PROBLEMS OF STOCKS

1 Introduction

1.1 General, 1.2, Characteristics of the problems of stocks, 1.3, Graphical representation, 1.4, Replenishment, 1.5, Replenishment delays.

— Arnold KAUFMANN, 1970, “Méthodes et modèles de la Recherche Opérationnelle”, Vol. I, 2.nd. edition, Dunod, Paris, p 165, Chapter IV, “Les Problèmes de stocks”

1.1 General

The supply of materials and equipment required for a manufacturing process, the customers’ orders, the reasonable availability of reserve parts incur varied problems. It is difficult to make a coherent and logical classification of the problems of stocks. The nature of demand should, however, be considered first:

- Determined (predictable with a certain accuracy);
- Random, but statistically stable;
- Random, but statistically unstable (seasonal);
- Unknown.

In stock problems, there can be constraints:

- Interactions between the various products;
- Limitations of means (volume, weight, financial availability, etc.).

Each time, an economic function will be defined to be optimized, which will often be, when demand is random, in the form of a mathematical expectation of global cost.

1.2 Characteristics of the problems of stocks

Given the variety of recognized stock problems in industrial practice or other areas, just a review of main cases will be done, to identify some simple concepts. The stock problems present themselves in the form of wait phenomena of a particular nature. Rather than assuming (as is done in the theory of queues) that units arrive one by one, it will be assume that arrivals relate to sets of units. The phenomena will be studied with support on probability, but in certain cases, otherwise frequent, in which variances are weak, deterministic models can be associated with them. All the problems of stocks include:

(1) A demand for certain articles, which is generally a random function of time, but may also be known and determined.

(2) The existence of a stock of the items to meet demand, which runs out and has to be replenished. The replenishment can be continuous, periodic or done at any intervals.

(3) Costs associated with these operations, investments, depreciation, insurance, various risks, storage, etc., and also one, more or less arbitrarily, assigned to stockout, which is essential in some problems. These costs allow to establish an economic function that we intend to optimize.

(4) Objectives to achieve or constraints involved as a consequence of the nature of the problem.

1.3 Graphical representation

In order to describe a problem of stocks, it is convenient to use the representation given in Figure 1, in which appear the initial stock, S_i, the final stock, S_f, the interval q separating them. In general, demand quantities are random, represented by steps. Often this path is replaced by a straight line or a curve which will give an easier analytical description of demand.

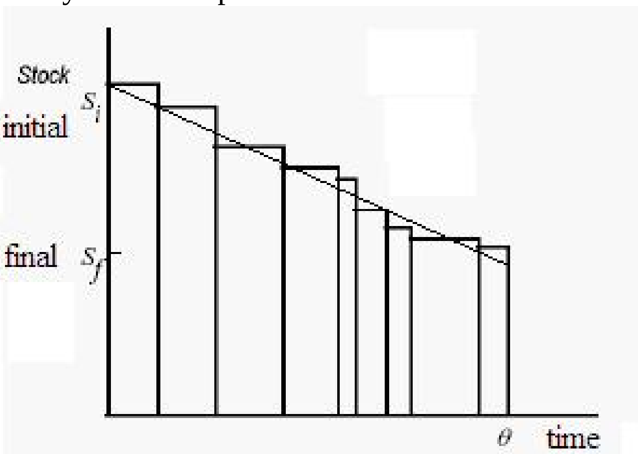

Figure 1

1.4 Replenishment

Suppose that the time interval between the issuance of the order to replenish and the reception is zero (negligible). Two main methods of basic inventory management are used. The first is called ***method by periods.*** A period T is established after which the replenishment is carried out systematically. This method has the drawback of risk of stockout and can lead to a costly management, but has the advantage of being automatic. The second may be termed a ***method of relaxation*** by analogy with physical phenomena of the same nature: the amount provided is constant, but the intervals T_1, T_2, T_3, ..., are no longer equal. There is no risk of stockout, the administration is generally less expensive, but not so easy to become systematic.

