

INTRODUCTION TO ERROR THEORY

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ABSTRACT

The measurement of a physical quantity can never be made with perfect accuracy, there will always be some error or uncertainty present. For any measurement there are an infinite number of factors that can cause a value obtained experimentally to deviate from the true (theoretical) value. Most of these factors have a negligible effect on the outcome of an experiment and can usually be ignored. However, some effects can cause a significant alteration, or error, in the experimental result. If a measurement is to be useful, it is necessary to have some quantitative idea of the magnitude of the errors. Therefore, when experimental results are reported, they are accompanied by an estimate of the experimental error, called the uncertainty. This uncertainty indicates how reliable the experimenter believes the results to be.

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1 Types of Errors

In order to determine the uncertainty for a measurement, the nature of the errors affecting the experiment must be examined. There are many different types of errors that can occur in an experiment, but they will generally fall into one of two categories: random errors or systematic errors.

1.1 Random errors

Random errors usually result from human and from accidental errors. Accidental errors are brought about by changing experimental conditions that are beyond the control of the experimenter; examples are vibrations in the equipment, changes in the humidity, fluctuating temperatures, etc. Human errors involve such things as miscalculations in analyzing data, the incorrect reading of an instrument, or a personal bias in assuming that particular readings are more reliable than others. By their very nature, random errors cannot be quantified exactly since the magnitude of the random errors and their effect on the experimental values is different for every repetition of the experiment. So statistical methods are usually used to obtain an estimate of the random errors in the experiment.

1.2 Systematic errors

A systematic error is an error that will occur consistently in only one direction each time the experiment is performed, i.e., the value of the measurement will always be greater (or lesser) than the real value. Systematic errors most commonly arise from defects in the instrumentation or from using improper measuring techniques. For example, measuring a distance using the worn end of a meter stick, using an instrument that is not calibrated, or incorrectly neglecting the effects of viscosity, air resistance and friction, are all factors that can result in a systematic shift of the experimental outcome. Although the nature and the magnitude of systematic errors are difficult to predict in practice, some attempt should be made to quantify their effect whenever possible. In any experiment, care should be taken to eliminate as many of the systematic and random errors as possible. Proper calibration and adjustment of the equipment will help reduce the systematic errors leaving only the accidental and human errors to cause any spread in the data. Although there are statistical methods that will permit the reduction of random errors, there is little use in reducing the random errors below the limit of the precision of the measuring instrument.

2 Statistical Methods

When several independent measurements of a quantity are made, an expected result to report for that quantity is represented by the average of the measurements. For a set of experimental data containing N elements, or measurements, given by $\{S_1, S_2, S_3, \dots, S_N\}$, the average \bar{S} , or expectation value $\langle S \rangle$, is calculated using the formula

$$\bar{S} \equiv \frac{1}{N} \sum_{i=1}^N S_i, \quad (2.1)$$

$$= \frac{S_1 + S_2 + S_3 + \dots + S_N}{N}. \quad (2.2)$$

The reason why it is more appropriately called expectation value lies beneath the fact that the average represents the closest approximation that is available to the true value of the quantity being measured. It is sometimes referred to as the best estimate of the true value.

The data $\{S_1, S_2, S_3, \dots, S_N\}$ are dispersed around the mean, or average. A measure of this dispersion is called the standard deviation and is given by

$$\Delta S \equiv \sqrt{\frac{1}{N-1} \left[\sum_{i=1}^N (S_i^2) - N \bar{S}^2 \right]}, \quad (2.3)$$

$$= \sqrt{\frac{(S_1^2 + S_2^2 + S_3^2 + \dots + S_N^2) - N \bar{S}^2}{N-1}}. \quad (2.4)$$

The smaller the standard deviation is the more closely the data is grouped about the mean. If there is a large number of normally distributed points (i.e. Gaussian distribution), statistical analysis shows that about 68.3% of them will fall within the interval between $S - \Delta S$ and $S + \Delta S$.

If the systematic errors have been reduced as far as possible, the random errors will dominate and hence will limit the accuracy of the final result.

Clearly there are two ways to reduce the effect of random errors and improve the accuracy of an experimental result, *i.e.*

- eliminate the majority of the random errors inherent in the experiment;
- obtain as many data points as is reasonable, thereby increasing N , and therefore reducing \bar{S} .

