Lecture 4: Selection.

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Much of the traditional literature on Truncation, Censoring, and Selection relies heavily on properties of the Normal distribution.

1 Normal Distribution

If X has a Standard Normal distribution its density is

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}x^2}, \quad -\infty < x < \infty.$$

Note that $\phi'(x) = -\phi(x)x$ and $\phi(-x) = \phi(x)$. The associated (cumulative) distribution function is

$$\Pr[X \leq x] = \int_{-\infty}^{x} \phi(t)dt = \Phi(x).$$

Note that $\Phi'(x) = \phi(x)$ and $\Phi(-x) = 1 - \Phi(x)$. Letting $Y = \mu + \sigma X$, we get

$$\Pr[Y \leq y] = \Pr[\mu + \sigma X \leq y] = \Pr[X \leq \frac{y-\mu}{\sigma}] = \int_{-\infty}^{\frac{y-\mu}{\sigma}} \phi(t)dt = \Phi(\frac{y-\mu}{\sigma}).$$

Applying Leibnitz' rule to the second to the integral above, the density of Y is

$$f(y) = \frac{1}{\sigma}\phi(\frac{y-\mu}{\sigma}) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}, \quad -\infty < y < \infty$$

2 Truncated Normal Distribution

Now suppose we condition on $Y \in A = [a_1, a_2]$, where $-\infty < a_1 < a_2 < \infty$. The probability of Y falling into this interval is $\Phi(\frac{a_2-\mu}{\sigma}) - \Phi(\frac{a_1-\mu}{\sigma})$. Thus the conditional density of Y is

$$f(y|A) = \frac{\frac{1}{\sigma}\phi(\frac{y-\mu}{\sigma})}{\Phi(\frac{a_2-\mu}{\sigma}) - \Phi(\frac{a_1-\mu}{\sigma})}, \quad a_1 \leq y \leq a_2$$

We want to derive the mean and variance of this distribution.

2.1 Moment Generating Function

The MGF is

$$M(t) = E[e^{tY}|Y \in A] = \frac{\int_{a_1}^{a_2} e^{ty} f(y)dy}{\Phi(\frac{a_2-\mu}{\sigma}) - \Phi(\frac{a_1-\mu}{\sigma})} = e^{\mu t+\sigma^2t^2/2}\frac{\Phi(\frac{a_2-\mu}{\sigma} - \sigma t) - \Phi(\frac{a_1-\mu}{\sigma} - \sigma t)}{\Phi(\frac{a_2-\mu}{\sigma}) - \Phi(\frac{a_1-\mu}{\sigma})}$$

The last equality follows from

$$\begin{aligned}
\frac{1}{\sigma\sqrt{2\pi}}\int_{a_1}^{a_2} e^{ty}e^{\frac{-1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}dy &= \frac{1}{\sigma\sqrt{2\pi}}\int_{a_1}^{a_2} e^{\frac{-1}{2\sigma^2}\{[y-(\sigma^2t+\mu)]^2-(\sigma^2t+\mu)^2+\mu^2\}}dy \\
&= e^{\frac{-1}{2\sigma^2}[\mu^2-(\sigma^2t+\mu)^2]}\frac{1}{\sigma\sqrt{2\pi}}\int_{a_1}^{a_2} e^{\frac{-1}{2}(\frac{y-\mu'}{\sigma})^2}dy \\
&= e^{\mu t+\sigma^2t^2/2}\int_{a_1}^{a_2}\frac{1}{\sigma}\phi(\frac{y-\mu'}{\sigma})dy \\
&= e^{\mu t+\sigma^2t^2/2}\left[\Phi(\frac{a_2-\mu'}{\sigma}) - \Phi(\frac{a_1-\mu'}{\sigma})\right].
\end{aligned}$$

where $\mu' = \sigma^2 t + \mu$.

2.2 Expected Value

Putting the MGF to work:

$$E[Y|Y \in A] = M'(t)|_{t=0} = \mu - \sigma\frac{\phi(\alpha_2) - \phi(\alpha_1)}{\Phi(\alpha_2) - \Phi(\alpha_1)}.$$

where $\alpha_k = \frac{a_k-\mu}{\sigma}$. Letting a_2 tend to infinity,

$$E[Y|Y > a_1] = \mu + \sigma\frac{\phi(\alpha_1)}{1 - \Phi(\alpha_1)} = \mu + \sigma\lambda(\alpha_1),$$

where $\lambda(\alpha) > 0$ is the hazard function. The hazard function of the Normal distribution is often called the inverse Mills ratio in the micro-econometrics literature.

