, B, C)	Pr(A, B, C)
T	T	T	0.3 X 0.5 X 0.8 = 0.12	0.4
T	T	F	0.3 X 0.5 X 0.2 = 0.03	0.1
T	F	T	0.3 X 0.5 X 0.7 = 0.105	0.35
T	F	F	0.3 X 0.5 x 0.3 = 0.045	0.15

Bayesian Networks - Classical

- What is the probability of node **C** given that node **A was observed to occur**?
  - Just sum the entries where **C = T**

Pr(C = t | A = t, B) = 0.75

A	B	C	Pr(A, B, C)	Pr(A, B, C)
T	T	T	0.3 X 0.5 X 0.8 = 0.12	0.4
T	T	F	0.3 X 0.5 X 0.2 = 0.03	0.1
T	F	T	0.3 X 0.5 X 0.7 = 0.105	0.35
T	F	F	0.3 X 0.5 x 0.3 = 0.045	0.15

Bayesian Networks - Classical

- What is the probability of node **C** given that node **A was observed to occur**?

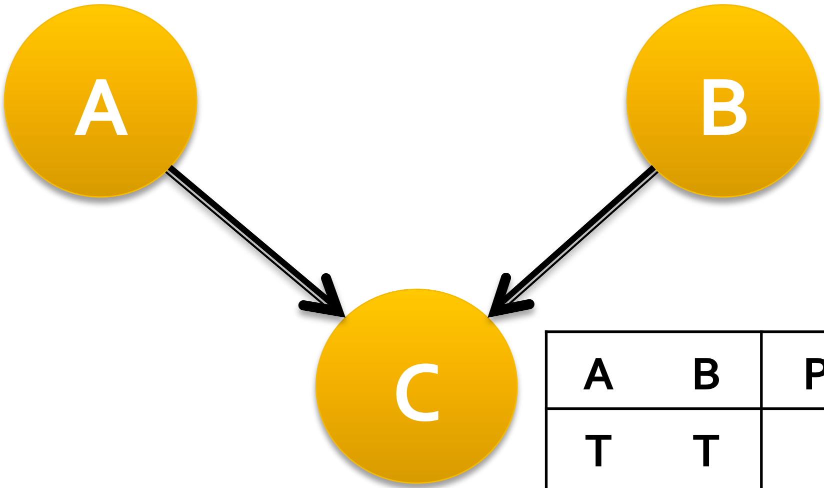
$$Pr(C = t | A = t, B) =$$

$$Pr(A = t) \sum_{b \in B} Pr(B = b) Pr(C = t | A = t, B = b)$$

Bayesian Networks – Quantum

Pr(A) = $\sqrt{0.3}e^{\theta 1}$

Pr(B) = $\sqrt{0.5}e^{\theta 2}$



A	B	P(C | A B)
T	T	$\sqrt{0.8}e^{\theta 3}$
T	F	$\sqrt{0.7}e^{\theta 4}$
F	T	$\sqrt{0.4}e^{\theta 5}$
F	F	$\sqrt{0.02}e^{\theta 6}$

Bayesian Networks - Quantum

- What is the probability of node **C** given that node **A** was observed to occur?
- The full joint distribution corresponds to the **superposition state**

$$|S\rangle = \sqrt{0.4}e^{\theta_1}|ABC\rangle + \sqrt{0.1}e^{\theta_2}|AB\bar{C}\rangle + \\ + \sqrt{0.35}e^{\theta_3}|A\bar{B}C\rangle + \sqrt{0.15}e^{\theta_4}|A\bar{B}\bar{C}\rangle$$

Bayesian Networks - Quantum

- What is the probability of node **C** given that node **A** was observed to occur?