3 Percent Errors

The standard deviation is a measure of the precision of an experiment: the smaller the ΔS , the greater the precision of the best estimate. One way to report the precision of the experimental value is through the use of the percent standard deviation given by

$$\Delta S\% = \frac{\Delta S}{\bar{S}} \times 100\%. \quad (3.1)$$

Unfortunately the average and standard deviation indicate nothing about the accuracy of the measurement, *i.e.* how close the experimental or average value is to the true value. In other words an experiment can yield extremely precise, consistent values without generating a result that is close to the true value (this type of result often occurs when the equipment has not been zeroed or calibrated properly and when other systematic errors have not been properly reduced). In many experiments, it is desirable to indicate the overall accuracy of the final experimental value by reporting some type of percent error. If the quantity measured has a standard or true (theoretical) value, then the accuracy of the experimental value is given by a ratio of the error to the true value:

$$\text{Error}\% = \frac{|\text{Experimental Value} - \text{Theoretical Value}|}{\text{Theoretical Value}} \times 100\%. \quad (3.2)$$

For many experiments, the true value of the quantity being measured is unknown. In this situation, it is often useful to compare two results obtained by different methods so that a percent difference can be obtained. For instance, if the two experimental values are represented by S_1 and S_2 , then the percent difference is defined by

$$|S_1 - S_2|\% \equiv \left| \frac{S_1 - S_2}{\left(\frac{S_1 + S_2}{2}\right)} \right| \times 100\%. \quad (3.3)$$

If an experiment is performed properly, with care taken to reduce the random and systematic errors as much as possible, then the percent errors will be correspondingly small.

The magnitude of the percent errors will depend heavily on the overall precision of the measuring instruments. This means that while in some cases an error of 5% or 10% might be acceptable, in other cases such an error would indicate a very poorly run experiment. Thus the success of an experiment, in terms of a percent error, can only be judged when the method and instrumentation of the experiment are taken into consideration.

4 Propagation of Errors

In many experiments, the quantities measured are not the quantities of final interest. Since all measurements have uncertainties associated with them, clearly any calculated quantity will have an uncertainty that is related to the uncertainties of the direct measurements. The procedure used to estimate the error for the calculated quantities is called the propagation of errors.

Consider the general case first. Suppose the variables A, B, C, \dots represent independent measurable quantities that will be used to obtain a value for some calculated quantity U . Since U is a function of A, B, C, \dots , it can be written as $U = f(A, B, C, \dots)$. The measurements of the quantities A, B, C, \dots yield estimates of the expectation values written as $\bar{A}, \bar{B}, \bar{C}, \dots$ and the associated uncertainties $\Delta A, \Delta B, \Delta C, \dots$ for each variable. To find the expectation value, or best estimate, for the quantity U , the expectation value of each measured variable is substituted into the equation for U :

$$\bar{U} = f(\bar{A}, \bar{B}, \bar{C}, \dots). \quad (4.1)$$

If the errors for A, B, C, \dots are independent, random, and sufficiently small, it can be shown that the uncertainty for U is given by

$$\Delta U = \sqrt{\left(\frac{\partial U}{\partial A}\right)^2 (\Delta A)^2 + \left(\frac{\partial U}{\partial B}\right)^2 (\Delta B)^2 + \left(\frac{\partial U}{\partial C}\right)^2 (\Delta C)^2 + \dots}, \quad (4.2)$$

where the partial derivatives are evaluated using the expectation values $\bar{A}, \bar{B}, \bar{C}, \dots$ as the values for the independent variables. The correct notation to express the final estimate for the calculated quantity U is given by

$$U = \bar{U} \pm \Delta U. \quad (4.3)$$

In the next three sections, §4.1-§4.3, we lighten our notation removing the "bar" signs and considering A, B, C, \dots as the expectation values for calculation purposes only. We restore the original notation in the examples given in §5.

