Chapter 4 - Duality Theory

- This chapter starts with a L.P. problem called the primal and then introduces another L.P. problem: the dual
- Duality theory deals with the relation between the two problems and uncovers the deeper structure of L.P.
- It is a powerful theoretical tool that has numerous applications, provides new geometric insights, and leads to another algorithm for L.P.: the dual simplex method.

4.1 Motivation

- Duality theory can be motivated as an outgrowth of the Lagrange multiplier method often used in calculus to minimize a function subject to equality constraints.

For example, in order to solve the problem

$$\text{Minimize } u^2 + y^2$$
$$\text{Subject to } u + y = 1$$

- We introduce a Lagrange multiplier p and form the Lagrangean $L(u, y, p)$ defined by

$$L(u, y, p) = u^2 + y^2 + p(1 - u - y)$$

- While keeping p fixed, we minimize the Lagrangean over all u and y, subject to no constraints

- This can be done by setting $\frac{\partial L}{\partial u} = \frac{\partial L}{\partial y} = 0$

- The optimal solution to the unconstrained problem is $u = y = \frac{p}{2}$

- The constraint $u + y = 1$ gives us the additional relation $p = 1$ and the optimal solution is thus $u = y = \frac{1}{2}$

Idea of the previous example:

- Instead of enforcing the hard constraint $u+y=$
  we allow it to be ~~violated~~ violated
- We violate the hard constraint by associating
  a price p (i.e. the Lagrange multiplier) with
  the amount $1-u-y$ by which it is violated
- This leads to the unconstrained minimization
  of $u^2+y^2+p(1-u-y)$
- When the price is properly chosen ($p=1$) the
  optimal solution to the constrained problem
  is also optimal for the unconstrained problem

Q: But how can this idea be applied to L.P.?
Any ideas?

- Associate a price variable with each constrai
- Start searching for prices under which the pres
  or absence of the constraints does not affect
  the optimal cost

- It turns out:
  - The right prices can be found by solving a new L.P. Problem: the $\boxed{\text{dual}}$ of the original.

- Consider the standard form problem:

$$\begin{aligned} &\text{Minimize: } c'u \\ &\text{Subject to: } Au = b \\ &\qquad\qquad\quad\; u \geq 0 \end{aligned}$$

- Which we call the $\boxed{\text{primal problem}}$ and let u^* be an optimal solution (assumed to exist)
- We introduce a $\boxed{\text{relaxed problem}}$ in which the constraint $Au = b$ is replaced by a penalty $p'(b - Au)$ where p is a price vector of the same dimension as b.
- We are then faced with the problem

$$\begin{aligned} &\text{Minimize: } c'u + p'(b - Au) \\ &\text{Subject to: } u \geq 0 \end{aligned}$$

- Let $g(p)$ be the optimal cost for the relaxed problem as a function of the price vector p.

- The relaxed problem allows for more options than those present in the primal problem, and we expect $g(p)$ to be no larger than the optimal primal cost $c'u^*$

- Indeed:

$$g(p) = \min_{u \geq 0} \left[c'u + p'(b - Au) \right] \leq c'u^* + p'(b - Au^*) = c'u^*$$

The 'last inequality' follows from the fact that u^* is a feasible solution to the primal problem and satisfies $Au^* = b$

$\therefore$ Each p leads to a lower bound $g(p)$ for the optimal cost $c'u^*$

- The problem:

  Maximize: $g(p)$

  Subject to: no constraints

can be then interpreted as a search for the tightest possible lower bound.

This is known as the $\boxed{\text{dual problem}}$

Main result in duality theory:

$\boxed{\text{The optimal cost in the dual problem is equal to the optimal cost in the primal problem}}$

In other words:

$\boxed{\text{When the prices are chosen according to an optimal solution for the dual problem, the option of violating the constraints is of no value}}$

Using the definition of $g(p)$ we have:

$$g(p) = \min_{u \geq 0} [u'u + p'(b - Au)]$$

$$= p'b + \min_{u \geq 0} (c' - p'A)u$$

Note that

$$\min_{u \geq 0} (c' - p'A)u = \begin{cases} 0, & \text{if } c' - p'A \geq 0' \\ -\infty, & \text{otherwise} \end{cases}$$

Q: Why is this so?

$\boxed{Q:}$ Why is $\min_{u \geq 0} (c' - p'A) u = -\infty$ if $c' - p'A < 0$

- First consider that $u \geq 0$ (and that u is a vec
- Then consider that if $c' - p'A < 0$ then t dot product between $\underline{c' - p'A}$ and $\underline{u}$ will ~~..~~ always be negative
- Because we are trying to minimize this means that there will always be a smaller possi value.
- Thus the $-\infty$

$\boxed{Q:}$ Why is $\min_{u \geq 0} (c' - p'A) u = 0$ if $c' - p'A \geq 0$'?

- First consider that $u \geq 0$ (and that u is a vec
- Then concider that if $c' - p'A \geq 0$ then the dot product between $\underline{c' - p'A}$ and $\underline{u}$ will have minimum value of $\underline{0}$ if the minimum value is assumed for either or both $\underline{c' - p'A}$ and/or $\underline{u}$
- I.e.

$$\underbrace{\underbrace{(c' - p'A)}_{=0} \cdot \underbrace{u}_{=0}}_{\text{And/or}} = 0$$

In maximizing $g(p)$ we only need to consider those values of p for which $g(p) \neq -\infty$

$\therefore$ The dual problem is the same as the L.P. problem:

Maximize: $p'b$

Subject to: $p'A \leq c'$

Summary: Construction of the dual of a primal problem

1) We have a vector of parameters (dual variables) p

2) For every p we have a method of obtaining a lower bound on the optimal primal cost.

3) The dual problem is a maximization problem that looks for the highest such lower bound.

4) For some vector p the lower bound is $-\infty$ and does not carry any useful information

5) Thus, we only need to maximize over those p that lead to nontrivial lower bounds.

4.2 The dual problem

- Let $\underline{A}$ be a matrix with rows $\underline{a_i'}$ and columns $\underline{A_j}$
- Given a primal problem with the struct shown on the ~~left~~ top, its dual is defined be the maximization problem shown on the ~~right.~~ bottom

Minimize: $c'u$

Subject to:
$$a_i' u \geq b_i \ , \ i \in M_1$$
$$a_i' u \leq b_i \ , \ i \in M_2$$
$$a_i' u = b_i \ , \ i \in M_3$$
$$u_j \geq 0 \ , \ j \in N_1$$
$$u_j \leq 0 \ , \ j \in N_2$$
$$u_j \text{ free} \ , \ j \in N_3$$

Primal

Respect

Maximize: $p'b$

Subject to:
$$p_i \geq 0 \ , \ i \in M_1$$
$$p_i \leq 0 \ , \ i \in M_2$$
$$p_i \text{ free} \ , \ i \in M_3$$
$$p' A_j \leq c_j \ , \ j \in N_1$$
$$p' A_j \geq c_j \ , \ j \in N_2$$
$$p' A_j = c_j \ , \ j \in N_3$$

Dual

Primal	Minimize	Maximize	Dual
Constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	Variables
Variables	≥ 0 ≤ 0 free	$\leq c_j$ $\geq c_j$ $= c_j$	Constraints

Table 4.1 - Relation between primal and dual variables and constraints

- A problem and its dual can be stated more compactly in matrix notation, if a particular form is assumed for the primal.

