E

Game Theory

Game Theory

Game theory addresses decision situations with two or more decision makers in competition.

In chapter 12 in the text on decision analysis, we discussed methods to aid the *individual* decision maker. All of the decision situations involved one decision maker. There were no *competitors* whose decisions might alter the decision maker's analysis of a decision situation. However, many situations do, in fact, involve several decision makers who compete with one another to arrive at the best outcome. These types of competitive decision-making situations are the subject of *game theory.* Although the topic of game theory encompasses a different type of decision situation than does decision analysis, many of the fundamental principles and techniques of decision making are the same. Thus, game theory is, in effect, an extension of decision analysis rather than an entirely new topic area.

Anyone who has played card games or board games is familiar with situations in which competing participants develop plans of action to win. Game theory encompasses similar situations in which competing decision makers develop plans of action to win. In addition, game theory consists of several mathematical techniques to aid the decision maker in selecting the plan of action that will result in the best outcome. In this module we will discuss some of these techniques.

Types of Game Situations

*A **two-person game** encompasses two players.*

*In a **zero-sum game** one player's gains represent another's exact losses.*

Competitive game situations can be subdivided into several categories. One classification is based on the number of competitive decision makers, called *players,* involved in the game. A game situation consisting of two players is referred to as a *two-person game.* When there are more than two players, the game situation is known as an *n-person game.*

Games are also classified according to their outcomes in terms of each player's gains and losses. If the sum of the players' gains and losses equals zero, the game is referred to as a *zero-sum game.* In a two-person game, one player's gains represent another's losses. For example, if one player wins $100, then the other player loses $100; the two values sum to zero (i.e., +$100 and −$100). Alternatively, if the sum of the players' gains and losses does not equal zero, the game is known as a *non-zero-sum game.*

The *two-person, zero-sum game* is the one most frequently used to demonstrate the principles of game theory because it is the simplest mathematically. Thus, we will confine our discussion of game theory to this form of game situation. The complexity of the *n-person* game situation not only prohibits us from demonstrating it but also restricts its application in real-world situations.

The Two-Person, Zero-Sum Game

Examples of competitive situations that can be organized into two-person, zero-sum games include (1) a union negotiating a new contract with management; (2) two armies participating in a war game; (3) two politicians in conflict over a proposed legislative bill, one attempting to secure its passage and the other attempting to defeat it; (4) a retail firm trying to increase its market share with a new product and a competitor attempting to minimize the firm's gains; and (5) a contractor negotiating with a government agent for a contract on a project.

The following example will demonstrate a two-person, zero-sum game. A professional athlete, Biff Rhino, and his agent, Jim Fence, are renegotiating Biff's contract with the Texas Buffaloes' general manager, Harry Sligo. The various outcomes of this game situation can be organized into a payoff table similar to the payoff tables used for decision analysis. The payoff table for this example is shown in Table E-1.

Table E-1
Payoff Table for Two-Person, Zero-Sum Game

Athlete/Agent Strategies	General Manager Strategies		
	A	*B*	*C*
1	$50,000	$35,000	$30,000
2	60,000	40,000	20,000

In a game situation it is assumed that the payoff table is known to all players.

The payoff table for a two-person game is organized so that the player who is trying to maximize the outcome of the game is on the left and the player who is trying to minimize the outcome is on the top. In Table E-1 the athlete and agent want to maximize the athlete's contract, and the general manager hopes to minimize the athlete's contract. In a sense, the athlete is an offensive player in the game and the general manager is a defensive player. In game theory it is assumed that the payoff table is known to both the offensive and the defensive player—an assumption that is often unrealistic in real-world situations and thus restricts the actual applications of this technique.

*A **strategy** is a plan of action followed by a player.*

A *strategy* is a plan of action to be followed by a player. Each player in a game has two or more strategies, only one of which is selected for each playing of a game. In Table E-1 the athlete and his agent have two strategies available, 1 and 2, and the general manager has three strategies, A, B, and C. The values in the table are the payoffs or outcomes associated with each player's strategies.

