```
1   0   A = 0      A <= 0.5        A >= 0
1   1   A = 1      A <= 1          A >= 1
```

Then, recalling that A can also only take zero-one values, it is clear that in each of the four possible cases the two linear constraints (A <= (B+C)/2 and A >= B+C-1) are equivalent to the single non-linear constraint (A=BC).

Returning now to our original non-linear constraint

- $z_t = x_{t-1}y_t$

this involves the three zero-one variables z_t, x_{t-1} and y_t and so we can use our general rule and replace this non-linear constraint by the two linear constraints

$$z_t \le (x_{t-1} + y_t)/2 \quad t=1,2,...,6$$
and $$z_t \ge x_{t-1} + y_t - 1 \quad t=1,2,...,6$$

Making this change transforms the non-linear integer program given before into an equivalent linear integer program.

Integer programming example 1996 MBA exam

A toy manufacturer is planning to produce new toys. The setup cost of the production facilities and the unit profit for each toy are given below. :

Toy	Setup cost (£)	Profit (£)
1	45000	12
2	76000	16

The company has two factories that are capable of producing these toys. In order to avoid doubling the setup cost only *one* factory could be used.

The production rates of each toy are given below (in units/hour):

	Toy 1	Toy 2
Factory 1	52	38
Factory 2	42	23

Factories 1 and 2, respectively, have 480 and 720 hours of production time available for the production of these toys. The manufacturer wants to know *which* of the new toys to produce, *where* and *how many* of each (if any) should be produced so as to maximise the total profit.

- Introducing 0-1 decision variables *formulate* the above problem as an integer program. (Do *not* try to solve it).
- Explain briefly how the above mathematical model can be used in production planning.

Solution

Variables

We need to decide whether to setup a factory to produce a toy or not so let f_{ij} = 1 if factory i (i=1,2) is setup to produce toys of type j (j=1,2), 0 otherwise

We need to decide how many of each toy should be produced in each factory so let x_{ij} be the number of toys of type j (j=1,2) produced in factory i (i=1,2) where x_{ij} >=0 and integer.

Constraints

- at each factory cannot exceed the production time available

$x_{11}/52 + x_{12}/38 <= 480$
$x_{21}/42 + x_{22}/23 <= 720$

- cannot produce a toy unless we are setup to do so

$x_{11} <= 52(480)f_{11}$
$x_{12} <= 38(480)f_{12}$
$x_{21} <= 42(720)f_{21}$
$x_{22} <= 23(720)f_{22}$

Objective

The objective is to maximise total profit, i.e.

maximise $12(x_{11}+ x_{21}) + 16(x_{12}+ x_{22}) - 45000(f_{11} + f_{21}) - 76000(f_{12} + f_{22})$

Note here that the question says that in order to avoid doubling the setup costs only one factory could be used. This is not a constraint. We can argue that if it is only cost considerations that prevent us using more than one factory these cost considerations have already been incorporated into the model given above and the model can decide for us how many factories to use, rather than we artificially imposing a limit via an explicit constraint on the number of factories that can be used.

The above mathematical model could be used in production planning in the following way:

- enables us to maximise profit, rather than relying on an ad-hoc judgemental approach
- can be used for sensitivity analysis - for example to see how sensitive our production planning decision is to changes in the production rates
- enables us to see the effect upon production of (say) increasing the profit per unit on toy 1
- enables us to easily replan production in the event of a change in the system (e.g. a reduction in the available production hours at factory one due to increased work from other products made at that factory)
- can use on a rolling horizon basis to plan production as time passes (in which case we perhaps need to introduce a time subscript into the above model)

Integer programming example 1995 MBA exam

A project manager in a company is considering a portfolio of 10 large project investments. These investments differ in the estimated long-run profit (net present value) they will generate as well as in the amount of capital required.

Let P_j and C_j denote the estimated profit and capital required (both given in units of millions of £) for investment opportunity j (j=1,...,10) respectively. The total amount of capital available for these investments is Q (in units of millions of £)

Investment opportunities 3 and 4 are mutually exclusive and so are 5 and 6. Furthermore, neither 5 nor 6 can be undertaken unless either 3 or 4 is undertaken. At least two and at most four investment