Quantum Adiabatic Evaluation of Trees

Luís Tarrataca
LNCC, Rio de Janeiro, Brazil

1 Introduction

A diverse number of relevant computational problems can be specified through NAND tree evaluation, *e.g.* from calculating the value of boolean expressions to determining the outcome of games.

Classical tree evaluation requires $O(N)$ time whilst randomized algorithms are capable of evaluating decision trees in $O(N^{0.753})$. Quantum computation allows for tree evaluation methods capable of running in $O(\sqrt{N})$ time [Farhi et al., 2008] in continuous-time. Discrete methods exist using $N^{\frac{1}{2}+o(1)}$ queries [Childs et al., 2009] for balanced and approximately balanced trees and arbitrary formulas depth d can be calculated using $O(\sqrt{N}\log^{d-1} N)$ queries [Ambainis, 2007].

Recently, advances in practical quantum annealers has led to a renewed interest in the range of potential applications of adiabatic quantum computation. In this work we consider how to tackle evaluation of NAND trees in an adiabatic context. Namely, is it possible to devise adiabatic methods based on the existing formulations for tree evaluation? What are the main advantages and disadvantages associated of such a procedure?

2 The adiabatic model

Schrödinger equation:

$$i\hbar\frac{d}{dt}|\psi(t)> = H(t)|\psi(t)>$$

Time-dependent Hamiltonian: map a state to an energy landscape

Adiabatic theorem:

$$H(s)|l;s> = E_l(s)|l;s>$$

(eigenvector l at time s; eigenvalue at time s)

$$E_0(s) \leq E_1(s) \leq \cdots \leq E_{N-1}(s)$$

$$g_{min} = \min_{0\leq s\leq 1}(E_1(s) - E_0(s)) \quad \text{If } g_{min} > 0$$

$$\lim_{T\to+\infty} |<l=0;s=1||\psi(T)>| = 1$$

A quantum system with a time-dependent Hamiltonian that is initially in a certain energy level tends to stay at the same level, provided that the Hamiltonian is evolved slowly enough.

Adiabatic computation:

$$\widetilde{H}(s) = (1-s)H_0 + sH_P$$

(H_0: An initial Hamiltonian with an easy to prepare ground-state; H_P: Problem-specific Hamiltonian whose ground-state encodes a solution to a problem)

Evolve the system according to the adiabatic theorem. Start with Hamiltonian H_P with an easy to prepare ground-state. Finish with the ground-state of H_P encoding the solution to a problem P.

3 Continuous tree evaluation

For $E \to 0^+$ the amplitudes of the nodes belonging to an extended graph of the tree, H, recurse down the tree in accordance with NAND logic [Farhi et al., 2008].

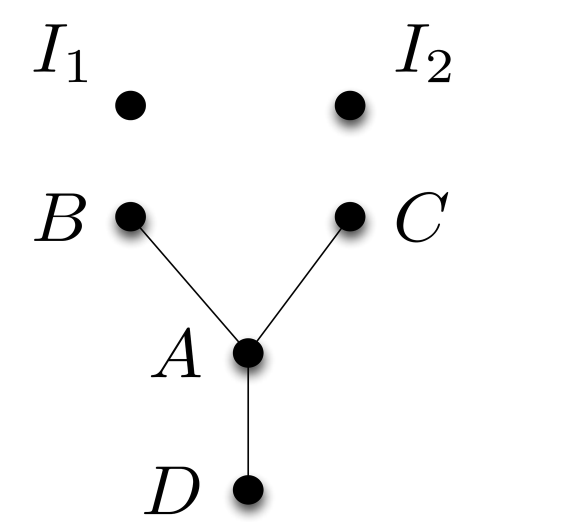

Fig 1: Extended graph of a $N=2$ binary tree.

x_1	x_2	$\frac{b}{a}$	$\frac{c}{a}$	$\lim_{E\to 0^+}\frac{a}{d}$	$\overline{x_1 \wedge x_2}$
0	0	$-\frac{1}{E}$	$-\frac{1}{E}$	0	1
0	1	$-\frac{1}{E}$	$\frac{E}{1-E^2}$	0	1
1	0	$\frac{E}{1-E^2}$	$-\frac{1}{E}$	0	1
1	1	$\frac{E}{1-E^2}$	$\frac{E}{1-E^2}$	$-\infty$	0

Tab 1: Relationships between inputs and amplitude ratios for $E \to 0^+$.