- We have, for example, the following pairs of primal and dual problems:

Minimize: $c'u$ → Maximize: $p'b$

Subject to: $Au = b$ → Subject to: $p'A \leq c'$

$u \geq 0$

And:

Minimize: $c'u$ → Maximize: $p'b$

Subject to: $Au \geq b$ → Subject to: $p'A = c'$

$p \geq 0$

Notes:

On the last pair: (According to Table 4.1)

- On the left: $Au \geq b$ implies on the right $p \geq 0$
- On the left: nothing is said about u_j, i.e. u_j is free, which implies on the right $p'A = c'$

Example 4.1.a

Transform the dual into the primal (see Table 4.1)

Maximize: $5p_1 + 6p_2 + 4p_3$

Subject to: p_1 free

$p_2 \geq 0$

$p_3 \leq 0$

$$-p_1 + 2p_2 \leq 1$$
$$3p_1 - p_2 \geq 2$$
$$3p_2 + p_3 = 3$$

These are the costs

Dual to Primal

Minimize: $1u_1 + 2u_2 + 3u_3$

Subject to: $-u_1 + 3u_2 = 5$

$-2u_1 - u_2 + 3u_3 \geq 6$

$u_3 \leq 4$

$u_1 \geq 0$

$u_2 \leq 0$

u_3 free

Example 4.1.b

Transform the primal into the dual:

Minimize: $-5u_1 - 6u_2 - 4u_3$

Subject to: u_1 free

$u_2 \geq 0$

$u_3 \leq 0$

$u_1 - 2u_2 \geq -1$

$-3u_1 + 2u_2 \leq 2$

$-3u_2 - u_3 = -3$

Primal to Dual

Maximize: $-p_1 - 2p_2 - 3p_3$

Subject to: $p_1 - 3p_2 = -5$

$2p_1 + p_2 - 3p_3 \leq -6$

$-p_3 \geq -4$

$p_1 \geq 0$

$p_2 \leq 0$

p_3 free

Q: Can you notice something important from the previous examples?

- The latter dual problem (4.1.b) is equivalent to the former primal problem (4.1.a)
- The only difference is that the first three constraints are multiplied by $\underline{-1}$.

$\therefore_1$ Indeed, we can have the following result:

Theorem 4.1 If we transform the dual into an equivalent minimization problem and then form its dual, we obtain a problem equivalent to the original

$\therefore_2$ $\boxed{\text{The dual of the dual is the primal}}$

<u>Theorem 4.2</u> Suppose that we have transformed a L.P. problem Π_1 to another L.P. problem Π_2, by a sequence of transformations of the following types:

<u>(a)</u> Replace a free variable with the difference of two nonnegative variables

<u>(b)</u> Replace an inequality constraint by an equality constraint involving a nonnegative slack variable.

<u>(c)</u> If some row of matrix $\underline{A}$ in a feasible standard form problem is a linear combination of the other rows, eliminate the corresponding equality constraint.

Then, the duals of $\underline{\Pi_1}$ and $\underline{\Pi_2}$ are equivalent, <u>i.e.</u> they are both <u>infeasible</u>, or they have the same <u>optimal cost</u>.

4.3 The duality theorem

Recall: the cost $g(p)$ of any dual solution provides a lower bound for the optimal cost. ~~We now~~

Lets show that this property is true in general:

Theorem 4.3 ($\boxed{\text{Weak duality}}$)

If u is a feasible solution to the primal problem and p is a feasible solution to the dual problem then

$$p'b \leq c'u$$

Proof: For any vectors u and p we define

$$u_i = p_i(a_i'u - b_i)$$

$$v_j = (c_j - p'A_j)u_j$$

Recall the dual definition of page 142 of the book?

- Suppose that u and p are primal and dual feasible, respectively

Again because of this...

- The definition of the dual problem requires the sign of p_i to be the same as the sign of $a_i'u - b_i$

Notes:

$$\begin{cases} p_i \geq 0 \\ a_i'u \geq b_i \end{cases} \iff \begin{cases} p_i \geq 0 \\ a_i'u - b_i \geq 0 \end{cases} \quad \therefore \; p_i \text{ and } a_i'u - b_i \text{ have the same sign}$$

- For the same reason, the sign of $c_j - p'A_j$ needs to be the same as the sign of u_j.

Notes:
$\begin{cases} u \geq 0 \\ c_j \geq p'A_j \end{cases} \Leftrightarrow \begin{cases} u \geq 0 \\ c_j - p'A_j \geq 0 \end{cases}$
$\therefore$ u and $c_j - p'A_j$ have the same sign

- Thus, primal and dual feasibility imply that:

$$u_i \geq 0 \quad , \forall i$$

$$v_j \geq 0 \quad , \forall j$$

- Notice that $\sum_i u_i = p'Au - p'b$

$$\sum_j v_j = c'u - p'Au$$

- If we add these two equalities and use the nonnegativity of u_i, v_j to obtain

$$0 \leq \sum_i u_i + \sum_j v_j = \cancel{p'Au} - p'b + c'u - \cancel{p'Au}$$

$$= c'u - p'b$$

$$\Leftrightarrow \quad 0 \leq c'u - p'b$$

$$\Leftrightarrow \quad c'u \geq p'b \quad \blacksquare$$

Corollary 4.1

(a) If the optimal cost in the primal is $-\infty$ then the dual problem must be infeasible

(b) If the optimal cost in the dual is $+\infty$, then the primal problem must be infeasible

Proof:

- Suppose that the optimal cost in the primal problem is $-\infty$ and that the dual problem has a feasible solution p.
- By weak duality: p satisfies $p'b \leq c'u$ for every primal feasible u.
- Taking the minimum over all primal feasible u we conclude that $p'b \leq c'u \leq -\infty$, i.e. $p'b \leq -\infty$ which is impossible

$\therefore$ The dual cannot have a feasible solution

Part (b) follows a symmetrical argument ∎

Another important corollary of the weak duality:

Corollary 4.2 Let $\underline{u}$ and $\underline{p}$ be feasible solutions to the primal and the dual, respectively. Suppose that $p'b = c'u$. Then, $\underline{u}$ and $\underline{p}$ are optimal solutions to the primal and the dual, respectively.