1.5 Replenishment delays

Suppose that the replenishment delay (time interval between issuing the order and reception) is independent of the amount ordered, i.e. constant and of duration t. Compare what would occur by either method. In the first (method by periods, T constant), the date of issue of order is known and it is necessary (to determine the quantity to be ordered) to extrapolate what was ordered in the interval $T - t$ preceding t; in certain cases, t can even be greater than T. In the second method (relaxation, several T_i), however, the quantity to order is constant, but the date of issue is unknown and has to be determined through extrapolation, which is sometimes insufficiently precise; in some cases, $t > T_i$. In general, the demand is known in probability. Sometimes, the delay is proportional to or a function of the order, which complicates the situation.

A method widely used for the management of stocks is to issue an order of constant size as soon as the stock reaches a critical value or *replacement level.* This may be called the *two-bin system* (“system of two boxes."). This method offers the advantage of a convenient management, but does not always guarantee against stockouts with sufficient probability.

2. Study of simple cases, proportional costs

2.1, First case: search for the economic (optimal) order quantity; **2.2**, Numerical example, **2.3**, Second case: EOQ with cost of shortage; **2.4**, Numerical example, **2.5**, Third case: random demand with loss on surplus and additional shortage cost (storage cost negligible) **2.6**, Numerical example; **2.7**, Search for the shortage cost; **2.8**, Resolution, **2.9**, Fourth case: random demand with costs of storage and shortage; **2.10**, Numerical example; **2.11**, Resolution by numerical calculation, **2.12**, Fifth case: known demand with storage cost proportional to the price of sale or purchase; **2.13**, Numerical example. (...)

Only the first and fourth cases will be briefly addressed below.

2.1 First case: the economic order quantity

Suppose parts of a certain model that are subject to constant demand, h parts per unit time, and stockout is not allowed. The parts are acquired in orders or lots. Suppose that a *fixed* cost of ordering, regardless of the number of parts, is c_L. [1]. The cost of storage of a part per unit time (day, for example) is c_S. The demand for a total q time interval, under study (*e.g.*, one year), is N. Assuming that all orders contain the same number of parts, n, the question is what value to give n so that the overall cost of ordering and storage of parts N is minimal (excluding the cost of the parts themselves). The number r of orders and the period T of replacement of the stock will also be determined.

The average level of the stock during a period T is $n/2$ (n in the beginning, 0 in the end). The storage cost during this period is thus $\frac{1}{2} n\, c_S\, T$. The total cost of an order is

$$c_L + \frac{1}{2} n c_S T \tag{1}$$

Moreover, it is

$$n = h\,T \tag{2}$$

and

$$r = \frac{N}{n} = \frac{q}{T} \tag{3}$$

The total cost for the time interval q is:

$$\begin{aligned} z &= \left(c_L + \frac{nT}{2} c_S\right) r = \left(c_L + \frac{nT}{2} c_S\right)\frac{N}{n} = \\ &= \frac{N}{n} c_L + \frac{NT}{2} c_S = \\ &= \frac{N}{n} c_L + \frac{q}{2} c_S n \end{aligned} \tag{4}$$

So, z depends on the variable n, the other parameters, N, q, c_L and c_S being known. The minimum z (obtained by differentiating or recalling that in the above form the two quantities must be equal)[2] occurs for

[1] L for “launch”.
[2] See Appendix.

$$n_0 = \sqrt{2\frac{N}{q}\frac{c_L}{c_S}} \qquad \{5\}$$

which is the optimum size sought. Substituting $n = n_0$ in $\frac{N}{n} = \frac{q}{T}$, we have

$$T_0 = \frac{n_0}{N} q = \sqrt{2\frac{N}{q}\frac{c_L}{c_S}}\frac{q}{N} = \sqrt{2\frac{q}{N}\frac{c_L}{c_S}} \qquad \{6\}$$

$$\left([T_0] = \sqrt{\frac{\mathrm{T}}{\mathrm{u}}\frac{\$}{\$/(\mathrm{u.T})}} = \mathrm{T}\right) \qquad \{6a\}$$

and, as total cost, from Eq. {4},

$$z_0 = \sqrt{2Nq\, c_L c_S} \qquad \{7\}$$

$$\left([z_0] = \sqrt{\mathrm{u\,T}\,\$\,\$/(\mathrm{u.T})} = \$\right) \qquad \{7a\}$$

