Zvi Drezner
Horst W. Hamacher
Editors

Facility Location

Applications and Theory

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Zvi Drezner · Horst W. Hamacher
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With 60 Figures and 20 Tables

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Contents

1 The Weber Problem*

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1.1 Introduction

The Weber problem discussed in this chapter has a long and convoluted history. Many players, from many fields of study, stepped on its stage, and some of them stumbled. The problem seems disarmingly simple, but is so rich in possibilities and traps that it has generated an enormous literature dating back to the seventeenth century, and continues to do so. Many of the people writing on this problem and its variations have had a basic difficulty: what to call it. As can be seen by perusing the references, some of the many names that have been used are: the Fermat problem, the generalized Fermat problem, the Fermat-Torricelli problem, the Steiner problem, the generalized Steiner problem, the Steiner-Weber problem, the Weber problem, the generalized Weber problem, the Fermat-Weber problem, the one median problem, the median center problem, the minisum problem, the minimum aggregate travel point problem, the bivariate median problem, and the spatial median problem.

The main object of this chapter is not a comprehensive history but rather an attempt to put into perspective the efforts of many people in different disciplines who struggled with various versions of this problem, often unaware that others had gone before them. Rather than being a drawback, the parallelism of these many efforts is not only a tribute to the enduring importance of this problem in several fields but has also resulted in a great variety of clever and inventive methods. The problem has given rise to an extraordinary number of generalizations, extensions and modifications. It would literally require volumes to do them justice; space permits only a brief and somewhat arbitrarily selected summary. Reviews of the Weber problem can be found in Wesolowsky (1993), Love et al. (1988) and Francis et al. (1992).

* Part of this chapter is based on the paper by Wesolowsky (1993). All excerpts from that paper are included with permission from Elsevier Science.

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1.1.1 Definition of the Weber Problem

We are to find the "minisum" point $(x^*,\ y^*)$ which minimizes the sum of weighted Euclidean distances from itself to n fixed points with co-ordinates $(a_i,\ b_i)$. The weights which are associated with the fixed points are denoted by w_i. One simple (and simplistic) scenario for the problem is that we wish to locate a warehouse and that the weights w_i are the costs per unit distance of shipping the requirements to customers located at the fixed points $(a_i,\ b_i)$; $(x^*,\ y^*)$ is then the warehouse location that minimizes the transportation cost. One can also view $(x^*,\ y^*)$ as a two-dimensional generalization of the simple (one-dimensional) median of n weighted values, and hence the name "spatial median".

The problem can be stated as:

$$\min_{x,y} \left\{ W(x,y) = \sum_{i=1}^{n} w_i d_i(x,y) \right\} \tag{1.1}$$

where $d_i(x,y) = \sqrt{(x-a_i)^2 + (y-b_i)^2}$ is the Euclidean distance between $(x,\ y)$ and $(a_i,\ b_i)$.

1.1.2 The Dual Formulation

A different approach to finding the location of the minisum or spatial median point (x^*, y^*) is to solve a problem that is dual to (1.1). For a discussion see Scott et al. (1995). Consider the programming problem:

$$\max_{U,V} \left\{ D(U,V) = -\sum_{i=1}^{n} (a_i u_i + b_i v_i) \right\} \tag{1.2}$$

subject to:

$$\sum_{i=1}^{n} u_i = 0$$

$$\sum_{i=1}^{n} v_i = 0$$

$$\sqrt{u_i^2 + v_i^2} \leq w_i$$

The maximum value of D in (1.2) is equal to the minimum value of W in (1.1). At optimality, the vectors $(u_i, v_i)^T$ "point" from the fixed points (a_i, b_i) to the minisum point (x^*, y^*), and thus any two such vectors can be used to solve for its location. A geometrical solution to the weighted three-point Weber problem and a historical review of the dual problem is found in Martini (1996).

1.2 History and Literature Review

The following is a brief history of the problem of finding the spatial median, (the minisum Euclidean distance point). W. Kuhn (1967) provided an excellent historical sketch. One of his sources was an article by M. Zacharias (1913). Other sources providing details of early solutions are Pottage (1983), Honsberger (1973), and Dörrie (1965).

Who actually first proposed the problem, or in what form, will probably never be known. It is usual to credit Pierre de Fermat (1601-1665) with proposing a basic form of the spatial median problem by issuing the challenge (Kuhn, 1967) "let he who does not approve of my method attempt the solution of the following problem: given three points in the plane, find a fourth point such that the sum of its distances to the three given points is a minimum". It is also usual to credit the Italian mathematician and student of Galileo, Evangelista Torricelli (1608-1647) with the solution (for details see Section 1.3.1 below). However, as usual, the history of the problem is a bit murky. Other mathematicians also worked on the problem at the time. Pottage (1983) mentions treatments by Cavalieri, Viviani and Roberval. Torricelli himself had several methods for this problem; see Honsberger (1973), and Dörrie (1965).