Proof: Let $\underline{u}$ and $\underline{p}$ be as in the statement of the corollary. For every primal feasible solution $\underline{y}$, the weak duality theorem yields $c'u = p'b \leq c'y$, which proves that $\underline{u}$ is optimal. The proof of optimality of $\underline{p}$ is similar ∎

The next theorem is the central result on L.P. duality

Theorem 4.4. (Strong duality) If a L.P. problem has an optimal solution, so does its dual, and the respective optimal cost are equal.

Proof: Consider the standard form problem:

$$\begin{aligned} \text{Minimize: } & c'u \\ \text{Subject to: } & Au = b \\ & u \geq 0 \end{aligned}$$

- Let $u_B = B^{-1}b$ be the corresponding vector of basic variables associated with an optimal solution u and an optimal basis B.
- When we apply the simplex method the reduced costs must be nonnegative and we obtain:

$$c' - c'_B B^{-1} A \geq 0'$$

(Check Definition 3. page 84 of the book)

Where c'_B - is the vector with the costs of the basic variables.

- Let us define a vector p - by letting $p' = c'_B B^{-1}$. Then

$$c' - c'_B B^{-1} A \geq 0'$$

$$\Leftrightarrow c' - p'A \geq 0'$$

$$\Leftrightarrow c' \geq p'A$$

This shows that p is a feasible solution to the dual problem:

$$\text{Maximize: } p'b$$

$$\text{Subject to: } p'A \leq c'$$

In addition,

$$\underbrace{\boxed{p'}}_{c'_B B^{-1}} b = c'_B \underbrace{\boxed{B^{-1} b}}_{u_B} = \boxed{c'_B \cdot u_B = c'u}$$

- It follows that p is an optimal solution (Corollary 4.
and the <u>optimal dual</u> cost is equal to the <u>optimal</u>
<u>primal</u> cost.

- If we are dealing with a general L.P. problem Π_1
that has an <u>optimal</u> solution, we first transf
it into an equivalent standard form problem Π_2
with the <u>same</u> optimal cost, and in which
the rows of matrix $\underline{A}$ are L.I. .

- Let $\underline{D_1}$ and $\underline{D_2}$ be the <u>duals</u> of $\underline{\Pi_1}$ and $\underline{\Pi_2}$
respectively.

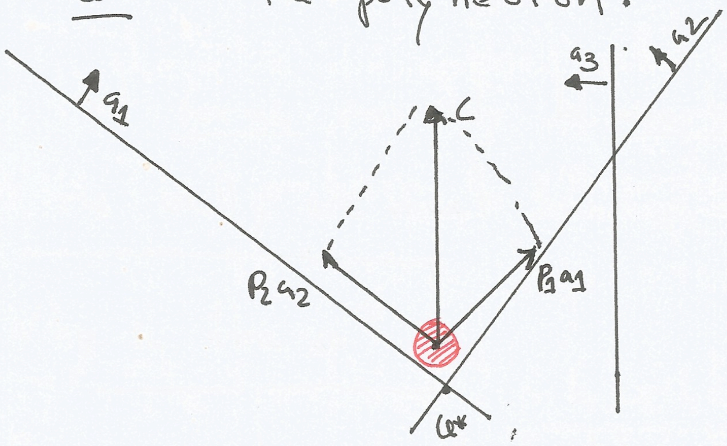
- By <u>Theorem 4.2</u>, the dual problems $\underline{D_1}$ and $\underline{D_2}$
have the same optimal cost.
- We have already proved that $\underline{\Pi_2}$ and $\underline{D_2}$ have the <u>same optimal cost</u>
(i.e. $p'b = c_B' B^{-1} b = c_B' x_B = c'x$)
- It follows that $\underline{\Pi_1}$ and $\underline{D_1}$ have the same
optimal cost:

$$\begin{tikzcd}
\Pi_1 \arrow[d, "\text{Equivalent}"'] & D_1 \\
\Pi_2 \arrow[r, "\text{Duality for standard form problems}"'] & D_2 \arrow[u, "\text{Duals of equivalent problems are equivalent}"']
\end{tikzcd}$$

Preceding proof shows that:

An optimal solution to the dual problem is obtained as a byproduct of the simplex method as applied to a primal problem in standard form.

Example 4.4 Consider a solid ball constrained to lie in a polyhedron defined by inequality constraints of the form $a_i' u \geq b_i$. If left under the influence of gravity, this ball reaches equilibrium at the lowest corner u^* of the polyhedron:



This corner is an optimal solution to the problem

Minimize: $c' u$

Subject to: $a_i' u \geq b_i \quad \forall i$

- Where $\underline{c}$ is a vertical vector pointing upwards.

- At equilibrium, gravity is counter-balanced by the forces exerted on the ball by the "walls" of the polyhedron. The latter forces are normal to the walls, that is, they are aligned with the vectors $\underline{a_i}$.

- We conclude that $c = \sum_i p_i a_i$, for some nonnegative coefficients $\underline{p_i}$, in particular, the vector p is a feasible solution to the dual problem:

$$\begin{aligned} \text{Maximize:} \quad & p'b \\ \text{Subject to:} \quad & p'A = c' \\ & p \geq 0 \end{aligned}$$

Given that forces can only be exerted by the walls that touch the ball, we must have $p_i = 0$, whenever $a_i' u^* > b$

$\boxed{Q:}$ Why is this so?

Because when $a_i' u^* > b_i$ then this means that the ball is not "touching" the "wall" of the hyperplane but is "above" it. Therefore that wall is not contributing for the counter-balance

- Consequently: $p_i(b_i - a_i' u^*) = 0 \quad \forall i$

- We therefore have:

$$\boxed{p'b} = \sum_i p_i \boxed{b_i} = \sum_i p_i \boxed{a_i'} \boxed{u^*} = \boxed{c'} u^*$$

- From corollary 4.2 it follows that p is an optimal solution to the dual, and the optimal dual cost is equal to the optimal primal cost

Recall that in a L.P. problem, exactly one of the following three possibilities will occur:

(a) There is an optimal solution.

(b) The problem is unbounded, i.e. optimal cost is $-\infty$

(c) The problem is infeasible

This leads to nine possible combinations for the primal and the dual

	Finite Optimum	Unbounded	Infeasi
Finite Optimum	Possible	Impossible	Imposs
Unbounded	Impossible	Impossible	Possi
Infeasible	Impossible	Possible	Possi

Table 4.2: The different possibilities for the primal an dual

- By the strong duality theorem, if one problem has an optimal solution, so does the other.
- Furthermore, the weak duality theorem i that if one problem is unbounded, the ot must be infeasible. This allows us to mark some of the entries in the table as "imposs
- Let look at an example where both probl are infeasible (Example 4.5)

Example 4.5 Consider the infeasible primal

$$\text{Minimize: } u_1 + 2u_2$$

$$\text{Subject to: } u_1 + u_2 = 1$$

$$2u_1 + 2u_2 = 3$$

Its dual is

$$\text{Maximize: } p_1 + 3p_2$$
$$\text{Subject to: } p_1 + 2p_2$$
$$p_1 + 2p_2 = 2$$

Which is also <u>infeasible</u>

<u>Complementary Slackness</u> ("folga complementar"?)