4.1 Addition and Subtraction

Suppose two quantities A and B are added and the uncertainties associated with each variable are ΔA and ΔB . It follows that

$$U = A + B \quad \longrightarrow \quad \frac{\partial U}{\partial A} = 1, \quad \frac{\partial U}{\partial B} = 1, \quad (4.4)$$

where equation (4.2) gives

$$\Delta U = \sqrt{(\Delta A)^2 + (\Delta B)^2}. \quad (4.5)$$

If the two values are subtracted then

$$U = A - B \quad \longrightarrow \quad \frac{\partial U}{\partial A} = 1, \quad \frac{\partial U}{\partial B} = -1, \quad (4.6)$$

and equation (4.2) gives same result as in the case of addition, *i.e.*

$$\Delta U = \sqrt{(\Delta A)^2 + (\Delta B)^2}. \quad (4.7)$$

Hence these equations can be generalized for any combination of addition and subtraction of any number of variables, *i.e.*

$$U = \pm A \pm B \pm C \pm \dots, \quad (4.8)$$

$$\Delta U = \sqrt{(\Delta A)^2 + (\Delta B)^2 + (\Delta C)^2 + \dots}. \quad (4.9)$$

4.2 Multiplication and Division

Suppose two quantities A and B are multiplied and the uncertainties associated with each variable are ΔA and ΔB . It follows that

$$U = AB \quad \longrightarrow \quad \frac{\partial U}{\partial A} = B, \quad \frac{\partial U}{\partial B} = A, \quad (4.10)$$

where equation (4.2) gives, after some algebraic re-arrangement,

$$\Delta U = U \sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2}. \quad (4.11)$$

Likewise if the two values are divided then

$$U = \frac{A}{B} \quad \longrightarrow \quad \frac{\partial U}{\partial A} = \frac{1}{B}, \quad \frac{\partial U}{\partial B} = -\frac{A}{B^2}, \quad (4.12)$$

and equation (4.2), after some algebraic re-arrangement, gives same result as in the case of multiplication, *i.e.*

$$\Delta U = U \sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2}. \quad (4.13)$$

Hence these equations can be generalized for any combination of multiplication and division of any number of variables, *i.e.*

$$U = \frac{AB \dots}{XY \dots}, \quad (4.14)$$

$$\Delta U = U \sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2 + \dots + \left(\frac{\Delta X}{X}\right)^2 + \left(\frac{\Delta Y}{Y}\right)^2 + \dots}. \quad (4.15)$$

4.3 Exponents and Roots

Suppose that a calculation involves the use of exponents or roots of a quantity A whose uncertainty is ΔA . It follows in the most general case that, $\forall \alpha \in \mathbb{R}$,

$$U = k A^\alpha, \quad (4.16)$$

$$\Delta U = k \alpha A^{\alpha-1} \Delta A. \quad (4.17)$$

5 Examples

5.1 Example: The Estimate of the Table Top Surface Area

The rectangular dimensions A and B of a table top are determined directly by estimating the left and right reading of each dimension as shown in Fig.1, where the error, or uncertainty, is given by half of the smallest reading unit on the meter stick. The smallest unit, in this example, is 1 mm , and therefore the uncertainty in reading is 0.5 mm .

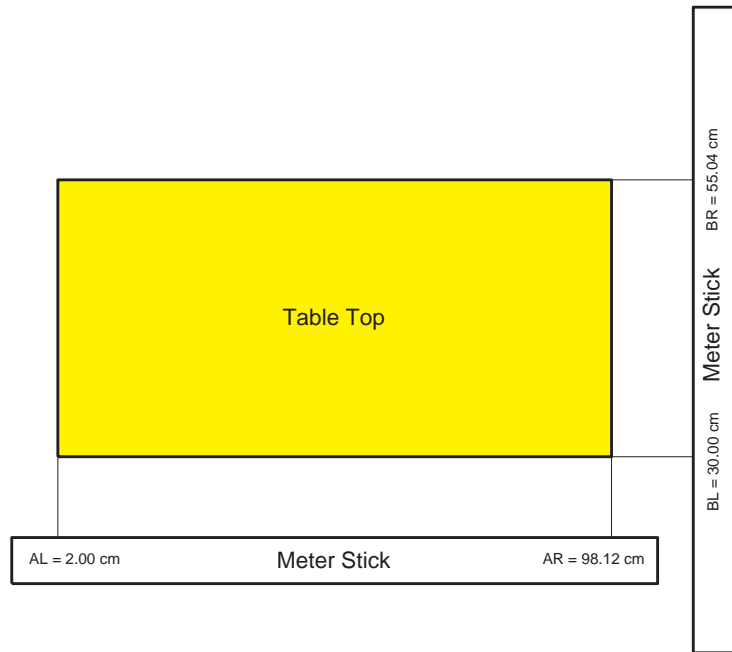


Figure 1: The smallest unit on the meter stick is 1 mm .

