ZIONTS, Stanley, 2004, **"Linear and integer programming",** Prentice-Hall Internat'l. ISBN: 0-13-536763-8. *Four types of problem structure:* **1. Resource allocation structure**

DOG FOOD PROBLEM

Later we shall make extensive use of a formulation having only equalities for our conceptual and theoretical framework. It can be shown that such a formulation is just as general as formulation (1.1). For solution purposes, any combination of equalities and inequalities that describes the problem may be used.

1.3 AN EXAMPLE

For illustrative purposes, we shall develop an example that will be used repeatedly to develop various aspects of linear programming. It is simplified, and we shall alter the problem as we proceed to illustrate various principles.

Suppose that a dog food manufacturer, Canine Products, Inc., produces two blends of dog food, Frisky Pup and Husky Hound. Two raw materials, cereal and meat, are available. Assuming the data given in Table 1.1 are applicable to the problem, the manufacturer wants to find a production mix that maximizes his profits. Frisky Pup dog food is a blend of 1 pound cereal and 1.5 pounds meat, and sells for 70 cents per 2.5 pound package. Husky Hound dog food is a blend of 2 pounds cereal and 1 pound meat and sells for 60 cents

Table 1.1

Data for Example Problem

	FRISKY PUP	HUSKY HOUND
Contents of finished package	2.5 lb.	3.0 lb.
Sale price per package	$ .70	$ .60
Raw materials usage per package		
Cereal	1.0 lb.	2.0 lb.
Meat	1.5 lb.	1.0 lb.
Purchase price of raw materials		
Cereal	$0.10/lb.	$0.10/lb.
Meat	$0.20/lb.	$0.20/lb.
Blending, packaging, and other variable costs per package	$ .14	$ .18

Resources available for production per month:

Raw materials

Cereal	240,000 lbs.
Meat	180,000 lbs.

Processing capacities

Husky Hound blending Husky Hound packaging Frisky Pup blending	sufficient for production of any product mix within above raw material availability.
Frisky Pup packaging	A maximum of 110,000 packages per month.

1. The marketing manager estimates that any feasible mix of the two blends, given the resource restrictions, can be sold at prices indicated above.

2. Canine Products has entered into long-term contracts with the suppliers of raw materials whereby every month Canine Products is required to purchase the quantities shown above.

3. Any unused quantity of raw materials left over at the end of a month is a total loss—it can neither be used in the following month nor can it be sold.

per 3 pound package. Cereal costs 10 cents per pound; meat costs 20 cents per pound. Husky Hound dog food costs 18 cents per package for packaging and Frisky Pup dog food costs 14 cents per package for packaging. Blending of the meat and cereal is accomplished automatically during the packaging. Frisky Pup requires the use of a special packaging machine.

In a one month period the company has available 240,000 pounds of cereal and 180,000 pounds of meat for which it has contracted at the above prices. If the meat or cereal is not completely used there is no alternate use for it. The special packaging machine for Frisky Pup dog food can package as many as 110,000 units per month. The Husky Hound packaging facility is of sufficient capacity to handle any mixture of products, given the raw material available. The marketing manager of Canine Products estimates that all of Canine production—whatever the mixture of products, given the raw materials available—will be sold at the indicated prices. Assuming that all of the variable costs of production are indicated above, what is the production plan that maximizes contribution to overhead and profits?

This problem is distinctly different from choosing a specific blend to accomplish a specific purpose, such as finding the lowest cost blend that satisfies certain nutritional requirements. That problem can also be formulated and solved as a linear programming problem, providing the constraints are linear.

To formulate the given problem, let x_1 and x_2 respectively represent the number of packages of Frisky Pup and Husky Hound dog foods produced in a month. The number of pounds of cereal required for production of Frisky Pup dog food is

$$1\,\frac{\text{pound}}{\text{package}} \cdot (x_1 \text{ packages}) = x_1 \text{ pounds}$$

and the number of pounds of cereal required for Husky Hound dog food is

$$2\,\frac{\text{pounds}}{\text{package}} \cdot (x_2 \text{ packages}) = 2x_2 \text{ pounds}$$

The total cereal required to produce an arbitrary program of production of x_1 packages of Frisky Pup and x_2 packages of Husky Hound is then

$$x_1 + 2x_2$$

Incorporating the cereal availability, we have the inequality constraint

$$x_1 + 2x_2 \leq 240{,}000 \tag{1.2}$$

In a similar manner, we determine that $1.5x_1$ and $1x_2$ pounds of meat are used to produce x_1 packages of Frisky Pup and x_2 packages of Husky Hound dog foods, respectively. Incorporating the meat availability for the month, we have the inequality constraint

$$1.5x_1 + x_2 \leq 180{,}000 \tag{1.3}$$

The packaging constraint for Frisky Pup is simply that, at most, 110,000 packages can be prepared in a given month, or

(1.4) $$x_1 \leq 110{,}000$$

Since the number of packages produced must be zero or positive, the following constraints must be satisfied:

(1.5) $$x_1 \geq 0, \; x_2 \geq 0$$

We now formulate the objective function. Our objective is to find the values of x_1 and x_2 which yield the maximum profit without violating the above constraints. Since a fixed cost is sunk, profits are maximized when we maximize the total contribution to overhead and profits (excess of selling price over marginal cost). We must therefore first determine the contribution of a unit of x_1 and of a unit of x_2. The computations are given in Table 1.2.