An <u>important</u> relation between <u>primal</u> and <u>dual</u> solution is provided by the <u>complementary slackness</u> <u>conditions</u>:

<u>Theorem 4.5</u> ((Complementary Slackness) Let $\underline{u}$ and $\underline{p}$ be feasible solutions to the primal and the dual problem, respectively. The vectors $\underline{u}$ and $\underline{p}$ are optimal solutions for the two respective problems if and only if:

$$p_i (a_i' u - b_i) = 0 \quad \forall i$$
$$(c_j - p' A_j) u_j = 0 \quad \forall j$$

Proof: In the proof of Theorem 4.3 we defined:

$$u_i = p_i (a_i' u - b_i)$$

$$v_j = (c_j - p' A_j) u_j$$

And noted that for u primal feasible and p dual feasible we have $u_i \geq 0$ and $v_j \geq 0$ $\forall i, \forall j$

In addition, we showed that

$$c'u - p'b = \sum_i u_i + \sum_j v_j$$

By the strong duality theorem (Theorems 4.4), if u and p are optimal, then $c'u = p'b$, which implies that $u_i = v_j = 0$ $\forall i, \forall j$

Q: Why is this so? Any ideas?

$$0 \leq \sum_i u_i + \sum_j v_j = \underbrace{c'u - p'b}_{\text{By strong duality}} = c'u - c'u$$

$$\iff 0 \leq \sum_i u_i + \sum_j v_j = 0 \Rightarrow \begin{cases} u_i = 0 & \forall i \\ v_j = 0 & \forall j \end{cases}$$

~~Since $u_i \geq 0$ and $v_j \geq 0$~~

- Conversely, if $u_i = v_j = 0$ $\forall i, \forall j$ then $c'u = p'b$ and Corollary 4.2 implies that u and p are optimal ▨

- The first complementary slackness condition is automatically satisfied by every feasible solution to a problem in standard form.
- If the primal problem $\boxed{\text{is not}}$ in standard form and has a restraint like $a_i' u \geq b_i$, the correspondi[ng] complementary slackness condition asserts that the variable p_i is zero unless the constraint is active. (a constraint that is not active can be removed without affecting the end result).
- If the primal problem is in standard form and a nondegenerate optimal basic feasible solution is known, the complementary slackness conditions determine a unique solution to the dual problem.

A geometric view

- We now delop a geometric view that allows us to vizualize pairs of primal and dual vectors without having to draw the dual feasible set

- We consider the primal problem:

$$\text{Minimize } c'u$$
$$\text{Subject to } a_i'u \geq b_i \quad , \quad i = 1, \dots, m$$

Where the dimension of u is equal to n. We assume that the vector a_i span $\mathbb{R}^n$.

- The corresponding dual problem:

$$\text{Maximize: } p'b$$
$$\text{Subject to: } \sum_{i=1}^{m} p_i a_i = c$$
$$p \geq 0$$

- Let I be a subset of $\{1, \dots, m\}$ of cardinality /size n such that the vectors $a_i \in I$ are linearly independent.

- The system $a_i'u = b_i$, $i \in I$ has a unique solution denoted by $\underline{u}^I$ which is a basic solution to the primal problem (cf. Definition 2.9)

- Assume that $\underline{u}^I$ is nondegenerate, that is, $a_i' u^I \neq b_i$ for $i \notin I$

- Let $\underline{p} \in \mathbb{R}^m$ be a dual vector (not necessarily dual feasible). Let us consider what is required for $\underline{u}^I$ and $\underline{p}$ to be optimal solutions to the primal and the dual problem, respectively. We need:

(a) $a_i' u^I \geq b_i \quad \forall i$ (primal feasibility)

(b) $p_i = 0 \quad \forall i \notin I$ (complementary slackness)

(c) $\boxed{\sum_{i=1}^{m}} p_i a_i = c$ (dual feasibility)

(d) $p \geq 0$ (dual feasibility)

Given ~~the~~ condition $\underline{b}$ and $\underline{c}$, we have

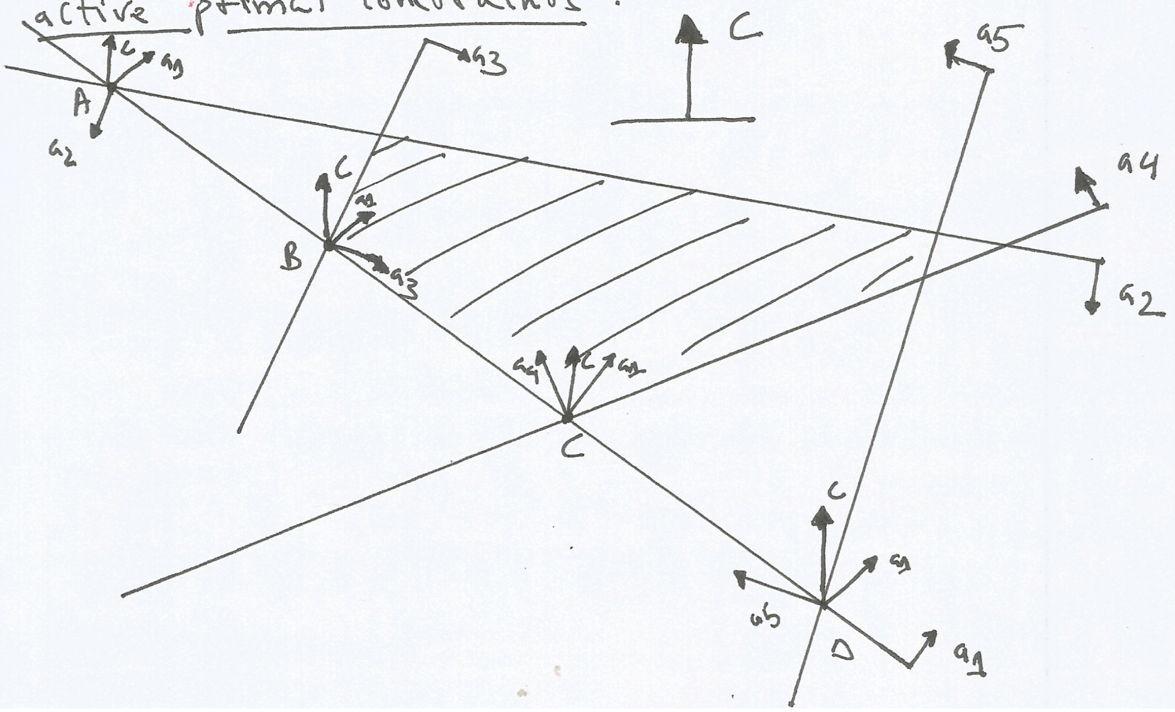
$$\boxed{\sum_{i \in I}} p_i a_i = c$$

- Since the vector $\underline{a_i}$, $i \in I$ are $\underline{L.\ I}$ the latter equation has a unique solution that we denote by $\underline{p}^I$

- These $\underline{L.\ I.}$ $\underline{a_i}$ vectors therefore form a basis for the dual problem (which is in standard form), an $\underline{p}^I$ is the associated $\underline{\text{basic}}$ solution.

