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TABLE 2.3 Nursing Staff Requirements

Period of the Day	*Number of Nurses Required*
8:00–12:00	140
12:00–16:00	120
16:00–20:00	160
20:00–24:00	90
24:00– 4:00	30
4:00– 8:00	60

2.3 SOME SCIENTIFIC APPLICATIONS OF LINEAR PROGRAMMING

The examples of Section 2.2 are simple versions of practical problems that arise in the work of business people, managers of industry, military planners, and other executives whose main concern is with routine operational matters. In addition to helping with the solution of problems from that sphere of activity, linear programming also finds numerous more abstract applications in engineering design and scientific research. This section presents some examples from those areas of application.

Curve Fitting

Values R_i are obtained from laboratory measurements of a certain physical quantity $R(t)$ at times t_i, $i = 1, \ldots, N$. Random experimental errors introduce noise in the measurements, but it is known theoretically that $R(t)$ depends on time t according to a quadratic relationship of the form

$$R(t) = at^2 + bt + c$$

The **parameters** a, b, and c are unknown and are to be estimated from the experimental measurements R_i. This kind of curve-fitting problem is often referred to as a **polynomial regression.** One approach to the problem is to find parameter values a, b, and c so as to minimize the absolute value of the largest discrepancy between the measured values R_i and corresponding theoretical values $R(t_i)$. For a given set of parameter values a, b, and c, the **deviation** of observation i from the value predicted by the **model function** $R(t)$ is defined as

$$D_i = R_i - [a(t_i)^2 + b(t_i) + c]$$

Using this definition, one way to formally state the mathematical problem of finding the best parameter values by minimizing the largest deviation is

$$\underset{a,b,c}{\text{minimize}}\ [\underset{i}{\text{maximum}}\ |D_i|]$$

The list of parameters beneath the word *minimize* means that the minimization is to be performed by varying the values of those parameters; similarly, the maximum is to be taken over the **absolute deviations** $|D_i|$, with a, b, and c held constant. Because we are minimizing the maximum absolute deviation, such a formulation is commonly called a **minimax** problem.

Finding values for the parameters a, b, and c to minimize the largest absolute deviation might not at first seem like a linear programming problem. Also, deciding on what decision variables to use does not appear to be as important an issue as it always was in the problems of Section 2.2. After all, the only things we can control are the values of the parameters, so they are natural choices as decision variables. There is, however, a dependent variable in this problem, namely, the largest deviation, which depends on the values of the parameters. Suppose we let

$$w = \underset{i}{\text{maximum}} \{|D_i|\}$$

and try to write a linear programming formulation involving that quantity.

Because w is the largest absolute deviation, it must satisfy each of the inequalities

$$w \geq |R_i - [a(t_i)^2 + b(t_i) + c]| \qquad i = 1, \ldots, N$$

Of course, for any given set of parameter values, one (or more) of the deviations D_i will be larger than the others, and so the corresponding inequality will actually hold with equality. The problem, then, is to find values for a, b, c, and w so that each of the inequalities is satisfied and so that w is as small as possible. In other words, we need to

$$\begin{aligned} &\text{minimize} \quad w \\ &\text{subject to} \\ &\qquad w \geq |R_i - [a(t_i)^2 + b(t_i) + c]| \qquad i = 1, \ldots, N \end{aligned}$$

Unfortunately, this problem is not a linear program as it stands because the absolute value of a linear function is not a linear function. However, we can convert it into a linear program by making use of the following elementary fact:

> For any w and y,
> $w \geq |y|$ if and only if $w \geq y$ and $w \geq -y$.

Using this fact, we can replace each of the nonlinear inequality constraints above with two linear inequality constraints. Then the preceding optimization problem is equivalent to the following linear program in the variables w, a, b, and c:

$$\begin{aligned} &\text{minimize} \quad w \\ &\text{subject to} \\ &\qquad w \geq (R_i - [a(t_i)^2 + b(t_i) + c]) \qquad i = 1, \ldots, N \\ &\qquad w \geq -(R_i - [a(t_i)^2 + b(t_i) + c]) \qquad i = 1, \ldots, N \end{aligned}$$

Nothing in the statement of the problem requires the parameters to be nonnegative, so a, b, and c are **free variables,** that is, variables unconstrained in sign. It would be harmless to include a nonnegativity constraint on the variable w, but it would also be superfluous because the above constraints already require w to be nonnegative.

Solving this linear program yields values for the parameters a, b, and c that minimize the largest absolute deviation, and an optimal value of w that is equal to that largest absolute deviation.