Propositional Logic – Basic Proofs

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Contents

Introduction

This document is a PDF version of one of my hobbies when I was *extremely* bored at some classes: come up and write proofs using Propositional Logic.

If you have any suggestion to make, please do not hesitate to mail

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Chapter 1

Rules

These are the base rules used in all demonstrations:

1	α	Prem (Premise rule)
2	β	Prem
3	γ	Prem
4	α	Rep, 1 (Repetition rule)
5	$\mid \ \gamma$	Hip (Introduction of Hypothesis)
6	$\mid \ \alpha$	Rei, 1 (Reiteration rule)
7	$\gamma \to \alpha$	I$\to$, (5,6)
8	α	E$\to$, (3,7)
9	$\alpha \land \beta$	I$\land$, (1,2)
10	β	E$\land$, 9
11	$\beta \lor \delta$	I$\lor$, 10
12	$\mid \ \beta$	Hip
13	$\mid \ \omega$	
14	$\mid \ \delta$	Hip
15	$\mid \ \omega$	
16	ω	E$\lor$, ((12, 13), (14, 15))
17	$\mid \ \neg\alpha$	Hip
18	$\mid \ \psi$	
19	$\mid \ \neg\psi$	
20	$\neg\neg\alpha$	I$\neg$, (17, (18,19))
21	α	E$\neg$, 20

Also,

$$\alpha \Leftrightarrow \beta \equiv (\alpha \to \beta) \land (\beta \to \alpha)$$

Chapter 2

The Basics

Three classic laws of thought: https://en.wikipedia.org/wiki/Three_classic_laws_of_thought

$P \lor \neg P$: Law of the excluded middle

Wikipedia page: https://en.wikipedia.org/wiki/Law_of_excluded_middle

1	$\quad \neg(P \lor \neg P)$	Hip
2	$\quad\quad P$	Hip
3	$\quad\quad P \lor \neg P$	I$\lor$, 2
4	$\quad\quad \neg(P \lor \neg P)$	Rei, 1
5	$\quad \neg P$	I$\neg$, (2, (3, 4))
6	$\quad P \lor \neg P$	I$\lor$, 5
7	$\quad \neg(P \lor \neg P)$	Rep, 1
8	$\neg\neg(P \lor \neg P)$	I$\neg$, (1, (6, 7))
9	$P \lor \neg P$	E$\neg$, 8

$P \Leftrightarrow \neg\neg P$: Law of double negation

1	$\mid\ P$	Hip
2	$\mid\ \mid\ \neg P$	Hip
3	$\mid\ \mid\ P$	Rei, 1
4	$\mid\ \neg\neg P$	I$\neg$, (2, (2, 3))
5	$P \rightarrow \neg\neg P$	I$\rightarrow$, (1, 4)
6	$\mid\ \neg\neg P$	Hip
7	$\mid\ P$	E$\neg$, 6
8	$\neg\neg P \rightarrow P$	I$\rightarrow$, (6, 7)
9	$P \Leftrightarrow \neg\neg P$	I$\Leftrightarrow$, (5, 8)

Chapter 3

Stoic Logic

Wikipedia page: https://en.wikipedia.org/wiki/Stoic_logic

Modus Ponens

Wikipedia page: https://en.wikipedia.org/wiki/Modus_ponens

1	$\quad (P \to Q) \land P$	Hip
2	$\quad P \to Q$	E$\land$, 1
3	$\quad P$	E$\land$, 1
4	$\quad Q$	E$\to$, (2, 3)
5	$(P \to Q) \land P \to Q$	I$\to$, (1, 4)

Modus Tollens

Wikipedia page: https://en.wikipedia.org/wiki/Modus_tollens

1	$\quad (P \to Q) \land \neg Q$	Hip
2	$\quad P \to Q$	E$\land$, 1
3	$\quad \neg Q$	E$\land$, 1
4	$\qquad P$	Hip
5	$\qquad Q$	E$\to$, (2, 4)
6	$\qquad \neg Q$	Rei, 3
7	$\quad \neg P$	I$\neg$, (4, (5, 6))
8	$(P \to Q) \land \neg Q \to \neg P$	I$\to$, (1, 7)

Modus Ponendo Tollens

Wikipedia page: https://en.wikipedia.org/wiki/Modus_ponnendo_tollens

1	$(\neg(P \land Q)) \land P$	Hip
2	$\neg(P \land Q)$	E$\land$, 1
3	P	E$\land$, 1
4	$\quad Q$	Hip
5	$\quad P$	Rei, 3
6	$\quad P \land Q$	I$\land$, (4, 5)
7	$\quad \neg(P \land Q)$	Rei, 2
8	$\neg Q$	I$\neg$, (4, (6, 7))
9	$(\neg(P \land Q)) \land P \to \neg Q$	I$\to$, (1, 8)

Modus Tollendo Ponens

Wikipedia page: https://en.wikipedia.org/wiki/Disjunctive_syllogism

1	$(P \lor Q) \land \neg P$	Hip
2	$P \lor Q$	E∧, 1
3	$\neg P$	E∧, 1
4	P	Hip
5	$\neg Q$	Hip
6	P	Rei, 4
7	$\neg P$	Rei, 3
8	$\neg\neg Q$	I¬, (5, (6,7))
9	Q	E¬, 8
10	Q	Hip
11	Q	Rep, 10
12	Q	E∨, (2, (4, 9), (10, 11))
13	$(P \lor Q) \land \neg P \to Q$	I→, (1, 12)