- For the vector $\underline{p}^I$ to be dual feasible, we also need it to be nonnegative (cf. Def 2.9)

- Once condition $\underline{b}$ is $\underline{\text{enforced}}$, $\underline{\text{feasibility}}$ of the resulting dual vector $\underline{p}^I$ is equivalent to $\underline{c}$ being a $\underline{\text{nonnegative linear ~~cob~~ combination}}$ of the vectors $\underline{a_i}$, $i \in I$, associated with the $\underline{\text{active primal constraints}}$.



From the previous figure:

- Consider a primal problem with two variables and five inequality constraints ($n=2$, $m=5$)
- Every two-element subset I of $\{1,2,3,4,5\}$ determines basic solution u^I and p^I of the primal and the dual, respectively

(Point A) - If $I=\{1,2\}$, u^I is primal infeasible (outside of feasible set) and p^I is dual infeasible, because c cannot be expressed as a nonnegative linear combination of the vectors a_1 and a_2.

(Point B) - If $I=\{1,3\}$, u^I is primal feasible and p^I is dual infeasible, because c cannot be expressed as a nonnegative linear combination of the vectors a_1 and a_3.

(Point C). If $I=\{1,4\}$, u^I is primal feasible and - p^I is dual feasible, because c can be expressed as a nonnegative linear combination of the vectors a_1 and a_4. In particular u^I and p^I are optimal

(Point D) - If $I=\{1,5\}$, u^I is primal infeasible and p^I is dual feasible

4.5 Standard form problems and the dual simplex method

- We concentrate on the case where the primal is in standard form
- We will develop the dual simplex method, which is an alternative to the simplex method
- We will also comment on the relation between the basic feasible solutions to the primal and the dual.
- In the proof of the strong duality theorem we considered the simplex method applied to a primal problem in standard form and defined a dual vector p by letting $p' = c'_B B^{-1}$
- We then noted that the primal optimality condition $c' - c'_B B^{-1} A \geq 0'$ (i.e. all the reduced costs being nonnegative) is the same as the dual feasibility condition $p'A \leq c'$
- We can thus think of the simplex method as an algorithm that maintains primal feasibility and works towards dual feasibility.

- This is called a primal algorithm.
- Alternative: start with a dual feasible solution and work towards ~~dual~~ primal feasibility. This is called a dual algorithm.
- This section presents a dual simplex method implemented in terms of the full tableau.
- We argue that it does indeed solve the dual problem, and show that it moves from one basic feasible solution of the dual problem to another.

The dual simplex method

- Let us consider a problem in standard form, under the usual assumption that the rows of the matrix A are L.I.
- Let B be a ~~m~~ basis matrix, consisting of the m L.I. columns of A, and consider the corresponding tableau:

$$\begin{array}{r|c|c|l}
 & & \overbrace{\qquad\qquad}^{\text{All Variables}} & \\
\hline
\text{Current Cost} \rightarrow & -c_B' B^{-1} b & \bar{c}' & \leftarrow \text{Reduced Costs} \\
\hline
\text{Basic Variables} \left\{ \right. & B^{-1} b & B^{-1} A & \\
\hline
\end{array}$$

- Or, in more detail:

$-c_B' x_B$	$\bar{c}_1$	$\dots$	$\bar{c}_n$
$x_{B(1)}$ $\vdots$ $x_{B(m)}$	$B^{-1}A_1$	$\dots$	$B^{-1}A_n$

- We do not require $B^{-1}b$ to be nonnegative, which means that we have a basic, but not necessarily feasible solution to the primal problem (cf. Definition 2.9, pp. 50 of the book)
- We assume that $\bar{c} \geq 0$, equivalently the vector $p' = c_B' B^{-1}$ satisfies $p'A \leq c'$ and we have a feasible solution to the dual problem.
- The cost of this dual feasible solution is $p'b = c_B' B^{-1} b$ (0^{th} row, 0^{th} column of the tableau) and also

$$p'b = c_B' B^{-1} b = c_B' x_B$$

which is the negative of the entry at the upper left corner of the tableau.
- If the inequality $B^{-1}b \geq 0$ happens to hold, we also have a primal feasible solution with the same cost, and optimal solutions to both problems have been found.

- If the inequality $B^{-1}b \geq 0$ fails to hold, we perform a $\boxed{\text{change of basis}}$: We find some l for which $u_{B(l)} < 0$ and consider the l^{th} row of the tableau called the pivot row

- This row is of the form $(u_{B(l)}, v_1, \dots, v_n)$ where v_i is the l^{th} component of $B^{-1}A_i$

- For each i with $v_i < 0$ (if such i exist), we form the ratio $\boxed{\dfrac{\overline{c_i}}{|v_i|}}$ and let j be an index for which this ratio is smallest, that is $v_j < 0$ and

(Equation 4.2) $$\frac{\overline{c_j}}{|v_j|} = \min_{\{i \mid v_i < 0\}} \frac{\overline{c_i}}{|v_i|}$$

- We call the corresponding entry v_j the pivot element. Note that u_j must be a nonbasic variable, since the j^{th} column in the tableau contains the negative element v_j.

- We then perform a <u>change of basis</u>:

  1) Column $\underline{A_j}$ <u>enters</u> the basis

  2) Column $\underline{A_{B(\ell)}}$ <u>exits</u> the basis.

- This change of basis (or <u>pivot</u>) is <u>effected</u> exactly as in the <u>primal simplex method</u>:

  1) We add each row of the <u>tableau</u> a multiple of the pivot row ...

  2) ... so that all entries in the pivot are set to <u>zero</u>, with the exception of the pivot element which is set to <u>1</u>.

- In particular, in order to set the <u>reduced cost</u> in the pivot column to <u>zero</u>, we multiply the pivot row by $\boxed{\frac{\bar{c}_j}{|v_j|}}$ and add it to zeroth row.

