

Goal programming

Introduction

Multicriteria decision making refers to situations where we have more than one objective (or goal) and these objectives conflict and we must somehow reach a decision taking them all into account. This contrasts with, for example, decision trees or linear programming, where we have a single objective – either optimise expected monetary value for decision trees or optimise a single linear objective in the case of linear programming. Here we consider goal programming, one technique used for multicriteria decision making.

Goal programming (GP)

To illustrate goal programming (GP) we consider the Two Mines problem.

The Two Mines Company own two different mines that produce an ore which, after being crushed, is graded into three classes: high, medium and low-grade. The company has contracted to provide a smelting plant with 12 tons of high-grade, 8 tons of medium-grade and 24 tons of low-grade ore per week. The two mines have different operating characteristics as detailed below.

Mine	Cost per day (£'000)	Production (tons/day)		
		High	Medium	Low
X	180	6	3	4
Y	160	1	1	6

How many days per week should each mine be operated to fulfil the smelting plant contract?

To solve this problem introduce variables

x = number of days per week mine X is operated

y = number of days per week mine Y is operated

with $x \geq 0$ and $y \geq 0$ and formulate the problem as a linear program:

minimise $180x + 160y$

subject to

$$6x + 1y \geq 12$$

$$3x + 1y \geq 8$$

$$4x + 6y \geq 24$$

$$x \leq 5$$

$$y \leq 5$$

$$x, y \geq 0$$

where we assume we can work no more than five days per week on each mine.

The solution to this linear program is:

$$x = 12/7 = 1.71$$

$$y = 20/7 = 2.86$$

with the value of the objective function being given by $180x + 160y = 180(12/7) + 160(20/7) = 765.71$

An Excel sheet for this problem is shown below.

	A	B	C	D	E	F	G
1		Cost	High	Medium	Low		
2	Mine X	180	6	3	4		
3	Mine Y	160	1	1	6		
4	Total	0	0	0	0		
5							
6							
7							
8							
9							
10							
11						Decision	
12						0	x
13						0	y
14							
15							
16							
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Extension to more objectives

Now although the solution shown above is the minimum cost solution it does raise a difficulty. If we adopt this solution we will be producing precisely as much medium and low-grade ore per week as we need. This is good. However we will be producing $6(12/7)+1(20/7) = 13.14$ tonnes of high-grade per week. As the contract is only for 12 tonnes we will somehow have to deal with this excess.

Hence we can see that the Two Mines company might well feel that whereas before we had a single objective problem:

- minimise cost

now we have a problem that has two (conflicting) objectives:

- minimise cost
- do not produce excess ore

where although excess ore might be taken as referring just to high-grade ore a moment's thought will reveal that if we move from our current solution to reduce the excess of high-grade ore we might find an excess of another grade of ore. Hence we need to consider all grades and their excess rather than just high-grade.

Goal programming is one approach to dealing with problems of this kind.

Goal programming formulation

To deal with these two objectives in our example problem via GP we need to introduce extra variables – these variables deal with the deviation from the goal for each objective. To proceed we need to decide a numeric goal for each objective.

We now have two objectives:

- minimise cost – our previously calculated minimum cost was 765.71 (per week). The company may consider that, in the interests of eliminating excess ore, they would be prepared to increase the cost that they incur, but they would not like this cost to increase beyond 800. This figure of 800 becomes our numeric cost goal.
- do not produce excess ore – which actually relates to the three grades of ore for which the respective goals are the amount that we need: 12, 8 and 24 tonnes respectively.

Let C^+ (≥ 0) represent the amount by which we deviate upward from our cost goal and C^- (≥ 0) represent the amount by which we deviate downward from our cost goal. Then we have an equation linking these new variables to our old variables:

$$180x + 160y = 800 + C^+ - C^-$$

This constraint, which must be an equality, says that whatever we decide in terms of production on mines X and Y (x and y respectively) the cost of that ($180x + 160y$) must equal the goal (800) adjusted by the deviation variables C^+ and C^- , the plus in front of C^+ in this equation indicating an upward movement (deviation) from the goal and the minus in front of C^- in this equation indicating a downward movement (deviation) from the goal.