Letting a_1 tend to minus infinity,

$$E[Y|Y < a_2] = \mu - \sigma\frac{\phi(\alpha_2)}{\Phi(\alpha_2)} = \mu - \sigma\lambda(-\alpha_2).$$

Letting a_2 tend to infinity as well, of course, we get $E[Y] = \mu$. Relative to this non-truncated case, truncation from below raises the mean $E[Y|Y > a_1] > E[Y]$. Truncation from above lowers the mean $E[Y|Y < a_2] < E[Y]$.

2.3 Variance

Putting the MGF to work again:

$$E[Y^2|Y \in A] = M''(t)|_{t=0} = \sigma^2 + \mu^2 + \sigma^2 \frac{\phi'(\alpha_2) - \phi'(\alpha_1)}{\Phi(\alpha_2) - \Phi(\alpha_1)} - 2\mu\sigma \frac{\phi(\alpha_2) - \phi(\alpha_1)}{\Phi(\alpha_2) - \Phi(\alpha_1)}.$$

Therefore,

$$\begin{aligned} Var[Y|Y \in A] &= E[Y^2|Y \in A] - E[Y|Y \in A]^2 \\ &= \sigma^2 \left\{ 1 - \frac{\alpha_2\phi(\alpha_2) - \alpha_1\phi(\alpha_1)}{\Phi(\alpha_2) - \Phi(\alpha_1)} - \left[\frac{\phi(\alpha_2) - \phi(\alpha_1)}{\Phi(\alpha_2) - \Phi(\alpha_1)} \right]^2 \right\} \end{aligned}$$

Letting a_2 tend to infinity,

$$\begin{aligned} Var[Y|Y > a_1] &= \sigma^2 \left\{ 1 + \frac{\alpha_1\phi(\alpha_1)}{1 - \Phi(\alpha_1)} - \left[\frac{\phi(\alpha_1)}{1 - \Phi(\alpha_1)} \right]^2 \right\} \\ &= \sigma^2 \left[1 + \alpha_1\lambda(\alpha_1) - \lambda(\alpha_1)^2 \right] = \sigma^2 \left[1 - \delta(\alpha_1) \right], \end{aligned}$$

where $\delta(\alpha) = \lambda(\alpha)[\lambda(\alpha) - \alpha]$. Notice that $\delta(\alpha) = \lambda'(\alpha)$. It can be shown that $0 < \delta(\alpha) < 1$. [Derivative is $\delta' = \lambda'[\lambda - \alpha] + [\lambda' - 1]\lambda = \lambda[(\lambda - \alpha)^2 + \lambda(\lambda - \alpha) - 1]$. Thus $\delta'(\alpha^*) = 0$ implies $1 > 1 - (\lambda - \alpha^*)^2 = \lambda(\lambda - \alpha^*) = \delta(\alpha^*)$. Since $\lim_{\alpha \to -\infty} \lambda(\alpha)\alpha = 0$ we have $\lim_{\alpha \to -\infty} \delta(\alpha) = 0$. To be completed.]

Letting a_1 tend to minus infinity,

$$\begin{aligned} Var[Y|Y < a_2] &= \sigma^2 \left\{ 1 - \frac{\alpha_2\phi(\alpha_2)}{\Phi(\alpha_2)} - \left[\frac{\phi(\alpha_2)}{\Phi(\alpha_2)} \right]^2 \right\} \\ &= \sigma^2 \left[1 - \alpha_2\lambda(-\alpha_2) - \lambda(-\alpha_2)^2 \right] = \sigma^2 \left[1 - \delta(-\alpha_2) \right]. \end{aligned}$$

Letting a_2 tend to infinity as well, of course, we get $Var[Y] = \sigma^2$. Relative to the non-truncated case, note how the variance shrinks toward zero with truncation either from above or from below.

3 Bivariate Normal

Suppose U_1 and U_2 are independent random variables, each drawn from the Standard Normal density $\phi(u)$.