Not everyone credits Fermat with originating the problem. Melzak (1983) says "This problem was proposed and solved by the Italian mathematician Battista Cavalieri (1598-1647); then it was proposed by the French mathematician Pierre Fermat (1601- 1655) and solved again by the Italian scientist Evarista Torricelli (1608-1647)." Zacharias (1913), on the other hand, credits Torricelli with both posing and solving the problem; hence the name "Torricelli point" (see Section 1.3.1). Other geometrical solutions and "rediscoveries" continued into the twentieth century (Honsberger, 1973).

In 1647, Cavalieri's "Exerciones Geometricae" showed that the three lines joining the Torricelli point to the vertices form angles of 120^o with each other. Another geometrical method of finding this "unweighted median" point was given by Simpson (Thomas Simpson (1710-1761)) in the "Doctrine and Application of Fluxions" (London 1750). (For details see Section 1.3.2 below.) Smith (1923) calls him "that strange mathematical genius". Simpson also suggested, as an exercise, generalizing the problem to include different weights.

The dual problem (see for example, Scott et al., 1995) also has early origins. The use of Simpson lines is already an implicit use of the dual. The Ladies Diary or Woman's Almanack (1755) contains the problem: "In the three sides of an equiangular field stand three trees, at the distances of 10, 12, and 16 chains from one another. To find the content of the field, it being the greatest the data will admit of?". This geometrical problem was stated in a more academic manner in Annales de Mathematiques Pures et Appliques,Vol. I (1810-11), on page 384, as "given any triangle, circumscribe the largest possible equilateral triangle about it."

It should be noted that for $n = 3$, finding the spatial median is equivalent to finding the shortest network (tree) spanning three points (see Winter, 1985). This latter problem has been popularized by Courant and Robbins (1941) as the Steiner (Jacob Steiner (1796-1863)) problem. However, as Kuhn (1967) says, "Although this very gifted geometer (Steiner) of the 19th century can be counted among the dozens of mathematicians who have written on the subject, he does not seem to have contributed anything new, either to its formulation or its solution."

In the twentieth century, the problem passed to those who claimed there was a use for it. Alfred Weber (1909) used a weighted three point version of the problem to depict industrial location minimizing transport cost; the three fixed points were two sources of materials with different weights and a weighted market location ("place of consumption") respectively. A mathematical appendix to his book, written by Georg Pick, gives a geometrical construction procedure. Pick refers to "an old apparatus which was invented by Varignon (see Section 1.3.4)", and uses the mechanical analogue for explanation as well as suggesting it for the solution of problems with $n \geq 3$. Tellier (1972) also obtained an explicit solution using trigonometry to find the optimum location, as well as discussing the conditions under which one of the points is optimum.

Very shortly after the English translation of Weber's book appeared, Eels (1930) published an article about the "unfortunate error" that "had repeatedly occurred in various publications of the United States Census Bureau" and "from this source ... spread to various books on sociology and population problems, and seems to have remained unchallenged for almost twenty years". Later authors like Schärlig (1973) have continued the lesson. It seems that the Census Bureau had been calculating the center of gravity for populations but attributing to it the property of the "point of minimum aggregate travel", which is the spatial median. Eels expounded on this error at considerable length, and then offered a solution of his own to the three point problem; however these points were restricted to form an isosceles triangle.

Eels' article unleashed a volume of correspondence so large that the editor, Ross (1930), had to abandon the "usual procedure to publish (the correspondence) in full". Professor Corrado Gini of the University of Rome and President of the Central Institute of Statistics of the Kingdom of Italy wrote referring to his article with Galvani (1929). He said that the problem had been fully discussed in his article and the Census errors noted. Also, contrary to Eels, he pointed out that "the designation of 'median point' should properly be given to the point of minimum aggregate travel, and not to the intersection of two orthogonal lines, which is not invariant with respect to the system of co-ordinates." This variation in definitions of median still continues. Hall (1988) says that "in two dimensions, the median minimizes the average rectangular distance and the mean minimizes the average squared distance". Professor Gini would have been upset.

The correspondence is also notable in that a Professor E.B. Wilson resolved the spatial median problem with $n = 3$ and references were given to yet more such re-discoveries. Even more intriguing was the mention of a Mr. Douglas E. Scates of Cincinnati Ohio, who "with his associates seem to be making progress toward establishing a working model for a general population". Scates (1933) did publish a method (without associates) in Metron; it used essentially trial and error. Other approximation (as opposed to iteration) methods were developed at the time.

It was Endre Vaszonyi Weiszfeld (1936) (a Hungarian mathematician, who wrote in French in a Japanese journal), now known as Andrew Vaszonyi, who provided a practical method for finding the spatial median or the Euclidean minisum point for large n and unequal weights (see Section 1.3.5 for details). This method is the iteration procedure (see equation (1.4)) which is the trick of partially separating out the point (x, y) from the extremum equations and using the result in an iterative way to improve the solution. His method, uniquely suited to the computer age, lay dormant and unknown until a series of rediscoveries in the late fifties and early sixties.

This method was first rediscovered by Miehle (1958), who was dealing with a more complex problem of link length minimization arising out of the Steiner problem. Miehle also has interesting photos of analogue machines. The most complete treatment was by Kuhn and Kuenne (1962), who gave the necessary and sufficient conditions for the optimum to be at a fixed point. A more recent paper on this topic is Juel and Love (1986). The same procedure was again proposed by Cooper (1963), who used it as part of his algorithm for location-allocation, and who, like Miehle, borrowed the basic idea from numerical analysis.