There are two comments to make here:

- it is standard notational practice in GP to use a plus superscript to indicate upward deviation from the goal and a negative superscript to indicate downward deviation from the goal, you will see that in defining C^+ and C^- we have adopted this standard.
- one subtle point here is that the way we have written our equation including $C^+ - C^-$ opens the possibility that when we come to numerically solve the problem we get an answer like $C^+=100$ and $C^-=120$, so both an upward and a downward deviation with the overall deviation being $C^+ - C^- = 100 - 120 = -20$; a downward deviation of 20. The simple answer is that, as our equation $180x + 160y = 800 + C^+ - C^-$ is currently written, there is nothing to prevent this happening. The reason why effects like this tend not to happen lies in the nature of the objective that we shall adopt later below in terms of deciding variable values.

We can now deal with our objective relating to production of excess ore. Letting H^+ , M^+ and L^+ (all ≥ 0) be the upward deviation for high, medium and low-grade respectively and H^- , M^- and L^- (all ≥ 0) be the downward deviation for high, medium and low-grade respectively we have the three equality equations:

$$6x + 1y = 12 + H^+ - H^-$$

$$3x + 1y = 8 + M^+ - M^-$$

$$4x + 6y = 24 + L^+ - L^-$$

One point to note here is that the above equations leave open the possibility that the company might decide not to supply all the ore for some particular grade (i.e. we have a downward deviation from the goal level where here the goal level relates to the amount that we are contracted to supply). If the company wishes to exclude this possibility then we simply set $H^- = M^- = L^- = 0$ (eliminate these variables from the problem).

Hence for our Two Mines problem where we wish to reconcile our conflicting objectives we have the equations:

$$\begin{aligned}
 180x + 160y &= 800 + C^+ - C^- \\
 6x + 1y &= 12 + H^+ - H^- \\
 3x + 1y &= 8 + M^+ - M^- \\
 4x + 6y &= 24 + L^+ - L^- \\
 x &\leq 5 \\
 y &\leq 5 \\
 x, y, C^+, C^-, H^+, H^-, M^+, M^-, L^+, L^- &\geq 0
 \end{aligned}$$

These equations are, given our variables, equations that must be satisfied.

To proceed there are a number of different approaches:

- to attach weights to each of the variables associated with upward/downward deviation and then to solve a single problem whose objective is to minimise this weighted deviation sum. This approach is sometimes known as **weighted goal programming**
- to decide priority levels for the goals (priority level 1 for the most important goal, then priority level 2 for the second most important goal, etc) and first satisfy priority one goals, then priority two goals, then..., so a sequence of related problems are solved. This approach is sometimes known as **sequential goal programming** or **preemptive goal programming** as priorities cannot be traded off against each other (unlike the weighted goal programming approach).

We shall illustrate both approaches.

Weighted approach

For the weighted approach suppose that we always wish to supply the amount of ore we are contracted to supply, so that $H^- = M^- = L^- = 0$ (i.e. these variables can be eliminated from the problem). This leaves variables C^+, C^-, H^+, M^+ and L^+ each of

which must be assigned a weight. This weight can only come from managerial consideration of the situation and, if necessary, starting with a set of weights and then revising them in the light of the solution obtained after solving the problem with those weights.

There is an important issue associated with weights here – namely that the equations that we are considering deal with different units – cost (£'000) and tonnes of high, medium and low-grade ore. What needs to be made clear to management in setting these weights is that they need to think in terms of the weight associated with **one percentage deviation** from the current goal for each variable.

Suppose, to proceed, that management have decided that the weights to be applied are:

Variable	Current goal	Weight for one per cent deviation from this goal	One per cent of goal
C^+	800	25	8
C^-	800	-20	8
H^+	12	40	0.12
M^+	8	10	0.08
L^+	24	3	0.24

Note the negative weight for C^- , this indicates that (as we shall be minimising, see below) we prefer to have a downward deviation for our cost goal – a natural desire.

Then our objective becomes:

$$\text{minimise } 25(C^+/8) - 20(C^-/8) + 40(H^+/0.12) + 10(M^+/0.08) + 3(L^+/0.24)$$

where in this objective $(C^+/8)$, for example, is the total percentage upward deviation from the cost goal which is weighted using 25 (see above).