*The **value of the game** is the offensive player's gain and the defensive player's loss in a zero-sum game.*

For our example, the athlete's strategies involve different types of contracts and the threat of a holdout and/or of becoming a free agent. The general manager's strategies are alternative contract proposals that vary with regard to such items as length of contract, residual payments, no-cut/no-trade clauses, and off-season promotional work. The outcomes are in terms of dollar value. If the athlete selects strategy 2 and the general manager selects strategy C, the outcome is a $20,000 gain for the athlete and a $20,000 loss for the general manager. This outcome results in a zero sum for the game (i.e., +$20,000 − 20,000 = 0). The amount $20,000 is known as the *value of the game.*

The best strategy for each player is his or her optimal strategy.

The purpose of the game for each player is to select the strategy that will result in the best possible payoff or outcome regardless of what the opponent does. The best strategy for each player is known as the *optimal strategy.* Next, we will discuss methods for determining strategies.

A Pure Strategy

*In a **pure strategy game** each player adopts a single strategy as an optimal strategy.*

When each player in the game adopts a single strategy as an optimal strategy, then the game is a *pure strategy* game. The value of a pure strategy game is the same for both the offensive player and the defensive player. In contrast, in a *mixed strategy* game, the players adopt a mixture of strategies if the game is played many times.

*With the **minimax criterion** each player seeks to minimize maximum possible losses; the offensive player selects the strategy with the largest of the minimum payoffs, and the defensive player selects the strategy with the smallest of the maximum payoffs.*

A pure strategy game can be solved according to the *minimax decision criterion.* According to this principle, each player plays the game to minimize the maximum possible losses. The offensive player will select the strategy with the largest of the minimum payoffs (called the *maximin* strategy), and the defensive player will select the strategy with the smallest of the maximum payoffs (called the *minimax* strategy). In our example involving the athlete's contract negotiation process, the athlete will select the maximin strategy from strategies 1 and 2, and the general manager will select the minimax strategy from strategies A, B, and C. We will first discuss the athlete's decision, although in game theory the decisions are actually made *simultaneously.*

To determine the maximin strategy, the athlete first selects the minimum payoff for strategies 1 and 2, as shown in Table E-2. The maximum of these minimum values indicates the optimal strategy and the value of the game for the athlete.

Table E-2
Payoff Table with Maximin Strategy

Athlete/Agent Strategies	General Manager Strategies			
	A	*B*	*C*	
1	$50,000	$35,000	$30,000	← Maximum of minimum payoffs
2	60,000	40,000	20,000	

The value $30,000 is the maximum of the minimum values for each of the athlete's strategies. Thus, the optimal strategy for the athlete is strategy 1. The logic behind this decision is as follows. If the athlete selected strategy 1, the general manager could be expected to select strategy C, which would minimize the possible loss (i.e., a $30,000 contract is better *for the manager* than a $50,000 or $35,000 contract). Alternatively, if the athlete selected strategy 2, the general manager could be expected to select strategy C for the same reason (i.e., a $20,000 contract is better for the manager than a $60,000 or $40,000 contract). Now since the athlete has anticipated how the general manager will respond to each strategy, he realizes that he can negotiate either a $30,000 or a $20,000 contract. The athlete selects strategy 1 in order to get the larger possible contract of $30,000, given the actions of the general manager.

Simultaneously, the general manager applies the minimax decision criterion to strategies A, B, and C. First, the general manager selects the maximum payoff for each strategy, as shown in Table E-3. The minimum of these maximum values determines the optimal strategy and the value of the game for the general manager.

Table E-3
Payoff Table with Minimax Strategy

Athlete/Agent Strategies	General Manager Strategies			
	A	*B*	*C*	
1	$50,000	$35,000	$30,000	← Minimum of maximum values
2	60,000	40,000	20,000	

The value $30,000 is the minimum of the maximum values for each of the general manager's strategies. Thus, the optimal strategy for the general manager is C. The logic of this decision is similar to that of the athlete's decision. If the general manager selected strategy A, the athlete could be expected to select strategy 2 with a payoff of $60,000 (i.e., the athlete will choose the better of the $50,000 and $60,000 contracts). If the general manager selected strategy B, then the athlete could be expected to select strategy 2 for a payoff of $40,000. Finally, if the general manager selected strategy C, then the athlete could be expected to select strategy 1 for a payoff of $30,000. Because the general manager has anticipated how the athlete will respond to each strategy, he realizes that either a $60,000, $40,000, or $30,000 contract could possibly be awarded. Thus, the general manager selects strategy C, which will result in the minimum contract of $30,000. In general, the manager considers the worst outcome that could result if a particular strategy were followed. Under the minimax criterion the general manager will select the strategy that ensures that he loses only the minimum of the maximum amounts that could be lost.