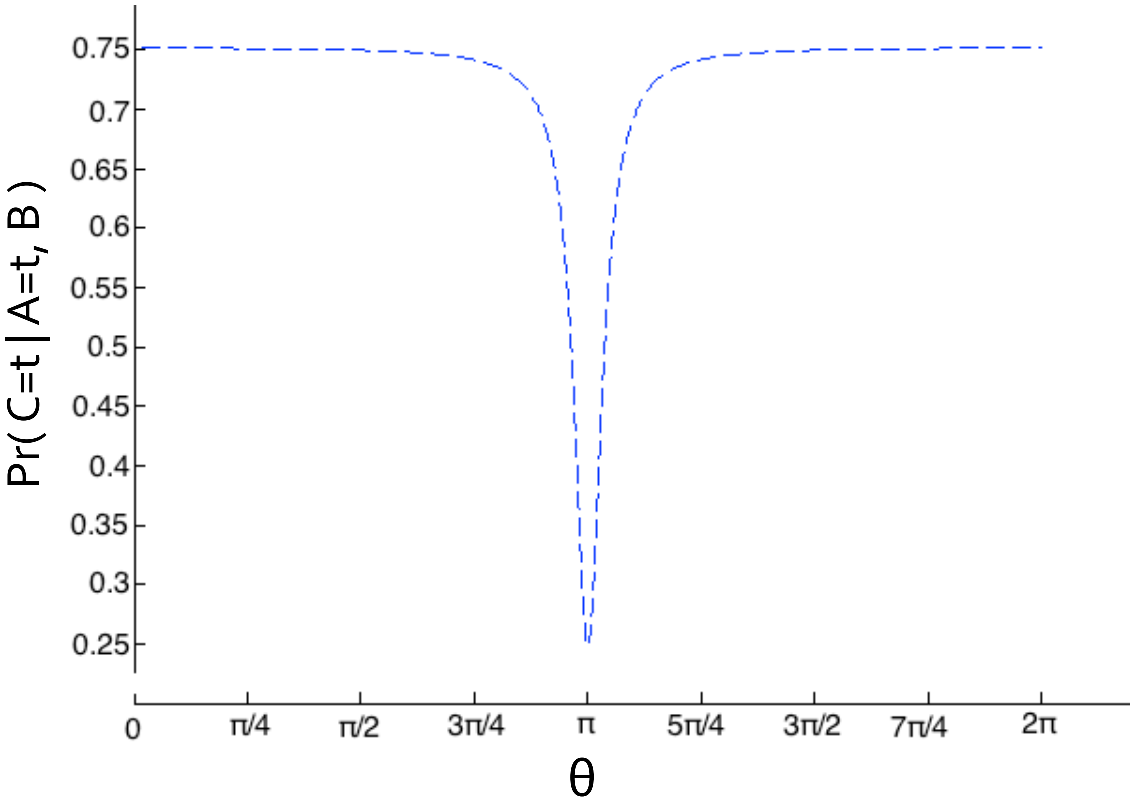
$$Pr(C = t | A = t, B) =$$

$$\left| P_{A=t} P_{B=t} P_{C=t|A=t,B=t} |S\rangle + P_{A=t} P_{B=t} P_{C=t|A=t,B=f} |S\rangle \right|^2$$

$$Pr(C = t | A = t, B) = 0.75 + 2\sqrt{0.4}\sqrt{0.35}\cos(\theta_1 - \theta_2)$$

Bayesian Networks - Quantum

- The quantum probability Pr(C=t | A=t, B) can be **anything!**



Bayesian Networks - Quantum

- The quantum probability Pr(C=t | A=t, B) can be **anything!**

- Parameters grow **exponentially** with the number of nodes!

$$|A_1 + A_2 + \cdots + A_N|^2 = \sum_{i=1}^{N} |A_i|^2 + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} |A_i||A_j| \cos(\theta_i - \theta_j)$$

Bayesian Networks - Quantum

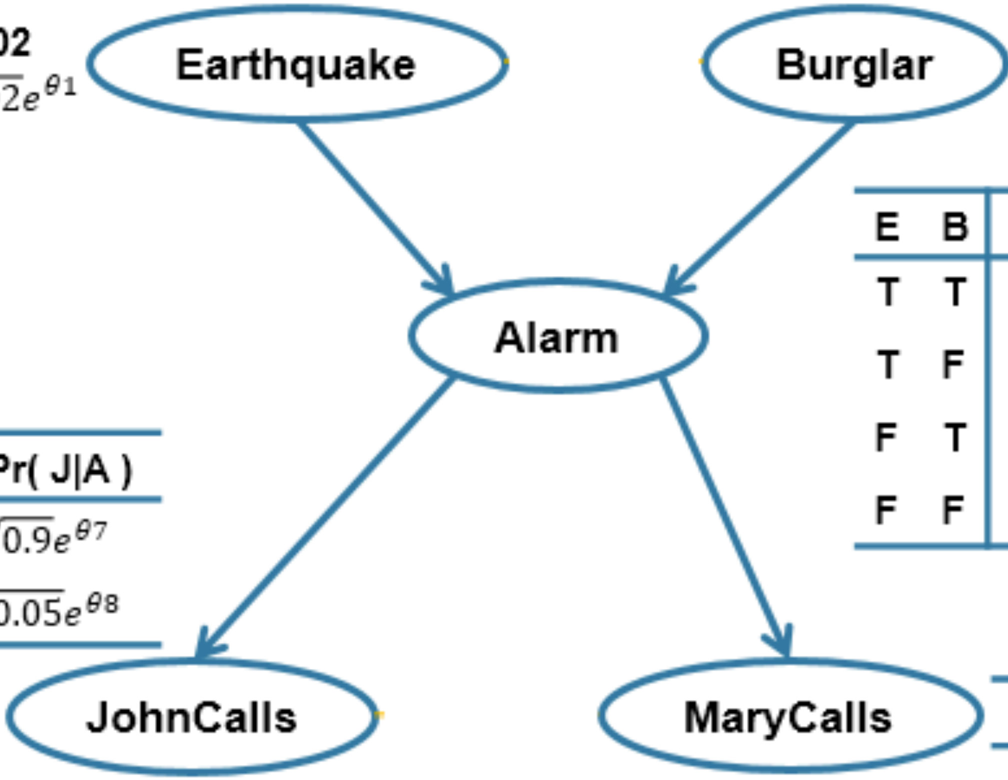
How do we automatically tune quantum parameters under Quantum Bayesian Networks?

