How to invert the Gaussian distribution

$$y = \mathrm{gauss}(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \qquad \{1\}$$

$$Y = \mathrm{Gauss}(x) = \int_{-\infty}^{x} \mathrm{gauss}(t)\,\mathrm{d}t = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left[-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right]\mathrm{d}t = \\ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-\mu)/\sigma} \exp\left(-\frac{1}{2}u^2\right)\mathrm{d}u = \Phi\left(\frac{x-\mu}{\sigma}\right) \qquad \{2\}$$

The function Φ is given by many programs (possibly computed by a *Simpson rule** numerical integration or a clever approximating formula):

Excel NORMDIST(x; μ; σ; cumulative=1); {1+ERF[0;2/SQRT(2)]}/2.

Fortran erf, such that $\Phi(x) = [1 + \mathrm{erf}(x/\sqrt{2})]/2$

Taking only the *standard* function Φ (for simplicity), the inversion can be made by the well-known Newton-Raphson algorithm. Let r be a "0, 1" (typical) random value:

$$\Phi(z) = r \qquad \{3\}$$

How to find z (the value of the Gaussian variable whose probability is r) ?

$$z = \Phi^{\mathrm{inv}}(r) \qquad \{4\}$$

Yes, but how do the various programs calculate Φ^{inv} ? (Usually, Φ^{inv} is called Φ^{-1}.) Possibly this way:

$$f(z) = \Phi(z) - r \overset{solve}{=} 0 \qquad \{5\}$$

So, which z solves this ?

The Newton-Raphson algorithm* says:

$$x_{new} = x - \frac{f(x)}{f'(x)} \qquad \{6\}$$

Our variable is z.

$$f'(z) = \frac{\mathrm{d}}{\mathrm{d}z}[\Phi(z) - r] = \frac{\mathrm{d}}{\mathrm{d}z}\int_{-\infty}^{z} \phi(t)\,\mathrm{d}t \overset{!}{=} \phi(z) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z^2\right) \qquad \{7\}$$

Indeed, r is constant (a given value).

Thus, we just need an initial guess, a first x (or z in our case), and the successive application of {5} and {7} in {6}. As these functions are "well-behaved", almost any value is a good initial guess (suggested, $z = 1$).

The Newton-Raphson algorithm is *general*, and solves any ("well-behaved") problem of this type.

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* See references among the innumerable ones.