Table 1.2

Computation of Per Unit Contribution to Overhead and Profits For Example

	FRISKY PUP	HUSKY HOUND
	x_1 / \$	x_2 / \$
I. Selling price per package	0.70	0.60
Marginal cost:		
Material cost		
Cereal*	—	—
Meat*	—	—
II. Total variable costs of blending and packaging	0.14	0.18
Contribution (I − II)	0.56	0.42

* In view of notes 2 and 3 to Table 1.1, cost of these materials is treated as fixed or sunk costs which are in total independent of the actual production.

We can now formulate our objective function as

(1.6) $$\text{Maximize Total Contribution } z = .56x_1 + .42x_2$$

Writing expressions (1.2) to (1.6), we have the complete linear programming problem, which is to

$$
\begin{array}{lll}
(1.6) & \text{Maximize } z = .56x_1 + .42x_2 & \\
(1.2) & \text{subject to:} & x_1 + 2x_2 \leq 240{,}000 \\
(1.3) & & 1.5x_1 + x_2 \leq 180{,}000 \\
(1.4) & & x_1 \leq 110{,}000 \\
(1.5) & & x_1, x_2 \geq 0
\end{array}
$$

(See results in the end.)

By explanation, any solution to the problem that we may find must be *feasible*. A solution is said to be feasible if it *satisfies* all the constraints. For example, the following, among many others, are feasible solutions:

$$x_1 = 100{,}000,\ x_2 = 0;\ x_1 = 1{,}954,\ x_2 = 76{,}950;$$
$$x_1 = 50{,}000,\ x_2 = 50{,}000;\ x_1 = 0,\ x_2 = 0$$

On the other hand, $x_1 = 115{,}000$, $x_2 = 10{,}000$; and $x_1 = 50{,}000$, $x_2 = 100{,}000$ are *not* feasible solutions because they violate one or more constraints.

Of all possible feasible solutions to the problem posed above, only a subset can fulfill the objective, namely, to maximize the objective function, in this case profits. That solution (or solutions) is called an *optimal feasible solution*. For some problems there may be no solutions that satisfy the constraints. No feasible solution is said to exist for such problems. The occurrence of no feasible solution generally indicates one of two conditions:

1. An error has been made in formulating a problem for which a feasible solution exists.
2. An inconsistent problem has been formulated in the sense that it is impossible to satisfy all constraints simultaneously.

Another anomalous kind of solution that may occur is an optimal solution that is infinite. This is called an *unbounded solution* and usually indicates that the problem has not been correctly formulated. Such a condition would occur, for our example, if unlimited supplies of meat and cereal were available at the indicated prices. Altering the formulation accordingly, the optimal solution would be to produce 110,000 packages of Frisky Pup dog food and an infinite amount of Husky Hound. Such a solution, aside from being nonsensical, is incorrect for the following reasons:

1. Only a finite amount of meat and cereal has been contracted for at the indicated price. In general, as greater and greater quantities are purchased, the price of the materials might be expected to rise.
2. We have not considered limitations on blending capacity and Husky Hound packaging capacity, which are presumably finite.
3. The marketing manager's assumption about selling all that can be produced cannot be extended beyond certain limits.

For the example problem, it can be shown that the unique optimal feasible solution would be to produce 60,000 packages of Frisky Pup and 90,000 packages of Husky Hound per month. The reader should attempt to verify the optimality of this solution by trial and error. Total contribution to overhead and profits is $71,400 per month for the optimal solution. Later, methods for computing the optimal solution in general will be introduced.

Resolution by Lindo

PROBLEM

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max 0.56 x1 + 0.42 x2
st
x1 + 2 x2 < 240
1.5 x1 + x2 < 180
x1 < 110
end
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RESULTS

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 LP OPTIMUM FOUND AT STEP      2

        OBJECTIVE FUNCTION VALUE

        1)      71.40000

  VARIABLE        VALUE          REDUCED COST
        X1        60.000000          0.000000
        X2        90.000000          0.000000


       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         0.000000          0.035000
        3)         0.000000          0.350000
        4)        50.000000          0.000000

 NO. ITERATIONS=       2


 RANGES IN WHICH THE BASIS IS UNCHANGED:

                          OBJ COEFFICIENT RANGES
 VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                    COEF          INCREASE         DECREASE
       X1        0.560000         0.070000         0.350000
       X2        0.420000         0.700000         0.046667

                          RIGHTHAND SIDE RANGES
      ROW         CURRENT        ALLOWABLE        ALLOWABLE
                    RHS           INCREASE         DECREASE
        2      240.000000       120.000000       100.000000
        3      180.000000        50.000000        60.000000
        4      110.000000         INFINITY        50.000000
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