2.2 Numerical example

A manufacturer receives an order for N = 120 000 parts, to be delivered in one year (q = 360 days). At what rate should he replenish his stock, if delay is not permissible in delivery ?

See “plate” http://web.ist.utl.pt/mcasquilho/compute/or/Fx-eoq.php . In the plate, the nomenclature is

Here		There	
N	demand in period	d	120e+3
c_L	setup cost	K	30e+3 $
(any)	purchase cost	c	1
c_S	holding cost	h	0,35 $/d × 360 d = 126 $

The demand in this case is at a constant rate. The costs are:

$$c_S = 0{,}35\ \$\ /\ \mathrm{day} \qquad c_L = 30\,000\ \$ \qquad \{8\}$$

We have:

$$n_0 = \sqrt{2\left(\frac{120000}{360}\right)\left(\frac{30\mathrm{E}3}{0{,}35}\right)} = 7559(,3)\ \text{parts} \qquad \{9\}$$

(Although it is not *a priori* important in this case, it should be numerically verified if n_0 is to be rounded down or up, examining the consequences in T_0 and, essentially, z_0).

$$T_0 = \frac{360 \times 7559}{120000} = 22{,}68\ \text{days} \qquad \{10\}$$

$$z_0 = \sqrt{2 \times 120000 \times 360 \times 30\mathrm{E}3 \times 0{,}35} = 952470 \qquad \{11\}$$

(This cost refers to one year.)

Another example, perhaps with more realistic data, is as follows (*in* Tavares *et al.* [1996], p 163), with its own nomenclature.

Annual demand	r = 1200 kg / year
Unit cost of purchase	C_1 = 20 $ / kg
Fixed cost of ordering	A = 15 $
Unit cost of possession	C_2 = 25 % of C_1 per year = 5 $ / kg-year

In the notation presented above (Kaufmann's):

Total demand (per year)	N = 1200 kg
Time span	q = 1 year
Fixed cost of ordering	A = 15 $
Unit purchase cost	C = 20 $ / kg
Fixed cost of ordering	c_L = 15 $
Cost of storage (per unit)	c_s= 25 % of C per year = 5 $ / kg-year

We find, as solutions to the various variables of interest:

$$n_0 = \sqrt{2\frac{1200(\text{kg})\times 15(\$)}{1(\text{yr})\times 5(\$/\text{kg-yr})}} = 84{,}9 \ \text{kg} \qquad \{12\}$$

$$T_0 = \sqrt{2\frac{q}{N}\frac{c_\ell}{c_s}} = \sqrt{2\frac{1(\text{yr})}{1200(\text{kg})}\frac{15(\$)}{5(\$/\text{kg-yr})}} = 0{,}0707 \ \text{year} = 25{,}5 \ \text{day} \qquad \{13\}$$

$$z_0 = \sqrt{2\times 1200(\text{kg})\times 1(\text{yr})\times 15(\$)\times 5(\$/\text{kg-yr})} = 424{,}3 \ \$ \qquad \{14\}$$

The annual cost of the material, not included in the model, is NC = 1200 × 20 = 24 000 $, so (after the optimization) the maintenance charges represent 424 / 24 000, or 1,8 % of that cost. Specifically, we would lead T_0 to a reasonable value (21 days, 28, 30, "1.st day of each month", etc.). In Figure 2 is plotted z depending on the size of the order n to monitor the increase of z for non-optimal values of n.