Rearranging, our objective is: $3.125C^+ - 2.5C^- + 333.333H^+ + 125M^+ + 12.5L^+$ and so we have the linear program:

$$\text{minimise } 3.125C^+ - 2.5C^- + 333.333H^+ + 125M^+ + 12.5L^+$$

subject to

$$180x + 160y = 800 + C^+ - C^-$$

$$6x + 1y = 12 + H^+$$

$$3x + 1y = 8 + M^+$$

$$4x + 6y = 24 + L^+$$

$$\begin{aligned}
 x &\leq 5 \\
 y &\leq 5 \\
 x, y, C^+, C^-, H^+, M^+, L^+ &\geq 0
 \end{aligned}$$

which, when we solve it, will give us values for x and y , the number of days to work each mine, that minimises the (total) weighted deviation from our goals.

Excel solution

Take the spreadsheet associated with this topic and look at Sheet A. You should see the problem we considered above set out as:

	A	B	C	D	E	F	G
1		Cost	High	Medium	Low		
2	Mine X	180	6	3	4		
3	Mine Y	160	1	1	6		
4	Total	0	0	0	0		
5	Goal	800	12	8	24		
6	Current LP left-hand side	0	0	0	0		
7							
8	Upward deviation weight	25	40	10	3		
9	Downward deviation weight	-20					
10							
11						Decision	
12	Upward deviation variable	0	0	0	0	0	x
13	Downward deviation variable	0				0	y
14							
15	Total deviation	0					

Here we have expressly given the weights (for one per cent deviation) both for upward and downward deviations as in cells B8 to E8 and B9.

Cells B12 to E12 and B13 contain the deviation variables we are trying to decide and cells F12 and F13 the variables (x and y respectively) associated with how many days per week to work each mine. Row 6 represents a rearrangement of the constraints in our LP so that cell B6, for example, is the left-hand side of the constraint $180x + 160y = 800 + C^+ - C^-$ when rearranged to have all variables collected together, i.e. it is $180x + 160y - C^+ + C^-$ which in our LP solution needs to be equal to 800, shown above in cell B5 where row 5 shows the goals for the problem.

To use Solver in Excel do Tools and then Solver. In the version of Excel I am using (different versions of Excel have slightly different Solver formats) you will get the Solver model as below:

	A	B	C	D	E	F	G
1		Cost	High	Medium	Low		
2	Mine X	180	6	3	4		
3	Mine Y	160	1	1	6		
4	Total	0	0	0	0		
5	Goal	800	12	8	24		
6	Current LP left-hand side	0	0	0	0		
7							
8	Upward deviation weight	25	40	10	3		
9	Downward deviation weight	-20					
10							
11						Decision	
12	Upward deviation variable	0	0	0	0	0	x
13	Downward deviation variable	0				0	y
14							
15	Total deviation	0					
16							
17							
18							
19							
20							
21							
22							
23							
24							
25							
26							
27							
28							
29							
30							
31							

Solver Parameters

Set Target Cell:

Equal To: Max Min Value of:

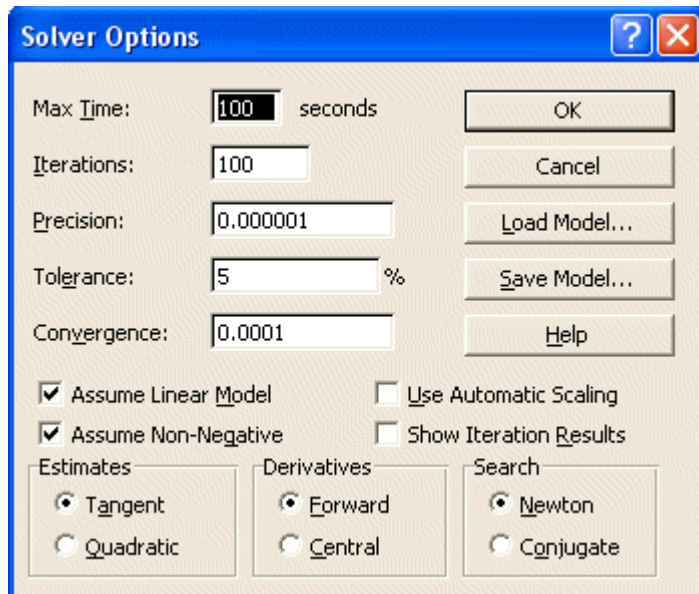
By Changing Cells:

Subject to the Constraints:

\$B\$6:\$E\$6 = \$B\$5:\$E\$5

\$F\$12:\$F\$13 <= 5

Here the objective function (total weighted deviation) is in cell B15 (ignore the use of \$ signs here – that is a technical Excel issue if you want to go into it in greater detail), which we wish to minimise by changing our variables. If you click on Options you will see:



where both the ‘Assume Linear Model’ and ‘Assume Non-Negative’ boxes are ticked – indicating we are dealing with a linear model with non-negative variables.

Solving using Solver we get:

	A	B	C	D	E	F	G
1		Cost	High	Medium	Low		
2	Mine X	180	6	3	4		
3	Mine Y	160	1	1	6		
4	Total	800	12.8	8	25.6		
5	Goal	800	12	8	24		
6	Current LP left-hand side	800	12	8	24		
7							
8	Upward deviation weight	25	40	10	3		
9	Downward deviation weight	-20					
10							
11						Decision	
12	Upward deviation variable	0	0.8	0	1.6	1.6	x
13	Downward deviation variable	0				3.2	y
14							
15	Total deviation	286.6667					

where the deviation variables show no deviation from our cost goal, i.e. the solution shown above of $x=1.6$ and $y=3.2$ has cost equal to the cost goal of 800. In this solution we over-produce high-grade ore (the 0.8 upward deviation variable) and also over-produce low-grade ore (the 1.6 upward deviation variable).

The advantage here is that once we have gone to the effort of generating a computerised solution approach (encoded the problem in Excel using Solver here) we can use it to investigate sensitivity, or to adjust the solution to one that we prefer.

Here, for example, it may be that management after considering the above solution decide that they wish to increase the weight associated with the over-production of high-grade ore to 50 (i.e. to increase the weight attached to the upward deviation variable H^+ from 40 to 50).

Making this change and resolving using Solver we get:

	A	B	C	D	E	F	G
1		Cost	High	Medium	Low		
2	Mine X	180	6	3	4		
3	Mine Y	160	1	1	6		
4	Total	880	12	8	29.3333		
5	Goal	800	12	8	24		
6	Current LP left-hand side	800	12	8	24		
7							
8	Upward deviation weight	25	50	10	3		
9	Downward deviation weight	-20					
10							
11						Decision	
12	Upward deviation variable	80	0	0	5.33333	1.3333333	x
13	Downward deviation variable	0				4	y
14							
15	Total deviation	316.6667					

showing that with this set of weights we avoid over-production of high-grade ore, but this is achieved by having a larger over-production of low-grade ore (the upward deviation variable in cell E12) and also an upward deviation on cost (the upward deviation variable of 80 in cell B12) so that the total cost has risen to 880.

Irrespective of what solution management prefer it is clear that GP provides them with a flexible tool to investigate the effect of differing weightings as they attempt to find a solution that satisfies their conflicting objectives.

Priority approach

For the priority approach we will again assume that we always wish to supply the amount of ore we are contracted to supply, so that $H^-=M^-=L^-=0$ (i.e. these variables can be eliminated from the problem). This leaves variables C^+ , C^- , H^+ , M^+ and L^+ .

As mentioned above in the priority approach we need to decide priority levels for the goals (priority level 1 for the most important goal, then priority level 2 for the second most important goal, etc) and first satisfy priority one goals, then priority two goals, then..., so that a sequence of related problems are solved.