4 Adiabatic tree spectrum

Hamiltonian spectrum:

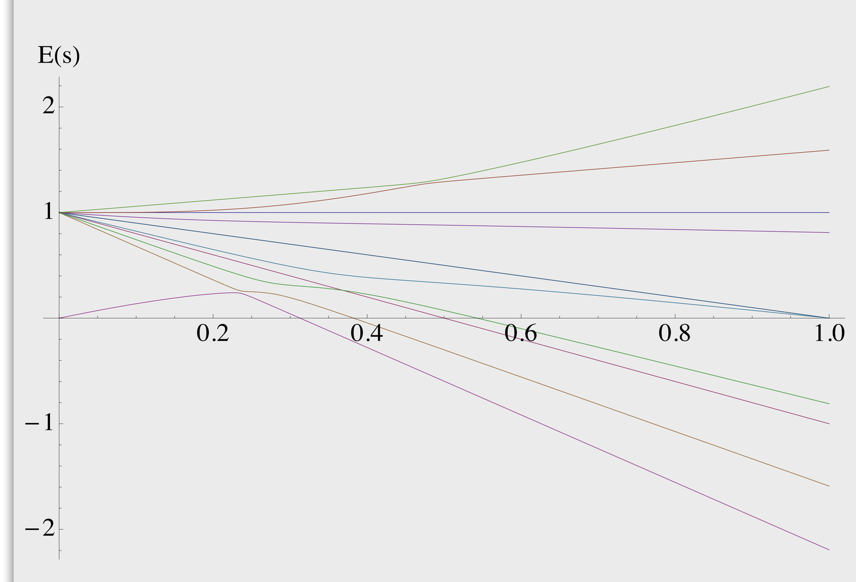

Fig 2: $\widetilde{H}(s) = sH_0 + (1-s)H$ for a randomly generated NAND tree ($N=4$).

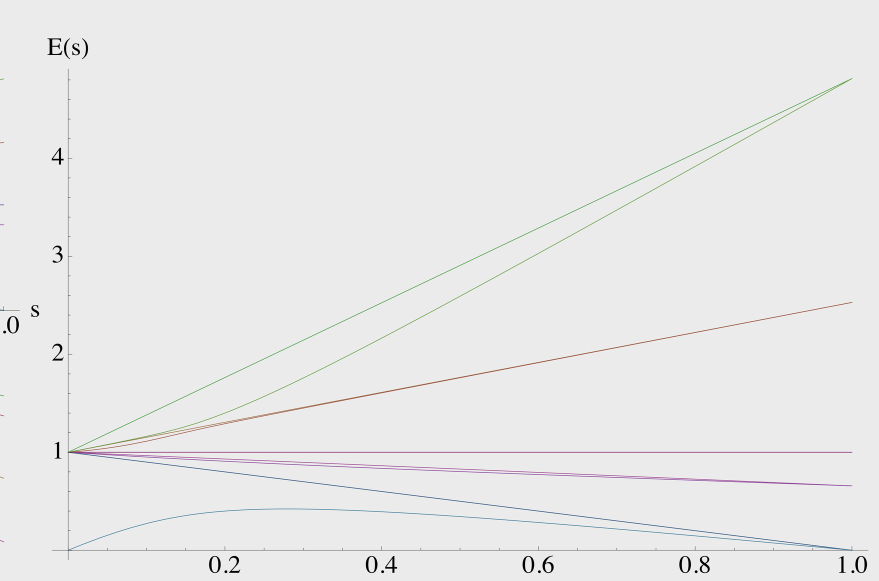

Fig 3: $\widetilde{H}(s) = sH_0 + (1-s)H^2$ for a randomly generated NAND tree ($N=4$).