Ostresh (1978a, 1978b) defined a slightly different step when an iteration falls on a fixed point. Katz (1974) showed that local convergence is generally linear. An extensive discussion of convergence was given by Morris (1981) and Brimberg (1989). An accelerated Weiszfeld algorithm was proposed by Drezner (1992, 1996). There have been many additional studies of the Weiszfeld algorithm when modified to apply to variations of the spatial median problem.

In the late sixties, computer programs for the optimization of non-linear functions started to be readily available in great abundance. Furthermore, even though its derivatives do not exist at the fixed points, $W(x, y)$ is convex (Love, 1967). However, the Weiszfeld iteration procedure is simple, elegant, and improves the solution at each step; it continued to be worked on.

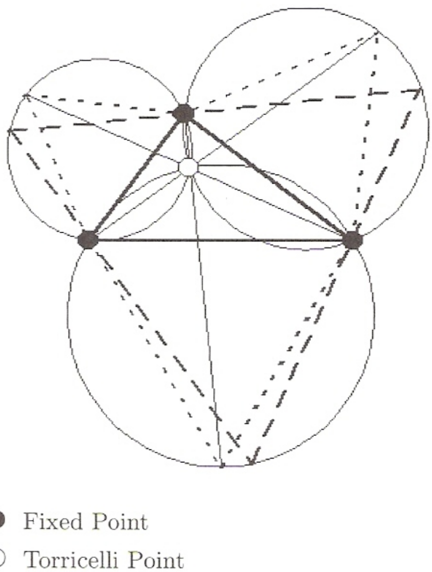
Applications of the spatial median problem have been both of the conjectural and the actual kind. Unfortunately, the problem is an integral part of more complex problems in many fields and therefore its applications are often "buried" and are not easily found by researchers from outside the field. Compounding this difficulty is the fact that there is a virtually perverse lack of consistency in naming the problem. We already discussed its incarnation as the "minimum of aggregate travel" among demographers and geographers, and its appearance in the economic theory of industrial location. Riveline (1967) reported the use of a version of the Varignon Frame as part of the analysis of optimum gallery location in French coal mines. Burstall, Leaver and Sussams (1962) used a simple Varignon Frame as an aid to the location of factories in London. It can be used in cluster analysis (Cooper, 1973) and in related statistical techniques. Overton (1983) gives several references for the problem's use in the physical application of discretizing minimal surfaces. Ostresh (1977) writes "it has application to the siting of steel mills and schools, houses of ill repute and hospitals." It is not clear if Ostresh was familiar with case studies on all of these applications.

1.3 Solution Procedures

1.3.1 Torricelli Point

A solution to the unweighted $n = 3$ problem attributed to Torricelli is as follows. The three points are joined by lines to make a triangle. Equilateral triangles are constructed on the sides with the vertices pointing outward. The three circles through the vertices of the equilateral triangles intersect in the spatial median point, which is labeled the "Torricelli Point" (sometimes called "Fermat point") in the example in Figure 1.1.

Fig. 1.1. Geometrical constructions on a problem with equal weights



1.3.2 The Simpson Lines

Simpson suggested that the outside vertices of the equilateral triangles proposed by Torricelli are used by joining these vertices to the opposite fixed points. These are known as Simpson Lines. The Simpson lines intersect at the minisum (Torricelli) point. See the example in Figure 1.1. Simpson lines are not to be confused with "Simson lines", named after R. Simson (1687-1768), who, to make matters worse, was not, according to Coxeter (1969), their discoverer.

Both the solutions by Torricelli and Simpson apply when the triangle formed by the fixed points has no angle greater than 120^o; if an angle were greater than 120^o they would give a point outside the triangle. Such a point can not be optimal. In this case, the fixed point associated with the angle which is $\geq 120^o$ is the optimum location.