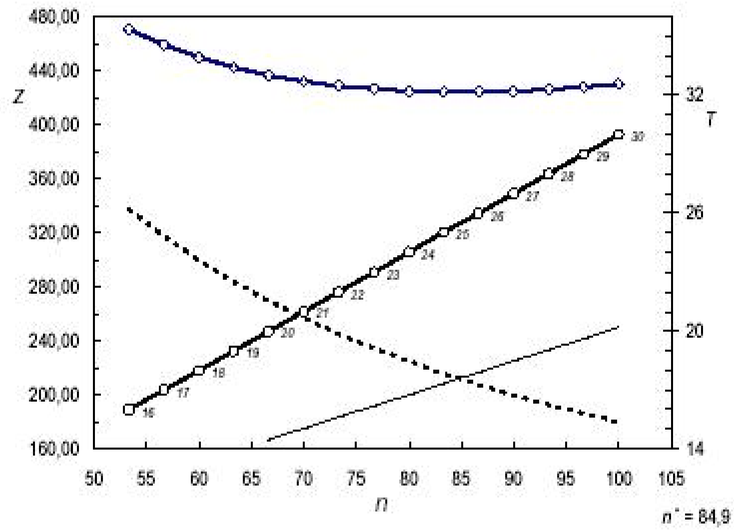

Figure 2

(2.3 ... 2.8)

2.9 Fourth case: random demand with costs of storage and shortage

Suppose that demand, for a certain time interval T, is random, where $p(r)$ is the probability of a total demand r on the interval T. The demand is discontinuous, but practically it can be assumed that its rate of change is constant. The parts retain their value in the range T, but the cost of storage per unit time, with the interest of capital they represent, has the value c_{s} (cost per unit of time). It is assumed that the shortage of a part results in a loss c_{p} per unit of time. Consider the following example.

A factory produces cranes and has several deposits in various parts of the country. Some spare parts are very expensive, but must be made available to customer in depots since the cranes should not be unavailable too long in case of failure. Let us consider one of these parts and determine the stock to place in a depot in order to minimize the expense of the cost of storage (including income from invested amounts) and of the cost of shortage (loss of a customer, borrowing another crane, etc..).

(1) *Average Stock* corresponding to situation *"a"*, *no-shortage:*

$$\bar{s}_a = \frac{1}{2}\left[s + (s - r)\right] = s - \frac{r}{2} \qquad \{15\}$$

(2) *Average Stock* corresponding to situation *"b"*, *shortage:*

$$\bar{s}_b = \left[\frac{1}{2}(s + 0)\right]\frac{s}{r} = \frac{s^2}{2r} \qquad \{16\}$$

(This refers to a fraction *s/r* of the period under consideration.)

(3) *Average shortage* corresponding to the situation *"b", shortage:*

$$\bar{p}_b = \left\{\frac{1}{2}\left[0 + (r - s)\right]\right\}\left(1 - \frac{s}{r}\right) = \frac{(r - s)^2}{2r} \qquad \{17\}$$

(This refers to the remaining fraction of the period.)

The mathematical expectation of the total cost of the stock will be:

$$\begin{aligned} z(s) &= c_{\mathrm{s}}\sum_{r=0}^{s}\left(s - \frac{r}{2}\right)p(r) + \\ &+ c_{\mathrm{s}}\sum_{r=s+1}^{\infty}\frac{s^2}{2r}\,p(r) + c_{\mathrm{p}}\sum_{r=s+1}^{\infty}\frac{(r - s)^2}{2r}\,p(r) \end{aligned} \qquad \{18\}$$

It can be shown that the minimum of *z(s)* occurs at a value s_0 such that

$$L(s_0 - 1) < r < L(s_0) \qquad \{19\}$$

with

$$r = \frac{c_\mathrm{p}}{c_\mathrm{s} + c_\mathrm{p}} = \frac{1}{1 + \frac{c_\mathrm{s}}{c_\mathrm{p}}} \qquad \{20\}$$

$$L(s_0) = p(r \le s_0) + \left(s_0 + \frac{1}{2}\right) \sum_{r=s_0+1}^{\infty} \frac{p(r)}{r} \qquad \{21\}$$

[Note also that $r = L(s_0)$ implies that both s_0 and $s_0 + 1$ correspond to optimum, while $r = L(s_0 - 1)$ implies optimal s_0 or $s_0 - 1$.] Of course, the determination of s_0 can be made directly numerically.