The ground-state of H does not convey information about the value of the tree. H^2 results in a remapping of the spectrum, but does H^2 exhibit the same behaviour as H?

$$<A|H^2|E> = \begin{cases} (-<B|-<C|-<D|)H|E> & = E(-b-c-d) \\ <A|HE|E> & = E<A|H|E> \quad = E^2 a \end{cases}$$

$$\Leftrightarrow E(-b-c-d) = E^2 a$$

$$\Leftrightarrow \frac{a}{d} = \frac{-1}{E + \frac{b}{a} + \frac{c}{a}} \qquad \square$$

5 Gap analysis

Minimum gap:

Practical computation does not allow for $T \to +\infty$

$$\tau(s) \gg \frac{||\frac{d}{ds}\widetilde{H}(s)||_2}{g(s)^2} \approx \frac{1}{g(s)^2} = \frac{1}{(E_1(s)-E_0(s))^2}$$

Tree gap analysis:

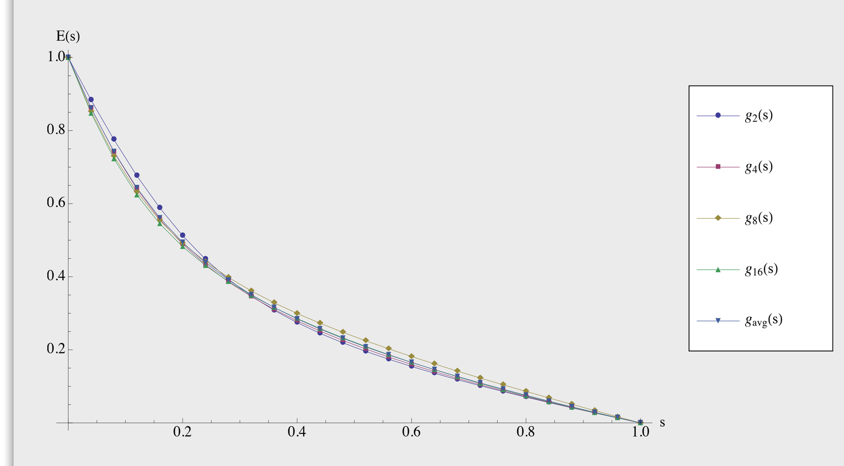

Fig 4: Numerical results for the average gap condition for all inputs consisting of 2, 4, 8 and 16 bits.

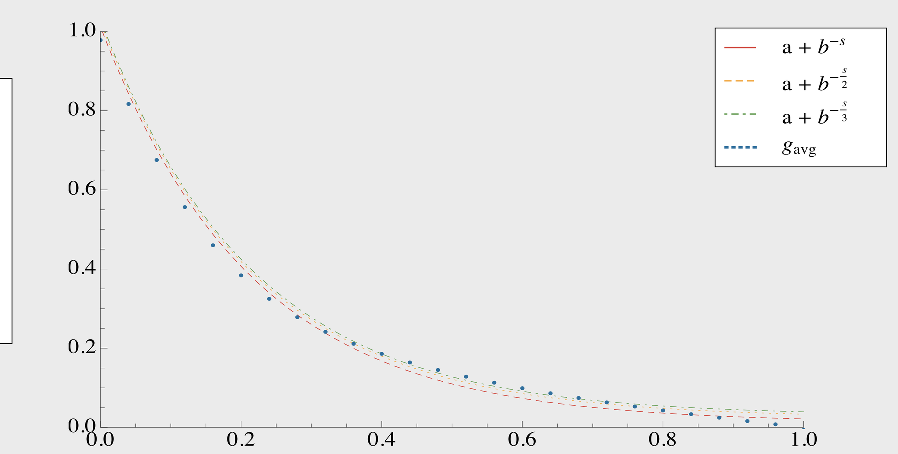

Fig 5: The fit for $g_{avg}(s)$ for based on: (i) functions $a+b^{-s}$, $a+b^{-\frac{s}{2}}$ and $a+b^{-\frac{s}{3}}$, where a and b are constants and s is the variable; and (ii) the average data points for the dimensions considered.

Performing local adiabatic evolution results in time:

$$t = \int_0^1 \frac{1}{g(s)^2}ds = \frac{\frac{1}{ab+1} + \log(ab+1) - \frac{1}{a+1} - \log(a+1)}{a^2 \log(b)} \Rightarrow O\left(\frac{N^4}{\log N^2}\right)$$

performance penalty of $N^3/(2\log N)$

6 Conclusions

1. Hard computational problems appear to have exponentially small gaps, and tree evaluation is no exception.

2. Not clear, or trivial, how to perform an adequate mapping for quantum adiabatic tree evaluation. Different Hamiltonian formulation may produce a gap yielding an execution time closer to the limit of $O(\sqrt{N})$.

7 References

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luis.tarrataca@gmail.com

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