1.3.3 The Dual Problem

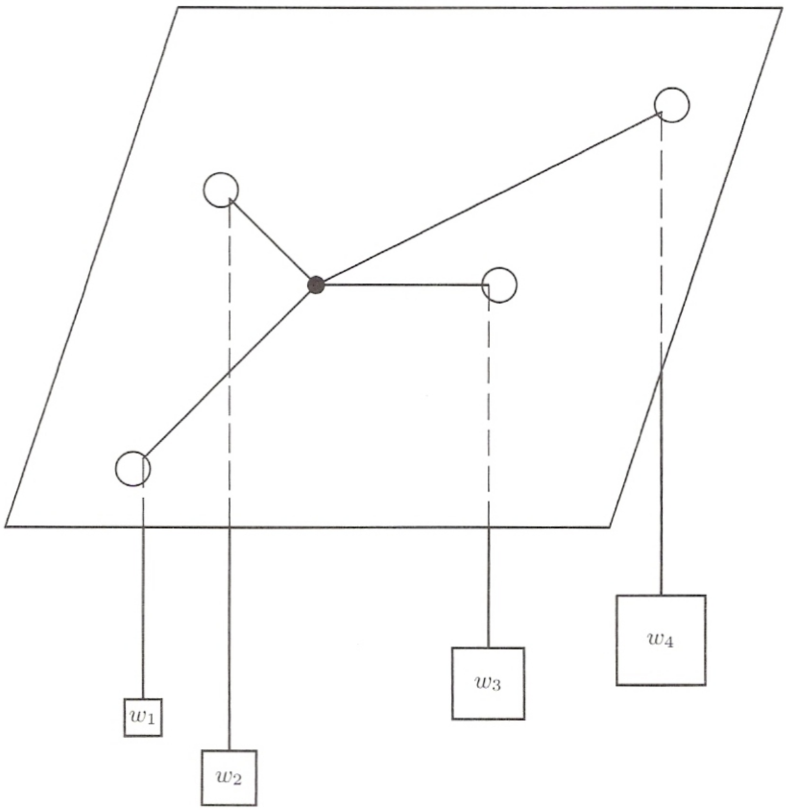
The dual problem is to find the largest equilateral triangle circumscribing a given triangle. The solution was given in Volume II (1811-12) by Rochat et al. (1811): "Thus the largest equilateral triangle circumscribing a given triangle has sides perpendicular to the lines joining the vertices of the given triangle to the point such that the sum of the distances to these vertices is a minimum ... One can conclude that the altitude of the largest equilateral triangle that can be circumscribed about a given triangle is equal to the sum of distances from the vertices of the given triangle to the point at which the sum of distances is a minimum." This problem is therefore a version of the dual for three unweighted fixed points (for this history of the dual, we are indebted to Kuhn (1976)). The equilateral triangle at optimality is given by the heavy-dotted triangle in Figure 1.1. It should be mentioned that this equilateral triangle was already incorporated into the constructions of Torricelli (Honsberger, 1973).

The formal statement of the dual dates back to just after the rediscoveries of the Weiszfeld procedure. Witzgall (1964) and Kuhn (1967) independently stated the problem in essentially the form in (1.2). White (1976) provided a Varignon Frame interpretation of the dual. Guccione and Gillen (1991) wrote a note on an economic interpretation of the dual wherein a transportation authority maximizes revenue and distances become rectilinear. Many other papers on the dual have appeared in the context of various generalized versions of our problem.

1.3.4 The Varignon Frame

Varignon proposed a mechanical analogue device which has actually been used in practice. The device yields valuable insights into the problem. Figure 1.2 shows a diagram of the device, which is called a Varignon Frame. A board is drilled with n holes corresponding to the co-ordinates of the n fixed points; n strings are tied together in a knot at one end, the loose ends are passed, one each, through the holes, and are attached to physical weights below the board which have the same magnitudes as the constants w_i. If the device were not subject to the ills of the physical world and there were no friction, the strings were infinitely thin, the holes infinitely small, and so on, then the final position of the knot would be at the optimum point $(x^*,\ y^*)$.

Fig. 1.2. The Varignon Frame



Why is the optimum point at the knot? Consider the force component, in the x direction, exerted by a single weight w_i on the knot in Figure 1.2; the knot is at co-ordinates (x, y). This force component in the x direction will be $\frac{w_i(x-a_i)}{d_i(x,y)}$ and will be to the left (have a negative sign) if $a_i > x$. It is evident that the sum of such components from all the weights is equal to the first expression in (1.3). Similarly, the sum of components in the negative y

direction is the second expression in (1.3). The resultant force vector exerted on the knot by all the strings is therefore zero at the same point (x^*, y^*) where conditions (1.3) are satisfied.

Note that if any one of the holes on the Varignon Frame is moved "out" on the line of its string with the knot, the optimum point is not affected; this is analogous to the property of the simple median that the points which do not coincide with the median can be "stretched out" or interchanged on any one side at a time without affecting the median.

The Varignon Frame provides an interpretation of the dual variables at (u^*, v^*): if the solution is not at a fixed point i, the vector $(u_i, v_i)^T$ is the negative of the force vector exerted on the knot by weight w_i.

1.3.5 The Weiszfeld Algorithm

The simplest and most commonly used technique to solve the Weber problem is called the "Weiszfeld procedure" (Weiszfeld, 1936).

If we differentiate and set the partial derivatives equal to zero to obtain the first order conditions for optimality we have:

$$\frac{\partial W(x,y)}{\partial x} = \sum_{i=1}^{n} \frac{w_i(x-a_i)}{d_i(x,y)} = 0$$
$$\frac{\partial W(x,y)}{\partial y} = \sum_{i=1}^{n} \frac{w_i(y-b_i)}{d_i(x,y)} = 0 \tag{1.3}$$

It can be shown that $W(x, y)$ is convex, so that (1.3) define a minimum. However, it is immediately evident that these derivatives do not exist when (x, y) coincides with fixed point i, because then $d_i(x, y) = 0$. Equations (1.3) can not, in general, be solved explicitly for (x, y) if $n > 3$.