(2.10)

2.11 Numerical resolution

Let c^s = 100 $ / month, c_p = 20 c_s = 2000 $ / month and use the following table of the probability function $p(r)$ observed for monthly consumption, r.

r	0	1	2	3	4	5	≥ 6
$p(r)$	0,1	0,2	0,2	0,3	0,1	0,1	0

See plate http://web.ist.utl.pt/mcasquilho/compute/or/Fx-inventoryRand.php

The calculations for s = 0, 1, 2, ..., seeking a minimum value of $z(s)$ provide (in the monetary unit $):

$$\begin{aligned} z_0 &= c_p \sum_{r=1}^{\infty} \frac{r}{2} p(r) = \\ &= 2000 \times \frac{1}{2}(1\times 0{,}2 + 2\times 0{,}2 + 3\times 0{,}3 + 4\times 0{,}1 + 5\times 0{,}1) = 2400 \end{aligned} \qquad \{22\}$$

and successively,

$$\begin{aligned} z_1 &= c_\mathrm{s} \sum_{r=0}^{1}\left(1 - \frac{r}{2}\right)p(r) + c_\mathrm{s}\sum_{r=2}^{\infty}\frac{p(r)}{2r} + c_\mathrm{p}\sum_{r=2}^{\infty}\frac{(r-1)^2}{2r}p(r) = \\ &= 100(1\times 0{,}1 + 0{,}5\times 0{,}2) + \\ &\quad +100(0{,}25\times 0{,}2 + 0{,}167\times 0{,}3 + 0{,}125\times 0{,}1 + 0{,}1\times 0{,}1) + \\ &\quad +2000(0{,}25\times 0{,}2 + 0{,}667\times 0{,}3 + 1{,}125\times 0{,}1 + 1{,}6\times 0{,}1) = 1077{,}25 \end{aligned} \qquad \{23\}$$

Assuming monotonicity, as $z_1 < z_0$ (cost is decreasing), we must continue the calculations (and so on until it starts to increase) to detect the optimum, i.e. *minimum*.

$$\begin{aligned} z_2 &= c_\mathrm{s} \sum_{r=0}^{2}\left(\mathbf{2} - \frac{r}{2}\right)p(r) + c_\mathrm{s}\sum_{r=3}^{\infty}\frac{2^2\, p(r)}{2r} + c_\mathrm{p}\sum_{r=3}^{\infty}\frac{(r-2)^2}{2r}p(r) = \\ &= 100(2\times 0{,}1 + 1{,}5\times 0{,}2 + 1\times 0{,}2) + \\ &\quad +100(0{,}667\times 0{,}3 + 0{,}5\times 0{,}1 + 0{,}4\times 0{,}1) + \\ &\quad +2000(0{,}167\times 0{,}3 + 0{,}5\times 0{,}1 + 0{,}9\times 0{,}1) = 479 \end{aligned} \qquad \{24\}$$

$$z_3 = c_s \sum_{r=0}^{3} \left(3 - \frac{r}{2}\right) p(r) + c_s \sum_{r=4}^{\infty} \frac{3^2\ p(r)}{2r} + c_p \sum_{r=4}^{\infty} \frac{(r-3)^2}{2r}\ p(r) =$$
$$= 100(3 \times 0{,}1 + 2{,}5 \times 0{,}2 + 2 \times 0{,}2 + 1{,}5 \times 0{,}3) +$$
$$+100(1{,}125 \times 0{,}1 + 0{,}9 \times 0{,}1) + 2000(0{,}125 \times 0{,}1 + 0{,}4 \times 0{,}1) = 290 \tag{25}$$