Solution of Game Theory Problems with QM for Windows

QM for Windows, which we use in the text to solve decision analysis problems, also has a module for solving game theory problems. The QM for Windows solution for our athlete/agent example is shown in Exhibit E-1. Notice that QM for Windows indicates the row minimums and column maximums, and provides the maximin solution in the upper right-hand corner of the solution table.

Exhibit E-1

Maximin & Minimax

Athlete-Team Salary Game Solution

	GM A	GM B	GM C	Row Minimum	Maximin
Athlete/Agent 1	50,000.	35,000.	30,000.	30,000.	30,000.
Athlete/Agent 2	60,000.	40,000.	20,000.	20,000.	
Column Maximum	60,000.	40,000.	30,000.		
Minimax			30,000.		
Value= 30000					

Dominant Strategies

A strategy is dominated, and can be eliminated, if all its payoffs are worse than the corresponding payoffs for another strategy.

We could have reduced the choices of the general manager if we had noticed that strategy C *dominates* strategies A and B. Dominance occurs when all the payoffs for one strategy are better than the corresponding payoffs for another strategy. In Table 3 the values $30,000 and $20,000 are both lower than the corresponding payoffs of $50,000 and $60,000 for strategy A and the corresponding payoffs of $35,000 and $40,000 for strategy B. Because strategy C dominates A and B, these two latter strategies can be eliminated from consideration altogether, as shown in Table E-4. If this had been done, strategy C could have been selected automatically without applying the minimax criterion. Thus, the most efficient approach is to first examine the payoff table for dominance in order to possibly reduce its size.

Table E-4
Payoff Table with Dominated Strategies Eliminated

Athlete/Agent Strategies	General Manager Strategies *A*	*B*	*C*
1	$50,000	$35,000	$30,000
2	60,000	40,000	20,000

The fact that the optimal strategy for each player in this game resulted in the *same payoff* game value of $30,000 is what classifies it as a pure strategy game. In other words, because strategy 1 is optimal for the athlete and strategy C is optimal for the general manager, a contract for $30,000 will be awarded to the athlete. Because the outcome of $30,000

In a pure strategy game the optimal strategy for each player results in the same payoff, called an ***equilibrium*** *or* ***saddle point.***

The equilibrium point in a game is simultaneously the minimum of a row and the maximum of a column.

The ***minimax criterion*** *will result in the optimal strategies only if both players use it.*

results from a pure strategy, it is referred to as an *equilibrium point* (or sometimes as a *saddle point*). A point of equilibrium is a value that is simultaneously the *minimum of a row* and the *maximum of a column,* as is the payoff of $30,000 in Table E-3.

It is important to realize that the minimax criterion results in the optimal strategy for each player as long as each player uses this criterion. If one of the players does not use this criterion, the solution of the game will not be optimal. If we assume that both players are logical and rational, however, then we can assume that this criterion will be employed.

If an equilibrium point exists, it makes the determination of optimal strategies relatively easy, as no complex mathematical calculations are necessary. However, as we mentioned earlier, if a game does not involve a pure strategy, then it is a mixed strategy game. We will discuss a mixed strategy game next.

A Mixed Strategy

A ***mixed strategy game*** *occurs when each player selects an optimal strategy that does not result in an equilibrium point when the minimax criterion is used.*

A *mixed strategy* game occurs when each player selects an optimal strategy and they *do not result in an equilibrium point* (i.e., the same outcome) when the maximin and minimax decision criteria are applied.