We can extract x from the first equation in (1.3) if we ignore its presence in $d_i(x, y)$ and we can do the same for y from the second equation in (1.3). The result can be formulated as an iterative procedure if we consider the extracted (x, y) to be a new iteration $(k + 1)$ and the (x, y) trapped in the distance term to be the old iteration (k). To be specific:

$$\left(x^{(k+1)}, y^{(k+1)}\right) = \left(\frac{\sum_{i=1}^{n} \frac{w_i a_i}{d_i(x^{(k)},y^{(k)})}}{\sum_{i=1}^{n} \frac{w_i}{d_i(x^{(k)},y^{(k)})}}, \frac{\sum_{i=1}^{n} \frac{w_i b_i}{d_i(x^{(k)},y^{(k)})}}{\sum_{i=1}^{n} \frac{w_i}{d_i(x^{(k)},y^{(k)})}} \right) \tag{1.4}$$

In other words, once we have a point $(x^{(k)}, y^{(k)})$, we can obtain the next, and hopefully better, point by substituting in the right hand side of (1.4).The trick of partially isolating the solution variables for the purpose of obtaining an iterative solution method is well known in modern numerical analysis and belongs to the class of procedures known as one-point iteration methods (Dahlquist and Björck, 1974, Chapter 6.) The term "one point" arises because only the current point is used to determine the succeeding one.

The Weiszfeld procedure has its quirks. Kuhn (1973) showed that it will fail if the iteration falls on a fixed point; also see Chandrasekaran and Tamir (1989). The following simple problem with five points gives the Weiszfeld algorithm a very hard time. Four points, each with a weight of 1, are placed at the corners of a square of side 1 centered at (0,0), and a fifth point (which is the optimal solution) with a weight of 4 is placed at (100,0). Applying the Weiszfeld algorithm on this problem starting at (50,0) reaches (90.44,0) after 100,000 iterations, (97.447,0) after 200,000 iterations, and (99.999887,0) (not even an accuracy of 10^{-4}) after a million iterations.

1.3.6 Other Iterative Methods

Austin (1959) started with a general n but with equal weights in the "point of minimum aggregate distance" tradition. He obtained the iteration equations but used them in a different way. He noted that for any starting point, they showed a centroid solution with weights inversely proportional to the distances. He then suggested that they be regarded as points projected from the starting point onto a circle of arbitrary radius around the starting point. The co-ordinates on the circle could be read graphically and their centroid point could be found. The next iteration would be halfway between this centroid point and the previous point. Austin briefly suggested a generalization to arbitrary weights (which was slightly incorrect). Seymour (1970) compared this method computationally with the Weiszfeld algorithm. However, it can be readily shown that this is a simple gradient method with fixed step size. While the Weiszfeld procedure is also gradient descent (Cooper and Katz, 1981), it has a variable step size of $1/\sum \frac{w_i}{d_i(x^{(k)},y^{(k)})}$.

Porter (1963) proposed a rather interesting method for solving for the point of minimum aggregate travel (with equal weights); he stated that the point lay between the centroid and a line perpendicular to the bisector separating the distribution into equal halves. Court (1964) showed that this method was incorrect. In his reply to Court, Porter (1964) wrote "Arnold Court has chased the point of minimum travel back into hiding, where it lies convulsed in helpless laughter at our inability to pin it down". However, with a little help from a computer, it could now indeed be pinned down.

There have also been numerous other iteration schemes proposed. Convergence of the Weiszfeld procedure is known to be slow in the vicinity of fixed points. To use standard nonlinear minimization techniques, one could eliminate the problem with derivatives at the fixed points by using a hyperbolic approximation; an example is $d_i^H(x,y) = \sqrt{(x-a_i)^2 + (y-b_i)^2 + \epsilon}$ where ϵ is very small (see Wesolowsky and Love (1972), Eyster, White and Wierwille

(1973)). Drezner (1986) proposed an interpretation to the value of ϵ. If the point has an area, then the average distance between the facility to a demand point resembles $d_i^H(x, y)$ with ϵ being proportional to the area of the demand point. All orders of derivatives are now always continuous. This approximation can be used with standard unconstrained optimization packages but can also be easily adapted to the Weiszfeld procedure. Unfortunately, using such an approximation may get the objective function close to optimum, but the actual point found may not be close to the true optimizing point in cases where the cost function is very "flat".

Many methods specifically adapted to the median problem or its generalizations have also been proposed. Vergin and Rodgers (1967) used gradient methods and Love (1969) applied convex programming to a problem in three dimensions. Over the years, still other iterative methods were proposed. These include: Seymour and Weindling (1975), Harris (1976), El-Shaieb (1978), Cooper and Katz (1981), Overton (1983) (for equal weights) and Rosen and Xue (1991). An algebraic programming method was given by Chandrasekaran and Tamir (1990).

1.3.7 Using a Lower Bound

Although the convergence of the solution may be "slow" in some problem configurations, the potential improvement in the sum of weighted distances in the succeeding iterations may be quite small. A method of dealing with this that was suggested by Love and Yeong (1981), is to have a continuously updated lower bound on $W(x, y)$ during iteration and stop the iterations once the difference between the value of the objective function and the lower bound is smaller than a given tolerance.

The basic idea of Love and Yeong (1981) is that the convex hull and the current value of the gradient determine an upper bound on the improvement that can be made. In effect, the upper bound on the possible improvement is the magnitude of the gradient times the distance from the current point to the farthest point in the convex hull.