For every $\underline{i}$, the new value of $\bar{c}_i$ is equal to

$$\bar{c}_i + v_i \frac{\bar{c}_j}{|v_j|}$$

which is nonnegative because of the way that $\underline{j}$ was selected (<u>cf. Eq 4.2</u>)

Example 4.7

Consider the tableau

		u_1	u_2	u_3	u_4	u_5
	0	2	6	10	0	0
$u_{B(1)} = u_4 =$	2	-2	4	1	1	0
$u_{B(2)} = u_5 =$	-1	4	$\boxed{-2}$	-3	0	1

→ pivot

1) Since $u_{B(2)} < 0$:

- We choose the 2nd row to be the pivot row

2) Negative entries of the pivot row are found in the 2nd and 3rd column, i.e.

$$j = \{2, 3\}$$

$$\frac{\bar{c}_j}{|v_j|} = \begin{cases} \boxed{\dfrac{\bar{c}_2}{|v_2|} = \dfrac{6}{|-2|} = 3} \\[2ex] \dfrac{\bar{c}_3}{|v_3|} = \dfrac{10}{|-3|} = \dfrac{10}{3} \end{cases}$$

Minimum value

∴ $\boxed{\begin{array}{l} u_2 \text{ enters the basis} \\ u_5 \text{ exits the basis} \end{array}}$

3) We multiply the pivot row by 3 and add it to the 0th row.

4) We multiply the pivot row by 2 and add it to the 1st row.

5) We then divide the pivot row by -2

- The new tableau is

	u_1	u_2	u_3	u_4	u_5
-3	14	0	1	0	3
$u_{B(1)} = u_4 =$ 0	6	0	-5	1	2
$u_{B(2)} = u_2 =$ $1/2$	-2	1	$3/2$	0	$-1/2$

Notes: This is because the dual is a maximization problem

- The cost has increased to 3 (Recall: 0^{th} row, 0^{th} column is the negative of the current cost)

- Furthermore, we now have $B^{-1}b \geq 0$ and an optimal solution has been found.

(end of the example)

- Note that the pivot element v_j is always chosen to be ~~nonnegative~~ negative, whereas the corresponding reduced cost $\bar{c}_j$ is nonnegative

- Let us temporarily assume that $\bar{c}_j$ is in fact positive

a) Then, in order to replace $\bar{c}_j$ by zero, we need to add a positive multiple of the pivot row to the 0^{th} row.

b) Then, in order to replace $\bar{c}_j$ by zero, we need to add a positive multiple of the pivot row to the 0^{th} row.

c) Since $u_{B(l)}$ is negative this has the effect of adding a negative quantity to the upper left corner (0^{th} row, 0^{th} column). Equivalently: the dual cost increases.

- Thus, as long as the reduced cost of every nonbasic variable is nonzero, the dual cost increases with each basis change, and no basis will ever be repeated in the course of the algorithm.

- It follows, that the algorithm must eventually terminate and this can happen in one of two ways:

  (a) We have $B^{-1}b \geq 0$ and an optimal solution.

  (b) All of the entries $v_1, \dots, v_n$ in the pivot row are nonnegative and we are therefore unable to locate a pivot element.

  Analogy to the primal simplex method: the optimal dual cost is $+\infty$ and the primal problem is infeasible (we are not going to prove this).

An iteration of the dual simplex method:

1. A typical iteration starts with the tableau associated with a basis matrix B and with all reduced costs nonnegative.

2. Examine the components of the vector $B^{-1}b$ in the 0^{th} column of the tableau. If they are all nonnegative, we have an optimal basic feasible solution, and the algorithm terminates; else choose some l such that $x_{B(l)} < 0$.

3. Consider the l^{th} row of the tableau, with elements $x_{B(l)}, v_1, \dots, v_n$ (the pivot row). If $v_i \geq 0 \ \forall i$ then the optimal dual cost is $+\infty$ and the algorithm terminates

4. For each i such that $v_i < 0$, compute the ratio $\frac{\bar{c}_i}{|v_i|}$ and let j be the index of a column that corresponds to the smallest ratio. The column $A_{B(l)}$ exits the basis and the column A_j takes its place.

5. Add to each row of the tableau a multiple of the l^{th} row (the pivot row) so that v_j (the pivot element) becomes 1 and all other entries of the pivot column become 0.

Let us now consider the possibility that the reduced cost $\bar{c}_j$ in the pivot column is <u>zero</u>:

- <u>In this case</u>: the 0^{th} row of the tableau does not change and the dual cost $\underline{c'_B B^{-1} b}$ remains the same

Notes:
The 0^{th} row does not change because the reduced cost $\bar{c}_j$ is already zero, <u>therefore</u> no operation is required and the 0^{th} row does not change

- If the dual cost $c'_B B^{-1} b$ remains the same it will never increase and the algorithm will never terminate (algorithm can cycle)

- This can be <u>avoided</u> by employing a suitable anticycling rule:

<u>Lexicographic pivoting rule for the dual simplex method</u>:

1.) Choose any row $\underline{l}$ such that $u_{B(l)} < 0$ to be the pivot row.

2.) Determine the index $\underline{j}$ of the entering column as follows: For each column with $v_i < 0$, divide all entries by $|v_i|$, and then choose the lexicographically smallest column. If there is a <u>tie</u> choose the one with the smallest index

If the dual simplex method is initialized so that every column of the tableau is lexicographically positive and if the above lexicographic pivoting rule is used, the method terminates in a finite number of steps (we will not see the proof).

Q: When should we use the dual simplex method?

- Suppose that we already have an optimal basis for some L. P. problem, and that we wish to solve the same problem for a different choice of the right-hand side vector b.

Notes:
Recall that L. P. problem:
- $Ax = b$
- $Bx_B = b$

- The optimal basis for the original problem may be primal infeasible under the new value of b.
(since it will have an impact on almost all of the 0^{th} column $B^{-1}b$)
- On the other hand: a change in b does not affect the reduced costs and we still have a dual feasible solution

Notes:
Recall that the reduced costs are
$c' - c'_B B^{-1} A$

- Recall that: Corollary 4.2 (pp. 148 of the book)
Let x and p be feasible solutions to the primal and the dual respectively, and suppose that $p'b = c'x$ (i.e. same cost). Then x and p are optimal solutions to the primal and the dual, respectively.

- Thus, instead of solving the new problem from scratch, it may be preferable to apply the dual simplex starting from the optimal basis for the original problem.

The geometry of the dual simplex method

- We ~~are~~ now present an alternative viewpoint based on geometric considerations
- We continue to assume that:
  - We are dealing with a problem in standard form
  - Matrix A has L.I. rows
  - B is a basis matrix with columns $A_{B(1)}, \dots, A_{B(m)}$
- B determines a basic solution to the primal problem with $u_B = B^{-1}b$. The same basis can also be used to determine a dual vector p by means of the equati

$$p' A_{B(i)} = c_{B(i)}, \quad i = 1, \dots, m$$

- Accordingly: There are m equations in m unknowns.

- Since the columns $A_{B(1)}, \dots, A_{B(m)}$ are L. I., there is a unique solution p. For such a vector p the number of L. I. active dual constraints is equal to the dimension of the dual vector, and it follows that we have a basic solution to the dual problem.

- In matrix notation, the dual basic solution p satisfies:

$$p'B = c'_B \quad \text{or} \quad p' = c'_B B^{-1}$$

- If p is also dual feasible, i.e. $p'A \leq c'$, then p is a basic feasible solution of the dual problem.