The following example will demonstrate a mixed strategy game. The Coloroid Camera Company (which we will refer to as Company I) is going to introduce a new camera into its product line and hopes to capture as large an increase in its market share as possible. In contrast, the Camco Camera Company (which we will refer to as Company II) hopes to minimize Coloroid's market share increase. Coloroid and Camco dominate the camera market, and any gain in market share for Coloroid will result in a subsequent identical loss in market share for Camco. The strategies for each company are based on their promotional campaigns, packaging, and cosmetic differences between the products. The payoff table, which includes the strategies and outcomes for each company (I = Coloroid and II = Camco), is shown in Table E-5. The values in Table E-5 are the *percentage* increases or decreases in market share for Company I.

Table E-5
Payoff Table for Camera Companies

Company I	Company II Strategies		
Strategies	*A*	*B*	*C*
1	9	7	2
2	11	8	4
3	4	1	7

The first step is to check the payoff table for any dominant strategies. Doing so, we find that strategy 2 dominates strategy 1, and strategy B dominates strategy A. Thus, strategies 1 and A can be eliminated from the payoff table, as shown in Table E-6.

Table E-6
Payoff Table with Strategies 1 and A Eliminated

Company I	Company II Strategies	
Strategies	*B*	*C*
2	8	4
3	1	7

Time Out for John von Neumann and Oskar Morgenstern

John Von Neumann first introduced the topic of game theory in 1928 at age 24 while on the faculty of the University of Berlin. He had received his doctorate in mathematics from the University of Budapest only two years earlier. He continued to develop the theory during the next 15 years. In 1944, while he was a member of the Institute for Advanced Study at Princeton University, as well as a consultant with the Navy Bureau of Ordinance and the Los Alamos Laboratory (working on the atomic bomb), his work culminated in the publication of the book *Theory of Games and Economic Behavior* with Oskar Morgenstern. Morgenstern was an economist who had arrived at Princeton in 1938 from his native Austria. Besides providing a mathematical basis for how strategies are developed in game situations, Von Neumann and Morgenstern's work also influenced other researchers, such as George Dantzig, in his development of linear programming, and Richard Bellman, in the development of dynamic programming.

Next, we apply the maximin decision criterion to the strategies for Company I, as shown in Table E-7. The minimum value for strategy 2 is 4%, and the minimum value for strategy 3 is 1%. The maximum of these two minimum values is 4%; thus, strategy 2 is optimal for Company I.

Table E-7
Payoff Table with Maximin Criterion

Company I Strategies	Company II Strategies	
	B	*C*
2	8	$\boxed{4}$
3	$\boxed{1}$	7

← Minimum of the maximum values

Now the minimax decision criterion is applied to the strategies for Company II in Table E-8. The maximum value for strategy B is 8%, and the maximum value for strategy C is 7%. Of these two maximum values, 7% is the minimum; thus, the optimal strategy for Company II is C.

Table E-8
Payoff Table with Minimax Criterion

Company I Strategies	Company II Strategies	
	B	*C*
2	8	4
3	1	7

← Minimum of the maximum values

Table E-9 combines the results of the application of the maximin and minimax criteria by the companies.

Table E-9
Company I and II Combined Strategies

Company I Strategies	Company II Strategies	
	B	*C*
2	8	$\boxed{4}$
3	1	7

In a mixed strategy game, players switch decisions in response to the decision of the other player and eventually return to the initial decisions, resulting in a closed loop.

From Table E-9 we can see that the strategies selected by the companies do not result in an equilibrium point. Therefore, this is not a pure strategy game. In fact, this condition will not result in any strategy for either firm. Company I maximizes its market share percentage increase by selecting strategy 2. Company II selects strategy C to minimize Company I's market share. However, as soon as Company I noticed that Company II was using strategy C, it would switch to strategy 3 to increase its market share to 7%. This move would not go unnoticed by Company II, which would immediately switch to strategy B to reduce I's market share to 1%. This action by Company II would cause Company I immediately to switch to strategy 2 to maximize its market share increase to 8%. Given the action of Company I, Company II would switch to strategy C to minimize Company I's market share increase to 4%. Now you will notice that the two companies are right back where they started. They have completed a closed loop, as shown in Table E-10, which could continue indefinitely if the two companies persisted.