Here, for the purposes of illustrating the approach, we shall assume that management consider that their priority levels are:

- priority level 1 – as little excess high-grade ore as possible
- priority level 2 – cost
- priority level 3 – as little excess combined medium-grade and low-grade ore as possible

Hence the first problem that we solve (a linear program) is:

minimise H^+

subject to

$$180x + 160y = 800 + C^+ - C^-$$

$$6x + 1y = 12 + H^+$$

$$3x + 1y = 8 + M^+$$

$$4x + 6y = 24 + L^+$$

$$x \leq 5$$

$$y \leq 5$$

$$x, y, C^+, C^-, H^+, M^+, L^+ \geq 0$$

Excel solution

Take the spreadsheet associated with this topic and look at Sheet B. You should see the problem we considered above set out as below where we also show the Solver model for Sheet B.

	A	B	C	D	E	F	G
1		Cost	High	Medium	Low		
2	Mine X	180	6	3	4		
3	Mine Y	160	1	1	6		
4	Total	0	0	0	0		
5	Goal	800	12	8	24		
6	Current LP left-hand side	0	0	0	0		
7							
8						Decision	
9	Upward deviation variable	0	0	0	0	0	x
10	Downward deviation variable	0				0	y
11							
12	Objective	0					
13							
14							
15							
16							
17							
18							
19							
20							
21							
22							
23							
24							
25							
26							
27							
28							

The objective for this Solver model is cell B12, which when you examine the spreadsheet you will find is equal to cell C9, i.e. the upward deviation variable H^+ . If you click Options in the Solver model you will also find that the ‘Assume Linear Model’ and ‘Assume Non-Negative’ boxes are ticked – indicating we are dealing with a linear model with non-negative variables.

Solving the linear program represented in the above Solver model we get:

	A	B	C	D	E	F	G
1		Cost	High	Medium	Low		
2	Mine X	180	6	3	4		
3	Mine Y	160	1	1	6		
4	Total	880	12	8	29.3333		
5	Goal	800	12	8	24		
6	Current LP left-hand side	800	12	8	24		
7							
8						Decision	
9	Upward deviation variable	80	0	0	5.33333	1.3333333	x
10	Downward deviation variable	0				4	y
11							
12	Objective	0					

Indicating that we can achieve zero excess high-grade ore (a zero objective value corresponding to a zero upward deviation variable for high-grade ore), but at the price of an upward deviation from our cost goal of 80 and an upward deviation from our low-grade ore goal of 5.333.

We can now move to our second priority level. This was cost so we have the new linear program:

minimise $C^+ - C^-$

subject to

$$180x + 160y = 800 + C^+ - C^-$$

$$6x + 1y = 12 + H^+$$

$$3x + 1y = 8 + M^+$$

$$4x + 6y = 24 + L^+$$

$$x \leq 5$$

$$y \leq 5$$

$$H^+ = 0$$

$$x, y, C^+, C^-, H^+, M^+, L^+ \geq 0$$

where we are trying to minimise deviation from our cost goal but where the upward deviation variable H^+ for high-grade ore is now constrained to be zero (the value it had in our solution calculated above at the first priority level).

Note the objective function here is minimise $C^+ - C^-$, i.e. $+ C^+ - C^-$, where the positive sign before C^+ indicates that we dislike upward deviations from our cost goal but the minus sign before C^- indicates that we like downward deviations from our cost goal. Had we wished to just to minimise deviation from our cost goal our objective would have been minimise $C^+ + C^-$.

Take the spreadsheet associated with this topic and look at Sheet C and its associated Solver model. You will see:

	A	B	C	D	E	F	G
1		Cost	High	Medium	Low		
2	Mine X	180	6	3	4		
3	Mine Y	160	1	1	6		
4	Total	0	0	0	0		
5	Goal	800	12	8	24		
6	Current LP left-hand side	0	0	0	0		
7							
8						Decision	
9	Upward deviation variable	0	0	0	0	0	x
10	Downward deviation variable	0				0	y
11							
12	Objective	0					
13							
14							
15							
16							
17							
18							
19							
20							
21							
22							
23							
24							
25							
26							
27							
28							

where the constraint relating to variable H^+ (cell C9) has been added and if you examine the objective cell B12 you will find that it is now equal to $B9 - B10$, i.e. $C^+ - C^-$. Solving we get:

	A	B	C	D	E	F	G
1		Cost	High	Medium	Low		
2	Mine X	180	6	3	4		
3	Mine Y	160	1	1	6		
4	Total	880	12	8	29.3333		
5	Goal	800	12	8	24		
6	Current LP left-hand side	800	12	8	24		
7							
8						Decision	
9	Upward deviation variable	80	0	0	5.33333	1.3333333	x
10	Downward deviation variable	0				4	y
11							
12	Objective	80					

which is actually the solution we had before, i.e. our previous solution that we obtained when we minimised upward deviation for high-grade ore also happened to minimise our cost goal.