$\therefore_1$ Basis matrix B is associated with:

- Basic solution to the primal problem
- Basic solution to the dual problem

$\therefore_2$ A basic solution to the primal which is primal feasible is a basic feasible solution to the primal

$\therefore_3$ A basic solution to the dual which is dual feasible is a basic feasible solution to the dual

We now have a geometric interpretation of the dual simplex method:

- At every iteration, we have a B.F.S. to the dual problem
- The B.F. Solutions obtained at any two consecutive iterations have $m-1$ L.I. active constraints in common (the reduced costs of the $m-1$ variables that are common to both bases are zero).
- Thus, consecutive B.F. Solutions are either adjacent or they coincide.

Example 4.8 Consider the following standard form problem and its dual:

Minimize: $u_1 + u_2$

Subject to: $u_1 + 2u_2 - u_3 = 2$

$u_1 - u_4 = 1$

$u_1, u_2, u_3, u_4 \geq 0$

Maximize: $2p_1 + p_2$

Subject to: $p_1 + p_2 \leq 1$

$2p_1 \leq 1$

$p_1, p_2 \geq 0$

The feasible set of the primal problem is 4-dimensional.

i.e. project into two axes

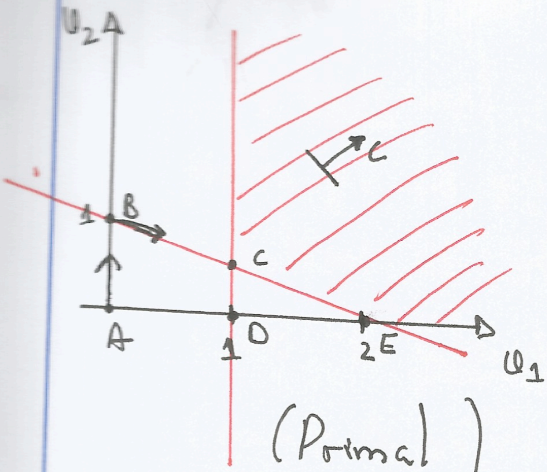
If we eliminate the variables u_3 and u_4, we obtain the equivalent problem:

Minimize: $u_1 + u_2$

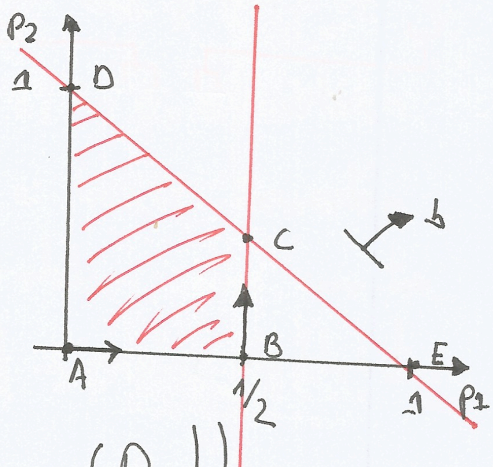
Subject to: $u_1 + 2u_2 \geq 2$

$u_1 \geq 1$

The feasible sets of the equivalent primal problem and of the dual are shown below:



(Primal)



(Dual)

- There is a total of five different bases in the standard form primal problem and five different basic solutions:
  - Points A, B, C, D, E in the primal figure (left)
  - " A, B, C, D, E in " dual " (right)
- Forexample:
  - If we choose the columns A_3 and A_4 to be the basic columns we have the infeasible primal basic solution $u = (0, 0, -2, -1)$ (Point A)

Basic Columns $A_3, A_4 \Rightarrow$ Basic Variables $u_3, u_3 \Rightarrow$ Nonbasic u_1, u_2

$$\begin{cases} u_1 + 2u_2 - u_3 = 2 \\ u_1 - u_4 = 1 \end{cases} \Leftrightarrow \begin{cases} -u_3 = 2 \\ -u_4 = 1 \end{cases} \Leftrightarrow \begin{cases} u_3 = -2 \\ u_4 = 1 \end{cases}$$

- The corresponding dual basic solution is obtained by letting $p'A_3 = c_3 = 0$ and $p'A_4 = c_4 = 0$ (the original cost was $(1, 1, 0, 0)$) which yields $p = (0, 0)$. This is a basic feasible solution of the dual problem and can be used to start the dual simplex method.
(point A of the dual)

- The associated initial tableau is:

$$
\begin{array}{r|c|cccc}
 & & u_1 & u_2 & u_3 & u_4 \\
\hline
 & 0 & 1 & 1 & 0 & 0 \\
\hline
u_3 = & -2 & -1 & \boxed{-2} & 1 & 0 \\
u_4 = & -1 & -1 & 0 & 0 & 1 \\
\end{array}
$$

- We carry out two iterations of the dual simplex:

$$
\begin{array}{r|c|cccc}
 & & u_1 & u_2 & u_3 & u_4 \\
\hline
 & -1 & 1/2 & 0 & 1/2 & 0 \\
\hline
u_2 = & 1 & 1/2 & 1 & -1/2 & 0 \\
u_4 = & -1 & \boxed{-1} & 0 & 0 & 1 \\
\end{array}
$$

$$
\begin{array}{r|c|cccc}
 & & u_1 & u_2 & u_3 & u_4 \\
\hline
 & -3/2 & 0 & 0 & 1/2 & 1/2 \\
\hline
u_2 = & 1/2 & 0 & 1 & -1/2 & 1/2 \\
u_1 = & 1 & 1 & 0 & 0 & -1 \\
\end{array}
$$

- This sequence corresponds to the path:

$$A \to B \to C$$

~~Notes:~~
~~1) $(0,0,-2,-1)$~~
~~2) $(1,1,0,0)$~~
~~3) $(1/2,1,1/2,-1)$~~

In either figure

- <u>In the primal figure:</u>
  - Path traces a sequence of <u>infeasible</u> basic solution until, at optimality, it becomes feasible

- <u>In the dual figure:</u>
  - Behaves exactly like the primal ($A \to B \to C$)
  - Moves through a sequence of dual B.F.S while at each step improving the cost function

Q: Why the sequence $A \to B \to C$?

1) We start out with $u = (0,0,-2,-1) \Rightarrow u_1 = u_2 = 0$ (Point A)
2) Tableau ② has basic variables $u_2 = 1$, $u_4 = 1 \Rightarrow u_1 = 0$, $u_2 = 1$ (Point B)
3) // 3 // // // $u_2 = 1/2$, $u_1 = 1$ (<u>Point</u> C)

- Having observed that the dual simplex method moves from one B.F.S. of the dual to an adjacent one, it may be tempting to say that the dual simplex method is simply the primal simplex method applied to the dual.

- However, the dual problem is not in standard form.

- If we were to convert it to standard form and the apply the primal simplex method, the resulting method is not necessarily identical for the dual simplex method.

- A more accurate statement is to simply say that the dual simplex method is a variant of the simplex method tailored to problems defined exclusively in terms of linear inequality constraints.