Table E-10
Payoff Table with Closed Loop

Company I Strategies	Company II Strategies	
	B	*C*
2	8	4
3	1	7

Several methods are available for solving mixed strategy games. We will look at one of them, which is analytical—the *expected gain and loss method.*

Expected Gain and Loss Method

*In the **expected gain and loss method,** a plan of strategies is determined by each player so that the expected gain of one equals the expected loss of the other.*

The *expected gain and loss method* is based on the principle that in a mixed strategy game a plan of strategies can be developed by each player so that the *expected gain* of the maximizing player or the *expected loss* of the minimizing player will be the same, regardless of what the opponent does. In other words, a player develops a plan of mixed strategies that will be employed regardless of what the opposing player does (i.e., the player is indifferent to the opponent's actions). As might be expected from its name, this method is based on the concept of expected values.

The mixed strategy game for the two camera companies, described in the previous section, will be used to demonstrate this method. First, we will compute the *expected gain* for Company I. Company I arbitrarily assumes that Company II will select strategy B. Given this condition, there is a probability of p that Company I will select strategy 2 and a probability of $1 - p$ that Company I will select strategy 3. Thus, if Company II selects B, the expected gain for Company I is

$$8p + 1(1 - p) = 1 + 7p$$

Next, Company I assumes that Company II will select strategy C. Given strategy C, there is a probability of p that Company I will select strategy 2 and a probability of $1 - p$ that Company I will select strategy 3. Thus, the expected gain for Company I given strategy C is

$$4p + 7(1 - p) = 7 - 3p$$

Previously, we noted that this method was based on the idea that Company I would develop a plan that would result in the same expected gain, regardless of the strategy that Company II selected. Thus, if Company I is indifferent to whether Company II selects strategy B or C, we equate the expected gain from each of these strategies:

$$1 + 7p = 7 - 3p$$

and

$$10p = 6$$
$$p = 6/10 = .60$$

Recall that p is the probability of using strategy 2, or the *percentage of time* strategy 2 would be employed. Thus, Company I's plan is to use strategy 2 for 60% of the time and to use strategy 3 the remaining 40% of the time. The expected gain (i.e., market share increase for Company I) can be computed using the payoff of either strategy B or C because the gain will be the same regardless. Using the payoffs from strategy B,

$$EG \text{ (Company I)} = .60(8) + .40(1) = 5.2\% \text{ market share increase}$$

To check this result, we will compute the expected gain if strategy C is used by Company II.

$$EG \text{ (Company I)} = .60(4) + .40(7) = 5.2\% \text{ market share increase}$$

Now we must repeat this process for Company II to develop its mixed strategy—except that what was Company I's expected *gain* is now Company II's expected *loss.* First, we assume that Company I will select strategy 2. Thus, Company II will employ strategy B for p percent of the time and C the remaining $1 - p$ percent of the time. The expected *loss* for Company II given strategy 2 is

$$8p + 4(1 - p) = 4 + 4p$$

Next, we compute the expected loss for Company II given that Company I selects strategy 3:

$$1p + 7(1 - p) = 7 - 6p$$

Equating these two expected losses for strategies 2 and 3 will result in values for p and $1-p$.

$$4 + 4p = 7 - 6p$$
$$10p = 3$$
$$p = 3/10 = .30$$

and

$$1 - p = .70$$

Because p is the probability of employing strategy B, Company II will employ strategy B 30% of the time and, thus, strategy C will be used 70% of the time. The actual expected loss given strategy 2 (which is the same as that for strategy 3) is computed as

$$\begin{aligned} EL \text{ (Company II)} &= .30(8) + .70(4) \\ &= 5.2\% \text{ market share loss} \end{aligned}$$

In the expected gain and loss method, the probability that each strategy will be used by each player is computed.

The mixed strategies for each company are summarized next.

Company I	***Company II***
Strategy 2: 60% of the time	Strategy B: 30% of the time
Strategy 3: 40% of the time	Strategy C: 70% of the time

The expected gain for Company I is a 5.2% increase in market share, and the expected loss for Company II is also a 5.2% market share. Thus, the mixed strategies for each company have resulted in an equilibrium point such that a 5.2% *expected* gain for Company I results in a simultaneous 5.2% *expected* loss for Company II.

It is also interesting to note that each company has improved its position over the one arrived at using the maximin and minimax strategies. Recall from Table E-4 that the payoff for Company I was only a 4% increase in market share, whereas the mixed strategy yields an expected gain of 5.2%. The outcome for Company II from the minimax strategy was a 7% loss; however, the mixed strategies show a loss of only 5.2%. Thus, each company puts itself in a better situation by using the mixed strategy approach.