Further works on such bounds include Elzinga and Hearn (1983), Juel (1984), and Drezner (1984). Wendell and Peterson (1984) derived a lower bound from the dual.

1.4 Properties of the Weber Problem

1.4.1 Optimality of a Fixed Point

What happens when one of the weights is larger than the sum of all the others? The knot in the Varignon frame will disappear down the hole of that weight. Is this the condition that the optimum location (x^*, y^*) be at a fixed point? Does one weight have to "overpower" all the others? No, all that is necessary is that over a hole, the net force exerted by the other weights is less than or equal to the weight on the string through that hole; this will guarantee that that hole is the optimum location even when the weight associated with that hole is very small. To see this, consider a board with only two holes, each with an equal weight. The knot will languish anywhere between them. If we now drill a new hole anywhere on a line between the two old ones, and attach a very small weight to the string through it, the new string will pull the knot to a position over its hole (but not down the hole). Since derivatives of $W(x, y)$ do not exist at the fixed points, we should expect strange behavior of the knot at these locations. The "hole" conditions, which are both necessary and sufficient for the optimum to occur at (a_r, b_r), are:

$$\left(\sum_{i \neq r} \frac{w_i(a_r - a_i)}{d_i(a_r, b_r)}\right)^2 + \left(\sum_{i \neq r} \frac{w_i(b_r - b_i)}{d_i(a_r, b_r)}\right)^2 \leq w_r^2 \tag{1.5}$$

1.4.2 How Likely is the Optimal Location on a Fixed Point?

Drezner and Simchi-Levi (1992) investigated the likelihood of the solution to be at a fixed point. Suppose n points are randomly generated in a disc. What is the probability that the optimal solution is one of the fixed points? Intuitively, one would think that this probability increases with n because the disc becomes denser and denser with fixed points and there is "less room" for location on non-fixed points. Drezner and Simchi-Levi (1992) analyzed the case where all weights are equal. They found that the probability that the optimal solution is on a fixed point is approximately $\frac{1}{n}$. That means that with 10 fixed points the probability is about 10%, but with 1,000 fixed points the probability is only 0.1% (or there is a 99.9% probability that the optimal location is *not* on a fixed point). Another interesting result in that paper concerns the difference between the optimal value of the objective function and the best value of the objective function on a fixed point. They showed that the best value of the objective function among all fixed points is expected to be $1+\frac{3}{n}$ times the optimal value. That means that when solving a problem with 1,000 fixed points, selecting the fixed point with the lowest value of the objective function is expected to be only 0.3% higher than the optimum.

One might conclude that for a problem with 1,000 fixed points one can evaluate the value of the objective function at the fixed points, pick the best one, and expected to be only 0.3% over the optimum. This naive approach is not a good one and illustrates the efficiency of the Weiszfeld algorithm. One iteration of the Weiszfeld algorithm requires about the same computing time as one evaluation of the value of the objective function. Therefore, this naive approach will require about the same time as 1,000 Weiszfeld iterations. However, for most problems of that size, the number of Weiszfeld iterations is only in the single digit.

Kuhn (1967) wrote "as for the statements of Courant and Robbins that the generalization of the problem to more than three points is a sterile generalization". (Actually, Courant and Robbins (1941) did not use the word "sterile". What they really said in reference to the four point unweighted problem is "This problem, which was also treated by Steiner, does not lead to interesting results. It is one of the superficial generalizations not infrequently found in mathematical literature"). Indeed, "sterile" seems to be a grossly inappropriate word given the number of generalizations, variations, modifications, extensions and downright mutations that the problem has given birth to. It would be a taxonomist's nightmare to attempt a consistent categorization, and in any case there is not the space available in this chapter. We will, however, attempt a brief sampler and refer the reader to recent literature reviews such as the survey of representative location problems by Brandeau and Chiu (1989) and the references in Love, Morris, and Wesolowsky (1988).

1.5 Other Distance Measures

1.5.1 Minimizing the Sum of Squared Euclidean Distances

In this case the problem is

$$\min_{x,y} \left\{ C(x,y) = \sum_{i=1}^{n} w_i d_i^2(x,y) \right\} \tag{1.6}$$

$C(x,y)$ is separable into a sum of two components, one containing only x, and one containing only y. By simple calculus we can show (White, 1971) that the optimum point $(x^\bullet, y^\bullet)$ is given by:

$$(x^\bullet, y^\bullet) = \left(\frac{\sum_{i=1}^{n} w_i a_i}{\sum_{i=1}^{n} w_i}, \frac{\sum_{i=1}^{n} w_i b_i}{\sum_{i=1}^{n} w_i} \right) \tag{1.7}$$

This point is known, of course, as the centroid or center of gravity. To give this point, our Varignon analogue machine must be modified somewhat. We must untie the knot joining the strings, and then tie a knot on each individual string large enough so that it can not pass through the hole. If we were fortunate enough to have a weightless board, the board would now balance on a needle point placed underneath at $(x^\bullet, y^\bullet)$. To put it another way, the centroid is a point such that if we draw any line through it, the weight×distance components projected onto the line would sum to the same value on either side of the point.