4.6 Farka's Lemma and linear inequalities

- Suppose that we wish to determine whether a given system of Liner inequalities is infeasible

- Using duality theory: We show that infeasibility of a given system of Linear inequalities is equivalent to the feasibility of another, related, system of linear inequalities.

- Consider a set of standard form constraints:

$$Au = b$$
$$u \geq 0$$

- Suppose that there exists some vector p such that

$p'A \geq 0'$ and $p'b < 0$

- Then, for any $u \geq 0$, we have $p'Au \geq 0$ and since $p'b < 0$, it follows that $p'Au \neq p'b$, i.e.

$$\begin{cases} p'Au \geq 0 \\ p'b < 0 \end{cases} \Rightarrow \cancel{p}'Au \neq \cancel{p}'b \Leftrightarrow Au \neq b$$

- We conclude that $Au \neq b \;\; \forall u \geq 0$

Q: But what does this mean? Any ideas?

∴ If we can find a vector p satisfying $p'A \geq 0'$ and $p'b < 0$, the standard form constraints cannot have any feasible solution. Such a vector p is called a certificate of infeasibility

- Farkas' lemma states that whenever a standard form problem is infeasible, such a certificate of infeasibility p is guaranteed to exist.

Theorem 4.6 (Farkas' lemma): Let $\underline{A}$ be a matrix of dimensions $\underline{m \times n}$ and let $\underline{b}$ be a vector in $\mathbb{R}^m$. Then, exactly one of the following two alternatives holds:

(a) There exists some $u \geq 0$ such that $Au = b$
(i.e. a solution exists)

(b) There exists some vector p such that $p'A \geq 0'$ and $p'b < 0$ (i.e. a solution does not exist

Proof:

- If there exists some $\underline{u \geq 0}$ satisfying $\boxed{Au = b}$, and if $\underline{p'A \geq 0'}$ then $p'\boxed{b} = \underbrace{\boxed{p'A}\,\boxed{u}}_{\geq 0} \geq 0$
$\geq 0 \quad \geq 0$

- Since $p'b \geq 0$ alternative b) of the theorem cannot hold

- Assume that there exists no vector $\underline{u \geq 0}$ satisfying $\underline{Au = b}$. Consider the pair of problems:

$$\begin{array}{llll} \text{Maximize:} & 0'u & \text{Minimize:} & p'b \\ \text{Subject to:} & Au = b & \text{Subject to:} & p'A \geq 0' \\ & u \geq 0 & & \end{array}$$

- Note that the first is the dual of the second.

- The maximization problem is infeasible since we are trying to maximize something that has always zero value $(0'.u = 0)$

↳

- Since the maximization problem is infeasible:

Minimization problem $\begin{cases} \text{Unbounded } (-\infty) \\ \text{Infeasible.} \end{cases}$

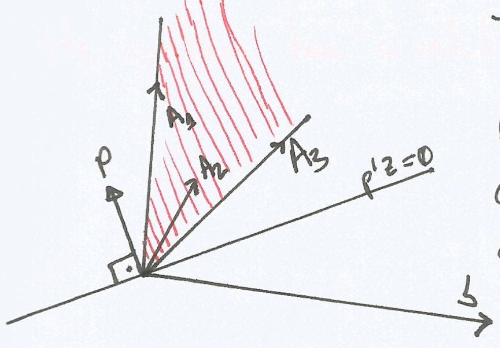
- Since $p=0$ is a feasible solution to the minimization problem (because with $p=0$ the only constraint $p'A \geq 0$ is satisfied) it follows that the minimization problem is unbounded.

- Therefore, there exists some p which is feasible, that is, $p'A \geq 0'$, and whose cost is negative, that is, $p'b < 0$
(since the problem is unbounded, i.e. cost is $-\infty$, the the function that we are trying to minimize need to be negative, i.e., $p'b<0$)

Geometric interpretation of Farkas' lemma

- Let $A_1, \dots, A_n$ be the columns of the matrix A and note that $Au = \sum_{i=1}^{n} A_i \cdot u_i$

- Therefore, the existence of a vector $u \geq 0$ satisfying $Au=b$ is the same as requiring that b lies in the set of all nonnegative linear combinations of the vectors $A_1, \dots, A_n$ which is the shaded region of the following image:



If the vector b does not belong to the set of all nonnegative combinations of $A_1, \dots, A_n$ (i.e. shaded area) then we can find an hyperplane $\{z \mid p'z=0\}$ that separates it from the set

- If $\underline{b}$ does not belong to the shaded region (in which case the first alternative in Farkas' lemma does not hold), we expect intuitively that we can find a vector p and an associated hyperplane $\{z \mid p'z = 0\}$ such that $\underline{b}$ lies on one side of the hyperplane while the <u>shaded region</u> lies on the other side.
- We then have $\begin{cases} p'b < 0 \\ \forall i, p'A_i \geq 0 \end{cases}$ ~~(one side of the hyperplane)~~ ~~(other side of the hyperplane)~~
- Therefore, the <u>second alternative</u> of Farkas' lemma <u>holds</u>.

- <u>Finally</u>, there is an equivalent statement of Farkas' lemma which is sometimes <u>more convenient</u>:

<u>Corollary 4.3</u> Let $A_1, \dots, A_n$ and $\underline{b}$ be given vectors and suppose that any vector p that satisfies $p'A_i \geq 0$ $i = 1, \dots, n$ must also satisfy $p'b \geq 0$. Then, $\underline{b}$ can be expressed as a <u>nonnegative</u> linear combination of the vector $A_1, \dots, A_n$ (i.e. belongs to the shaded area, therefore the system has a solution).

Theorem 4.7 Suppose that the system of linear inequalit[ies] $Au \leq b$ has at least one solution and let d be some scalar. Then, the following are equivalent:

(a) Every feasible solution to the system $Au \leq b$ satisfies $c'u \leq d$

(b) There exists some $p \geq 0$ such that $p'A = c'$ and $p'b \leq d$.

Proof: Consider the following pair of problems:

Maximize: $c'u$ | Minimize: $p'b$
Subject to: $Au \leq b$ | Subject to: $p'A = c'$, $p \geq 0$

- Note that the first is the dual of the second
- If the system $Au \leq b$ has a feasible solution and if every feasible solution satisfies $c'u \leq d$, then the first problem has an optimal solution and the optimal cost is bounded above by d.
- By the strong duality theorem: the 2^{nd} problem also has an optimal solution p whose cost is bounded above by d
- This optimal solution satisfies $\begin{cases} p'A = c' \\ p \geq 0 \\ p'b \leq d \text{ (bounded above by } d) \end{cases}$

— Conversely, if some p satisfies $\begin{cases} p'A = c' \\ p \geq 0 \\ p'b \leq d \end{cases}$ then the weak duality theorem asserts that every feasible solution to the first problem must also satisfy $c'x \leq d$ $\blacksquare$