Bayesian Networks - Quantum

- The Burglar / Alarm network

Pr(E) = 0.02
QPr(E) = $\sqrt{0.02}e^{\theta_1}$

Pr(B) = 0.01
QPr(B) = $\sqrt{0.01}e^{\theta_2}$



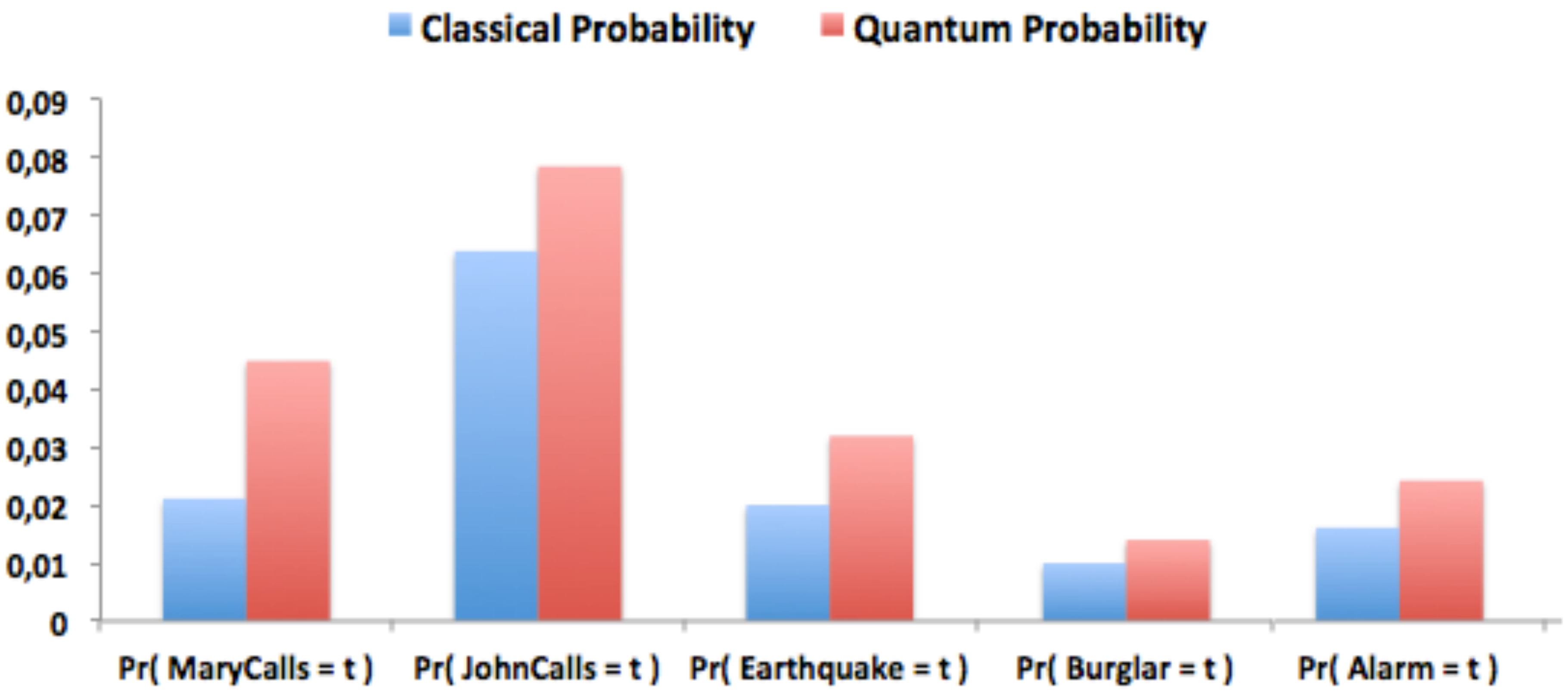
E	B	Pr(A|E, B)	QPr(A|E, B)
T	T	0.9	$\sqrt{0.9}e^{\theta_3}$
T	F	0.3	$\sqrt{0.3}e^{\theta_4}$
F	T	0.2	$\sqrt{0.2}e^{\theta_5}$
F	F	0.01	$\sqrt{0.01}e^{\theta_6}$