1.5.2 Minimizing the Sum of Rectilinear Distances

Rectilinear, rectangular, or Manhattan distances are distances that are often used to approximate travel in a grid. In this case the distance between the facility and a demand point i is given by $d_i^R(x,y) = |x - a_i| + |y - b_i|$. Minimizing the sum of weighted distances to find the optimum location $(x^\diamond, y^\diamond)$ now becomes:

$$\min_{x,y} \left\{ R(x,y) = \sum_{i=1}^{n} w_i d_i^R(x,y) \right\} \tag{1.8}$$

This problem is easy to solve because the objective function is separable, as was the one in the preceding problem. For example, to find $x^\diamond$ we minimize $\sum w_i |x - a_i|$. As can easily be verified by examining the piecewise linear derivative of $\sum w_i |x - a_i|$, this is done by finding the median of the weighted a_i's, or, in other words, the midpoint of the weights w_i arranged along the x-axis at their corresponding locations a_i. This median may be some value a_i, or a range that includes two adjacent values. To summarize, $(x^\diamond, y^\diamond)$ is found by finding the median of the weighted a_i values and the median of the weighted b_i values respectively. This solution is sometimes called the co-ordinatewise median (Rousseeuw and Leroy, 1987).

1.5.3 *p*-Norm Distances

A generalization of Euclidean distance is the ℓ_p distance, which is given by

$$\ell_{pi} = \sqrt[p]{|x - a_i|^p + |y - b_i|^p},$$

for the demand point i and a facility (x, y). It can depict a wide variety of distance measures; the Euclidean distance is the special case $p = 2$ and rectilinear distance is $p = 1$. Love and Morris (1979) showed that ℓ_p distances can be used to approximate road distances. Weber problems with respect to p-norms have been studied, for example, in Brimberg (1989) and in Brimberg and Love (1995). For solving these location problems Weiszfelds's algorithm (see Section 1.3.5) can be adapted; and this is mostly done by using a hyperbolic approximation. Linear convergence of this method could be shown for $1 \leq p \leq 2$, while for $p > 2$ counterexamples for convergence have been found, see Morris (1981), Brimberg and Love (1992), and Brimberg and Love (1993). Note that the "hole" conditions of the Varignon frame for the Weber problem with Euclidean distance (see equation (1.5)) can also be generalized to p-norm distances, see Juel and Love (1981); hence optimal solutions at demand points can easily be found.

Recently, Ortega, Mesa and Sánchez (2000) proposed an iterative method for solving p-norm location problems with $1 < p < 2$. Their approach is based on an approximation of round norms by block norms which they use to develop an iterative linear programming approach.

1.5.4 Block Norms and Polyhedral Gauges

If the unit ball of a norm γ is a polyhedral set, the norm is called a block norm and it is called a polyhedral gauge if symmetry is not required any more. Examples for block norms are the rectilinear norm, or the Chebychef norm, both having polyhedral unit balls with four vertices. The halflines starting at the origin and passing through the vertices of the polyhedral gauge are called fundamental directions. Such distances can be used, for example, to approximate road networks in a planar setting. Location problems with polyhedral gauges can be formulated as linear programs, see Ward and Wendell (1985) and are therefore easily solvable. If we draw the fundamental directions starting at each of the demand points we get a partition of the plane into polyhedral cells, and the objective function is linear on each cell. In particular, Durier and Michelot (1985) showed that there always exists an optimal solution at a cell vertex.

1.5.5 Other Distance Metrics

In addition to the basic generalizations mentioned above, many other types of distances have been investigated. Examples are:

- central metrics (Perreur and Thisse, 1974),
- distance functions based on altered norms (Love and Morris (1979) and Love, Truscott and Walker (1985)),
- weighted one-infinity norms (Ward and Wendell, 1980),
- mixed norms (Planchart and Hurter (1975) and Hansen, Perreur and Thisse (1980)),
- block and round norms (Thisse, Ward and Wendell, 1984),
- mixed gauges (Durier and Michelot, 1985),
- asymmetric distances (Hodgson, Wong and Honsaker, 1987).
- weighted sums of order p (Brimberg and Love (1995), Üster (1999))

Note that there are many techniques to solve locations problems with arbitrary gauges, among them for example a grid-approximation technique, see Carrizosa and Puerto (1995), or a primal-dual approach, see Michelot (1993). Hansen et al. (1985) developed a geometrical Branch-and-Bound algorithm, which was improved and applied by Plastria (1992). In a new approach, Carrizosa et al. (2000) use polyhedral gauges to approximate any other gauge in a planar location problem, for example, by using the sandwich approximation algorithm of Burkard, Hamacher and Rote (1991).

The space in which location can take place has also been generalized. Love (1969) extended the problem to three dimensions. Also, the earth's surface is approximately planar only on a small scale. Drezner and Wesolowsky (1978), Aly, Kay and Litwhiler (1979), and Drezner (1985) are among those using spherical distances. Wesolowsky (1983) and Plastria (1995) give a review of spherical location problems. More general spaces have also been employed, for example Eckhardt (1980) used Banach spaces.