This approach assumes that the game is repetitive and will be played over a period of time so that a strategy can be employed a certain percentage of that time. For our example, it can be logically assumed that the marketing of the new camera by Company I will require a lengthy time frame. Thus, each company could employ its mixed strategy.

The QM for Windows solution of this mixed strategy game for the two camera companies is shown in Exhibit E-2.

Exhibit E-2

Game Theory Results

Coloroid vs. Camco Game Solution

	A	B	C	Row Mix
1	9.	7.	2.	0.
2	11.	8.	4.	0.6
3	4.	1.	7.	0.4
Column Mix--->	0.	0.3	0.7	
Value of game (to row)	5.2			

Problems

1. The Army is conducting war games in Europe. One simulated encounter is between the Blue and Red Divisions. The Blue Division is on the offensive; the Red Division holds a defensive position. The results of the war game are measured in terms of troop losses. The following payoff table shows Red Division troop losses for each battle strategy available to each division. Determine the optimal strategy for both divisions and the number of troop losses the Red Division can expect to suffer.

Blue Division Strategies	Red Division Strategies		
	A	*B*	*C*
1	1,800	2,000	1,700
2	2,300	900	1,600

2. The Baseball Players Association has voted to go on strike if a settlement is not reached with the owners within the next month. The players' representative, Melvin Mulehead, has two strategies

(containing different free agent rules, pension formulas, etc.); the owners' representative, Roy Stonewall, has three counterproposals. The financial gains in millions of dollars from each player strategy given each owner strategy are shown in the following payoff table.

Player Strategies	Owner Strategies		
	A	*B*	*C*
1	15	9	11
2	7	20	12

a. Determine the initial strategy for the players and for the owners.
b. Is this a pure or a mixed strategy game? Explain.

3. Mary Washington is the incumbent congresswoman for a district in New Mexico, and Franklin Truman is her opponent in the upcoming election. Since Truman is seeking to unseat Washington, he is on the offensive, and she hopes to minimize his gains in the polls. The following payoff table shows the possible percentage point gains for Truman given the political strategies available to each politician.

Franklin Truman Strategies	Mary Washington Strategies	
	A	*B*
1	7	3
2	6	10

a. Determine the optimal political strategy for each politician and the percentage gain in the polls Franklin Truman can expect.
b. Solve this problem using the computer.

4. Edgar Allan Melville is a successful novelist who is negotiating a contract for a new novel with his publisher, Potboiler Books, Inc. The novelist's contract strategies encompass various proposals for royalties, movie rights, advances, and the like. The following payoff table shows the financial gains for the novelist from each contract strategy.

Novelist Strategies	Publisher Strategies		
	A	*B*	*C*
1	$ 80,000	$120,000	$90,000
2	130,000	90,000	80,000
3	110,000	140,000	100,000

a. Does this payoff table contain any dominant strategies?
b. Determine the strategy for the novelist and the publisher and the gains and losses for each.

5. Two major soft drink companies are located in the Southeast—the Cooler Cola Company and Smoothie Soft Drinks, Inc. Cooler Cola is the market leader, and Smoothie has developed several marketing strategies to gain a larger percentage of the market now belonging to Cooler Cola. The

following payoff table shows the gains for Smoothie and the losses for Cooler given the strategies of each company.

Smoothie Strategies	Cooler Cola Strategies		
	A	*B*	*C*
1	10	9	3
2	4	7	5
3	6	8	–4

Determine the mixed strategy for each company and the expected market share gains for Smoothie and losses for Cooler Cola.

6. Tech is playing State in a basketball game. Tech employs two basic offenses—the shuffle and the overload; State uses three defenses—the zone, the man-to-man, and a combination zone and man-to-man. The points Tech expects to score (estimated from past games) using each offense against each State defense are given in the following payoff table.

Tech Offenses	State Defenses		
	Zone	*Man-to-Man*	*Combination*
Shuffle	72	60	83
Overload	58	91	72

Determine the mixed strategy for each team and the points Tech can expect to score. Interpret the strategy probabilities.