Let left and right readings of A and B be respectively (see Fig.1)

$$AL = 2.00 \pm 0.05\text{ cm}, \quad AR = 98.12 \pm 0.05\text{ cm}, \quad (5.1)$$

$$BL = 30.00 \pm 0.05\text{ cm}, \quad BR = 55.04 \pm 0.05\text{ cm}. \quad (5.2)$$

It is easy to see that the two dimensions A and B , and their uncertainties ΔA and ΔB , are determined by equations (4.8) and (4.9) applied to the left and right reading values. Indeed, we have

$$\bar{A} = 98.12 - 2.00 = 96.12 \text{ cm}, \quad \Delta A = \sqrt{(0.05)^2 + (0.05)^2} \simeq 0.07 \text{ cm}, \quad (5.3)$$

$$\bar{B} = 55.04 - 30.00 = 25.04 \text{ cm}, \quad \Delta B = \sqrt{(0.05)^2 + (0.05)^2} \simeq 0.07 \text{ cm}, \quad (5.4)$$

We usually write the above results in a more appropriate form for an estimate as follows

$$A = 96.12 \pm 0.07 \text{ cm}, \quad B = 25.04 \pm 0.07 \text{ cm}. \quad (5.5)$$

The next step is to calculate the expectation value for the surface area $S = AB$ and the associate uncertainty ΔS as provided by equation (4.15), *i.e.*

$$\bar{S} = 96.12 \times 25.04 \simeq 2407 \text{ cm}^2, \quad (5.6)$$

$$\Delta S = 2407 \sqrt{\left(\frac{0.07}{96.12}\right)^2 + \left(\frac{0.07}{25.04}\right)^2} \simeq 7 \text{ cm}^2. \quad (5.7)$$

And again, this can be written in a more compact form, *i.e.*

$$S = 2407 \pm 7 \text{ cm}^2. \quad (5.8)$$

5.2 Example: The Estimate of the Acceleration due to Gravity

Suppose the acceleration due to gravity can be calculated from experimental data using the following equation

$$g = \frac{2x\ell}{ht^2}, \quad (5.9)$$

where the estimate measurements are given by

$$\begin{aligned} x &= \bar{x} \pm \Delta x = 74.11 \pm 0.05 \text{ cm}, \\ \ell &= \bar{\ell} \pm \Delta \ell = 100.1 \pm 0.2 \text{ cm}, \\ h &= \bar{h} \pm \Delta h = 1.10 \pm 0.1 \text{ cm}, \\ t &= \bar{t} \pm \Delta t = 3.708 \pm 0.003 \text{ s}. \end{aligned} \quad (5.10)$$

Hence $\bar{g} \simeq 981 \text{ cm s}^{-2}$. Instead, the uncertainty here is compound and therefore given by the combination of equation (4.15) and (4.17), *i.e.*

$$\Delta g = \bar{g} \sqrt{\left(\frac{\Delta x}{\bar{x}}\right)^2 + \left(\frac{\Delta \ell}{\bar{\ell}}\right)^2 + \left(\frac{\Delta h}{\bar{h}}\right)^2 + \left(\frac{\Delta(t^2)_{t=\bar{t}}}{\bar{t}^2}\right)^2}, \quad (5.11)$$

$$= \bar{g} \sqrt{\left(\frac{\Delta x}{\bar{x}}\right)^2 + \left(\frac{\Delta \ell}{\bar{\ell}}\right)^2 + \left(\frac{\Delta h}{\bar{h}}\right)^2 + \left(\frac{2 \Delta t}{\bar{t}}\right)^2}, \quad (5.12)$$

$$= 981 \sqrt{\left(\frac{0.05}{74.11}\right)^2 + \left(\frac{0.2}{100.1}\right)^2 + \left(\frac{0.1}{1.10}\right)^2 + \left(\frac{0.006}{3.708}\right)^2}, \quad (5.13)$$

$$\simeq 9 \text{ cm s}^{-2}. \quad (5.14)$$

This can be written in a more compact form, *i.e.*

$$g = 981 \pm 9 \text{ cm s}^{-2}. \quad (5.15)$$

Acknowledgements

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