1.6 Multiple Facilities

One of the earliest and most straight-forward generalizations was to add more new facilities. This was done in two basically different ways. One was to simply add additional facilities, with given interactions between themselves and the demand points. The other one is to assign each demand point one facility for service (location-allocation models).

1.6.1 The Multifacility Model

If we have m new facilities and the weight between facility j and demand point i is w_{ij} and between facility j and s is v_{js}, then we have:

$$\min_{(x_j,y_j)j=1,\dots,m} \left\{ \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\sqrt{(x_j-a_i)^2+(y_j-b_i)^2} + \sum_{j=1}^{m-1}\sum_{s=j+1}^{m} v_{js}\sqrt{(x_j-x_s)^2+(y_j-y_s)^2} \right\} \qquad (1.9)$$

The facilities thus have predetermined shipments to the demand points and to each other. Miehle (1958), a "co-rediscoverer" of the Weiszfeld procedure, considered a problem of this type.

The multifacility Weber problem is a convex optimization problem with a nondifferentiable objective function since two objects (demand points or facilities) may coincide, i.e., have the same location.

For block norm distances and polyhedral gauges, (1.9) can be formulated as a linear programming problem (Ward and Wendell, 1985). However, the dominating set result (compare Section 1.5.4) transfers to the multifacility case only for polyhedral gauges with at most 4 fundamental directions (see Hansen, Perreur and Thisse (1980) and Michelot (1987) for a counterexample for general gauges). Efficient algorithms are available particularly for the case of rectilinear distances, see, for example, Dax (1986).

For general distance metrics, hyperbolic approximations of the objective function can be used to avoid nondifferentiability (Wesolowsky and Love (1972), Eyster, White and Wierwille (1973)), or primal-dual methods may be applied (Idrissi, Lefebvre and Michelot, 1989). Many authors developed solution methods for Euclidean distances (see, for example, Calamai and Charalambous (1980), Rado (1988) and Xue, Rosen and Pardalos (1996) who recently gave a polynomial time dual algorithm). Interior point and related methods were succesfully applied by Andersen (1996) and for gauge distances by Fliege (2000) who also proved that his method has polynomial complexity.

Other solution concepts are based on coincidence conditions to identify points of nondifferentiablility (see Fliege (1997) for a very general discussion), or use special treatment for points of coincidence (Overton, 1983).

When the problem is viewed as a cost minimizing location problem, it is static in the sense that the parameters are assumed to be constant over the decision horizon. When there are anticipated changes such as those in weight or locations of demand points and if re-locations of the facility are possible, the problem becomes "dynamic" rather than static. Erlenkotter (1981) summarized different approaches to dynamic location.

1.10 Epilogue

The Weber problem is the cornerstone of locational analysis. It is the first location problem ever posed, and gave rise to numerous extensions and models. It is the root of the tree spanning location models. Every model discussed in this book can be traced back to the Weber problem.

The Weber problem was extended by considering other environments (such as networks, or the globe) leading to a variety of distance measures. Objectives considered in many models include the minisum (the original Weber objective) as well as the minimax, maximin (used for modeling obnoxious facility location where being close is a detriment), competitive (when attempting to attract as much demand as possible from competing facilities) and composites of these measures. Many models assume stochastic or dynamic rather than deterministic demand. Barriers to travel or constraints are considered in many models as well. Extensions to the location of multiple facilities abound. Most of these models (location-allocation or by other terminologies p-median or p-center models) assume that demand is serviced by the closest facility, leading to difficult combinatorial problems of simultaneously allocating demand to facilities and locating the facilities (location-allocation models).

All these extensions are summarized in the LOLA classification scheme (see Chapter 8). Each model is classified by five characteristics describing its uniqueness in the vast ocean of location models.

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2 Continuous Covering Location Problems

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2.1 Introduction

A location problem arises whenever a question is raised like

– where are we going to put the thing(s) ?

The next two questions then immediately follow:

1. which places are available ?
2. on what basis do we choose ?

2.1.1 Locational Space

The answer to first question determines the locational space. We have a *continuous location problem* when this space is described by way of continuous variables, usually coordinates.

In most applications this space is either planar — just think of an integrated circuit, a piece of paper, your desktop, a shopfloor, a piece of land or a country (if not too large) — or on a sphere, when considering a really large region like a continent or even the whole earth; in these cases we need two coordinates to describe a position, and our locational space is two-dimensional. For positioning within a building, underwater or in the air it will also be necessary to take height or depth into account, so a third coordinate will be needed. Certain more theoretical frameworks may even call for more dimensions. Problems with one dimension also occur when we are locating on a line (which might be straight, curved and/or broken), such as a single stretch of highway, waterway or railway.

Continuous location problems also assume one cannot give an exhaustive list of all individual available places, as is the case in discrete location problems which are also discussed in another chapter. Here we deal with the somewhat more vague situation where we do not really know which sites are available, but rather that these are 'all over the place' and we want to find out where to look for good candidates. Thus, continuous location models can be considered as *site generating*, (Love et al., 1988), and will always have some geometrical flavour.

What we do have to take into account, however, is that in order to be eligible sites must come from some *feasible region(s)*. Hopefully this is described