We can now consider the third priority level which was as little excess combined medium-grade and low-grade ore as possible. This implies that we minimise the combined (unweighted) deviation for medium and low-grade ore. The linear program for this problem is:

$$\begin{aligned} &\text{minimise} && M^+ + L^+ \\ &\text{subject to} && \\ &&& 180x + 160y = 800 + C^+ - C^- \\ &&& 6x + 1y = 12 + H^+ \\ &&& 3x + 1y = 8 + M^+ \\ &&& 4x + 6y = 24 + L^+ \\ &&& x \leq 5 \\ &&& y \leq 5 \\ &&& H^+ = 0 \\ &&& C^+ - C^- = 80 \\ &&& x, y, C^+, C^-, H^+, M^+, L^+ \geq 0 \end{aligned}$$

where we have added a constraint specifying that the objective at the second priority level ($C^+ - C^-$) must retain the minimum value (80) we calculated above.

Take the spreadsheet associated with this topic and look at Sheet D and its associated the Solver model. You will see:

	A	B	C	D	E	F	G
1		Cost	High	Medium	Low		
2	Mine X	180	6	3	4		
3	Mine Y	160	1	1	6		
4	Total	0	0	0	0		
5	Goal	800	12	8	24		
6	Current LP left-hand side	0	0	0	0		
7							
8						Decision	
9	Upward deviation variable	0	0	0	0	0	x
10	Downward deviation variable	0				0	y
11	Cost goal (second priority level)	-10					
12							
13	Objective	0					

Solver Parameters

Set Target Cell: Solve

Equal To: Max Min Value of: Close

By Changing Cells: Guess

Subject to the Constraints:

\$B\$11 = 80

\$B\$6:\$E\$6 = \$B\$5:\$E\$5

\$C\$9 = 0

\$F\$9:\$F\$10 <= 5

Options

Reset All

Help

where the constraint relating to variable $C^+ - C^- = 80$ has been added using cell B11 and the constraint that B11=80 in the Solver model. If you examine the objective cell B13 you will find that it is now equal to $D9+E9$, i.e. $M^+ + L^+$. Solving we get:

	A	B	C	D	E	F	G
1		Cost	High	Medium	Low		
2	Mine X	180	6	3	4		
3	Mine Y	160	1	1	6		
4	Total	880	12	8	29.3333		
5	Goal	800	12	8	24		
6	Current LP left-hand side	800	12	8	24		
7							
8						Decision	
9	Upward deviation variable	90	0	0	5.33333	1.3333333	x
10	Downward deviation variable	10				4	y
11	Cost goal (second priority level)	80					
12							
13	Objective	5.333333					

where an effect we referred to before appears – we have both an upward and downward deviation variable – upward of 90 and downward of 10, equivalent to an

upward of 80. This is not what we wanted but the resolution of this problem is easy – simply put into the spreadsheet an upward deviation variable of 80 and a downward deviation variable of 0, you will get:

	A	B	C	D	E	F	G
1		Cost	High	Medium	Low		
2	Mine X	180	6	3	4		
3	Mine Y	160	1	1	6		
4	Total	880	12	8	29.3333		
5	Goal	800	12	8	24		
6	Current LP left-hand side	800	12	8	24		
7							
8						Decision	
9	Upward deviation variable	80	0	0	5.33333	1.3333333	x
10	Downward deviation variable	0				4	y
11	Cost goal (second priority level)	70					
12							
13	Objective	5.333333					

which we can sensibly interpret and which has precisely the same objective function value and also satisfies the constraints we imposed on the first and second level priorities. The technical explanation of why this difficulty appeared here is that the underlying linear program was degenerate (there were two or more solutions with the same objective function value).