Chapter 4

Paradoxes of material implication

Wikipedia page: https://en.wikipedia.org/wiki/Paradoxes_of_material_implication

$(P \wedge \neg P) \to Q$: Paradox of entailment

1	$\quad P \wedge \neg P$	Hip
2	$\quad\quad \neg Q$	Hip
3	$\quad\quad P$	E$\wedge$, 1
4	$\quad\quad \neg P$	E$\wedge$, 1
5	$\quad \neg\neg Q$	I$\neg$, (2, (3, 4))
6	$\quad Q$	E$\neg$, 5
7	$(P \wedge \neg P) \to Q$	I$\to$, (1, 6)

$$P \to (Q \to P)$$

1	P	Hip
2	$\quad Q$	Hip
3	$\quad P$	Rei, 1
4	$Q \to P$	I$\to$, (2, 3)
5	$P \to (Q \to P)$	I$\to$, (1, 4)

$\neg P \to (P \to Q)$

Wikipedia page: https://en.wikipedia.org/wiki/Principle_of_explosion

1	$\quad \neg P$	Hip
2	$\quad\quad P$	Hip
3	$\quad\quad\quad \neg Q$	Hip
4	$\quad\quad\quad P$	Rei, 2
5	$\quad\quad\quad \neg P$	Rei, 1
6	$\quad\quad \neg\neg Q$	I$\neg$, (3, (4, 5))
7	$\quad\quad Q$	E$\neg$, 6
8	$\quad P \to Q$	I$\to$, (2, 7)
9	$\neg P \to (P \to Q)$	I$\to$, (1, 8)

Alternatively, using the already proven Paradox of Entailment:

1	$\quad \neg P$	Hip
2	$\quad\quad P$	Hip
3	$\quad\quad \neg P$	Rei, 1
4	$\quad\quad P \land \neg P$	I$\land$, (2, 3)
5	$\quad\quad (P \land \neg P) \to Q$	Teo
6	$\quad\quad Q$	E$\to$, (4, 5)
7	$\quad P \to Q$	I$\to$, (2, 6)
8	$\neg P \to (P \to Q)$	I$\to$, (1, 7)

$P \to (Q \lor \neg Q)$

1	$\mid P$	Hip
2	$\mid \mid \neg(Q \lor \neg Q)$	Hip
3	$\mid \mid \mid Q$	Hip
4	$\mid \mid \mid Q \lor \neg Q$	I$\lor$, 3
5	$\mid \mid \mid \neg(Q \lor \neg Q)$	Rei, 2
6	$\mid \mid \neg Q$	I$\neg$, (3, (4, 5))
7	$\mid \mid Q \lor \neg Q$	I$\lor$, 6
8	$\mid \mid \neg(Q \lor \neg Q)$	Rei, 2
9	$\mid \neg\neg(Q \lor \neg Q)$	I$\neg$, (2, (7, 8))
10	$\mid (Q \lor \neg Q)$	E$\neg$, 9
11	$P \to (Q \lor \neg Q)$	I$\to$, (1, 10)

Alternatively, using the Law of the Excluded Middle:

1	$\mid P$	Hip
2	$\mid Q \lor \neg Q$	Teo
3	$P \to (Q \lor \neg Q)$	I$\to$, (1, 2)

$(P \to Q) \vee (Q \to R)$

1	$Q \vee \neg Q$	Teo
2	$\mid Q$	Hip
3	$\mid \mid P$	Hip
4	$\mid \mid Q$	Rei, 2
5	$\mid P \to Q$	I$\to$, (3, 4)
6	$\mid (P \to Q) \vee (Q \to R)$	I$\vee$, 5
7	$\mid \neg Q$	Hip
8	$\mid \mid Q$	Hip
9	$\mid \mid \mid \neg R$	Hip
10	$\mid \mid \mid Q$	Rei, 8
11	$\mid \mid \mid \neg Q$	Rei, 7
12	$\mid \mid \neg\neg R$	I$\neg$, (9, (10, 11)
13	$\mid \mid R$	E$\neg$, 12
14	$\mid Q \to R$	I$\to$, (7, 13)
15	$\mid (P \to Q) \vee (Q \to R)$	I$\vee$, 14
16	$(P \to Q) \vee (Q \to R)$	E$\vee$, (1, (2, 6), (7, 15))

$$\neg(P \to Q) \to (P \land \neg Q)$$

1	$\neg(P \to Q)$	Hip
2	$\quad \neg P$	Hip
3	$\quad \neg P \to (P \to Q)$	Teo
4	$\quad P \to Q$	E$\to$, (2, 3)
5	$\quad \neg(P \to Q)$	Rei, 1
6	$\neg\neg P$	I$\neg$, (2, (4, 5))
7	P	E$\neg$, 6
8	$\quad Q$	Hip
9	$\quad Q \to (P \to Q)$	Teo
10	$\quad P \to Q$	E$\to$, (8, 9)
11	$\quad \neg(P \to Q)$	Rei, 1
12	$\neg Q$	I$\neg$, (8, (10, 11))
13	$P \land \neg Q$	I$\land$, (7, 12)
14	$\neg(P \to Q) \to (P \land \neg Q)$	I$\to$, (1, 13)