$$z_4 = c_s \sum_{r=0}^{4} \left(4 - \frac{r}{2}\right) p(r) + c_s \sum_{r=5}^{\infty} \frac{4^2\ p(r)}{2r} + c_p \sum_{r=5}^{\infty} \frac{(r-4)^2}{2r}\ p(r) =$$
$$= 100(4 \times 0{,}1 + 3{,}5 \times 0{,}2 + 3 \times 0{,}2 + 2{,}5 \times 0{,}3 + 2 \times 0{,}1) +$$
$$+100(1{,}6 \times 0{,}1) + 2000(0{,}1 \times 0{,}1) = 301 \tag{26}$$

It is now simultaneously $z_3 < z_2$ and $z_3 < z_4$, i.e., $z_2 > z_3 < z_4$, so the minimum has been found, with $z^* = 290$ \$. See Figure 3.

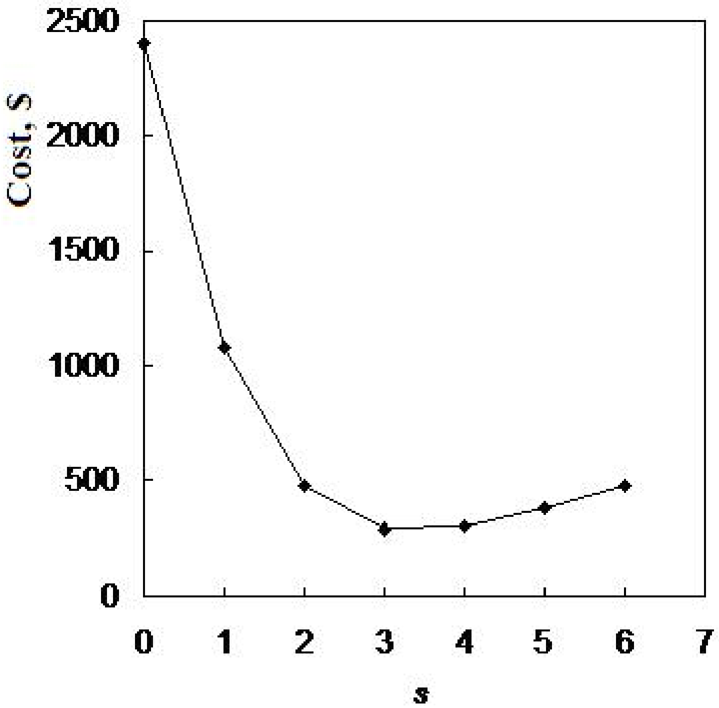

Figure 3

Appendix

$$y = ax + \frac{b}{x} \tag{27}$$

$$y' = a - \frac{b}{x^2} \stackrel{?}{=} 0 \tag{28}$$

$$a = \frac{b}{x^2} \quad \Rightarrow \quad x_0 = +\sqrt{\frac{b}{a}} \tag{29}$$

In the EOQ, it is $a = \frac{1}{2} q c_S$ and $b = N c_L$, so $n_0 = 2\sqrt{\frac{1}{2} \frac{N c_L}{q c_S}} = \sqrt{2 \frac{N}{q} \frac{c_L}{c_S}}$.

$$y_0 = a\sqrt{\frac{b}{a}} + b\sqrt{\frac{a}{b}} = 2\sqrt{ab} \qquad \{30\}$$

In the EOQ, $z_0 = 2\sqrt{\frac{1}{2} q c_S N c_L} = \sqrt{2 N q \, c_L c_S}$.

$$y_0'' = 2\frac{b}{x^3} = 2b\left(\sqrt{\frac{a}{b}}\right)^3 = 2ab\sqrt{a} > 0 \quad \text{(minimum)} \qquad \{31\}$$

The minimum occurs coincident with the intersection of the straight line with the hyperbola, where both contributions are $\sqrt{ab}$, i.e., $z_0/2$.

❖