A	Pr(J|A)	QPr(J|A)
T	0.9	$\sqrt{0.9}e^{\theta_7}$
F	0.05	$\sqrt{0.05}e^{\theta_8}$

A	Pr(M|A)	QPr(M|A)
T	0.7	$\sqrt{0.7}e^{\theta_9}$
F	0.01	$\sqrt{0.01}e^{\theta_{10}}$

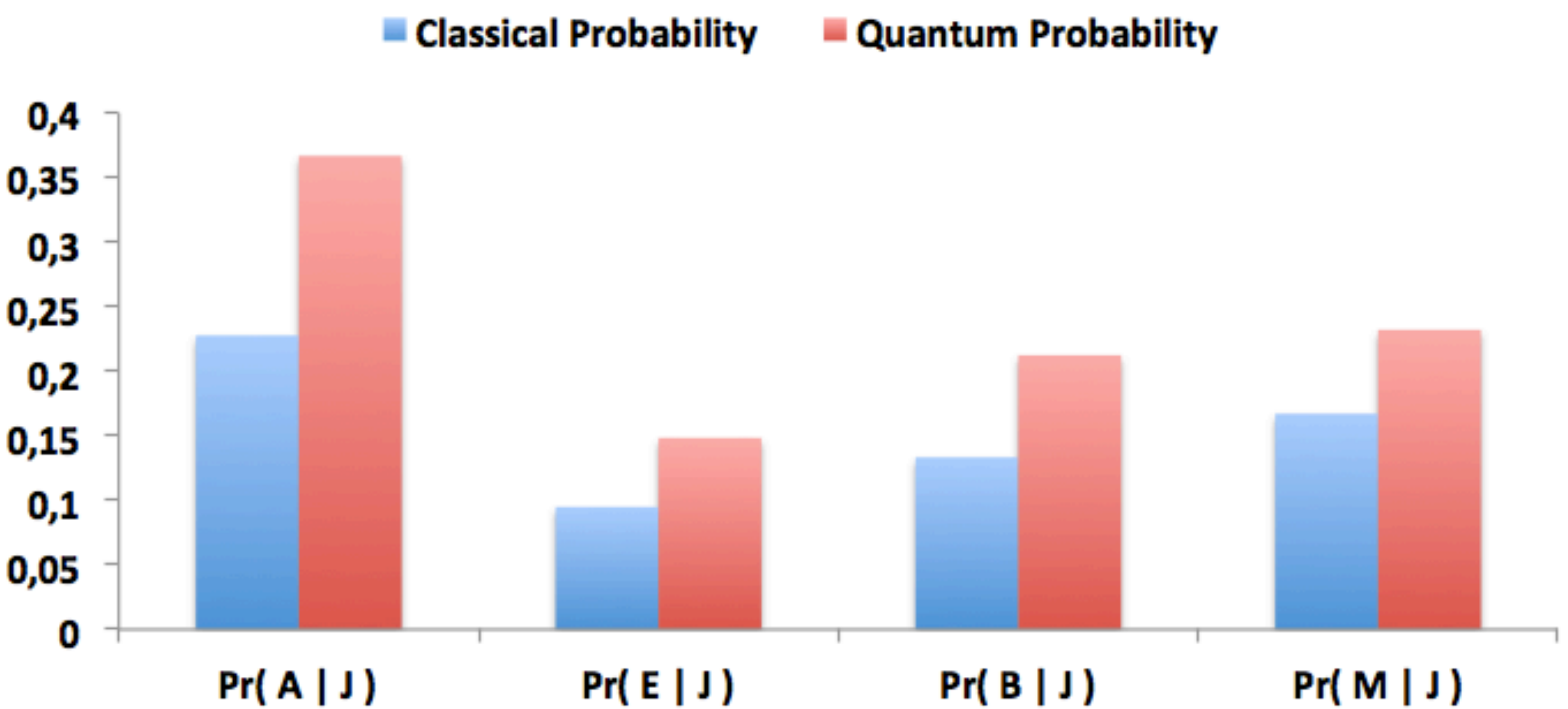
Bayesian Networks - Quantum

- The Burglar / Alarm network



Bayesian Networks - Quantum

- The Burglar / Alarm network



THANK YOU!

QUESTIONS?

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