Chapter 2 - The geometry of linear programming

Notes:

This chapter introduces:

Notes:
Polyhedron = Poliedro in portuguese

Polyhedron is a set described by a finite number of linear equality and inequality constraints

The feasible set is a polyhedron

2.1 - Polyhedra and convex sets

Definition 2.1 A polyhedron is a set that can be described in the form $\{u \in \mathbb{R}^n \mid Au \geq b\}$, where A is an $m \times n$ matrix and b is a vector in $\mathbb{R}^m$

Recall that:

- Feasible set can be described by inequality constraints of the form $Au \geq b$
- Therefore, the feasible set is a polyhedron

Definition 2.2 A set $S \subset \mathbb{R}^n$ is <u>bounded</u> if there exists a constant k such that the absolute value of every component of every element of S is less than or equal to k

Examples:

- Consider $S \subset \mathbb{R}^3$, $S = \{(1,2,3), (4,5,6), (7,8,9)\}$ then S is bounded by $k = 9$
- Consider $S \subset \mathbb{R}^3$, $S = \{(-10,0,1), (-9,1,2), (-8,2,3)\}$ then S is bounded by $k = 10$

Definition 2.3 Let a be a nonzero vector in $\mathbb{R}^n$ and let b be a scalar

<u>(a)</u> The set $\{u \in \mathbb{R}^n \mid a'u = b\}$ is called a <u>hyperplane</u>

<u>(b)</u> The set $\{u \in \mathbb{R}^n \mid a'u \geq b\}$ is called a <u>halfspace</u>

Note that a polyhedron is equal to the intersection of a finite number of half-spaces.

Q: Why is this so?

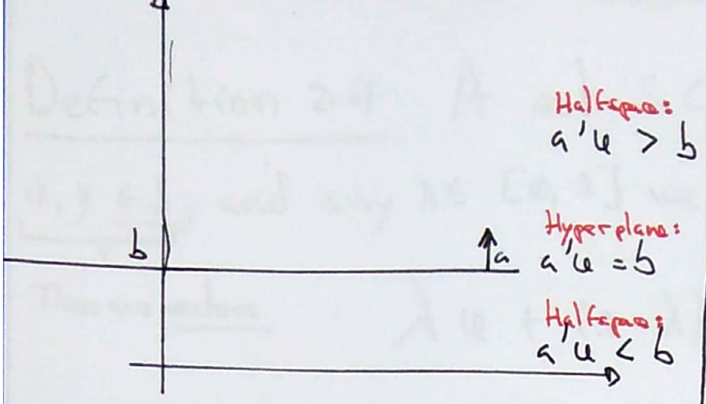
- Polyhedron := set of the form $\{u \in \mathbb{R}^n \mid Au \geq b\}$
- A is the matrix where the constraints are represented
- Halfspace := set of the form $\{u \in \mathbb{R}^n \mid a'u \geq b\}$

  Recall that a is an inequality constraint
- Therefore, A contains multiple a_i'

Q: How can this be viewed graphically?

- Hyperplane := set $\{u \in \mathbb{R} \mid a'u = b\}$
- Notice what this means:

  A set of vectors u that when multiplied by a' have value b
- This b value can be viewed in a 2D space as a constant line



Notes:

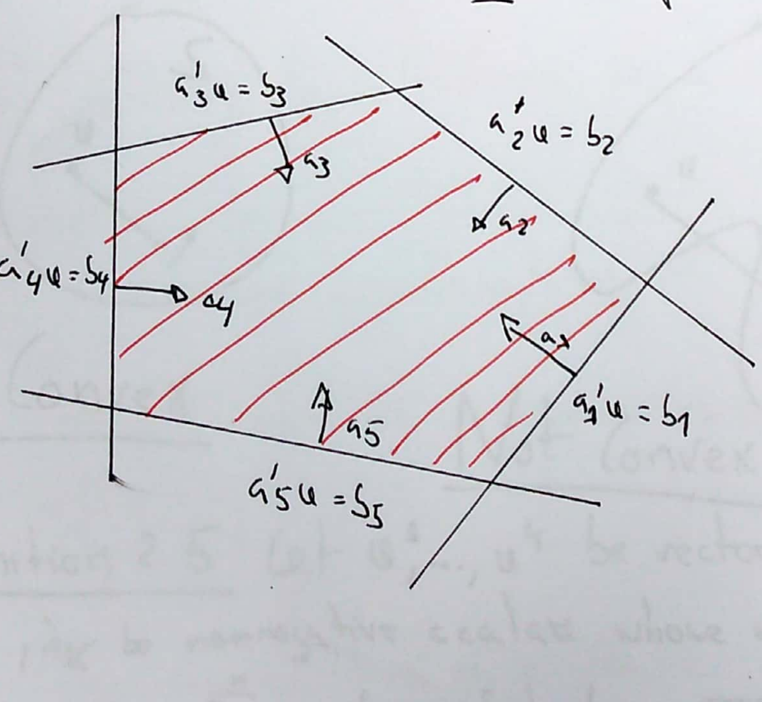
- Vector $\underline{a}$ is always perpendicular to the hyperplane
- If $\underline{u}$ and $\underline{y}$ belong to the same hyperplane:

$$\begin{cases} a'u = b \\ a'y = b \end{cases} \iff a'u = a'y \iff a'\underbrace{(u-y)}_{\text{but this is also a vector}} = 0$$

- Therefore $\underline{a}$ is orthogonal to any direction vector confined to the hyperplane

- If we consider vector in $\mathbb{R}^2$ then the equations $\underline{a'u = b}$ can be viewed as lines

- Consider the polyhedron $\{u \mid a_i' u \geq b_i,\ i = 1, \dots, 5\}$ that is the intersection of $\underline{5}$ halfspaces



Definition 2.4 A set $S \subset \mathbb{R}^n$ is $\boxed{\text{convex}}$ if for any $u, y \in S$, and any $\lambda \in [0, 1]$ we have

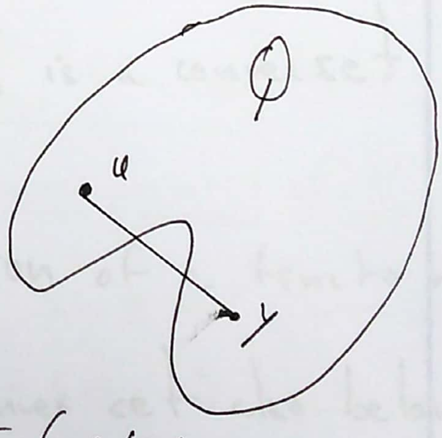
These are vectors

$$\lambda u + (1-\lambda) y \in S$$

∴ A set is convex if the segment joining any two of its elements is contained in the set

$\boxed{\text{Q}}$: How can this be visualized?

Convex



Not Convex

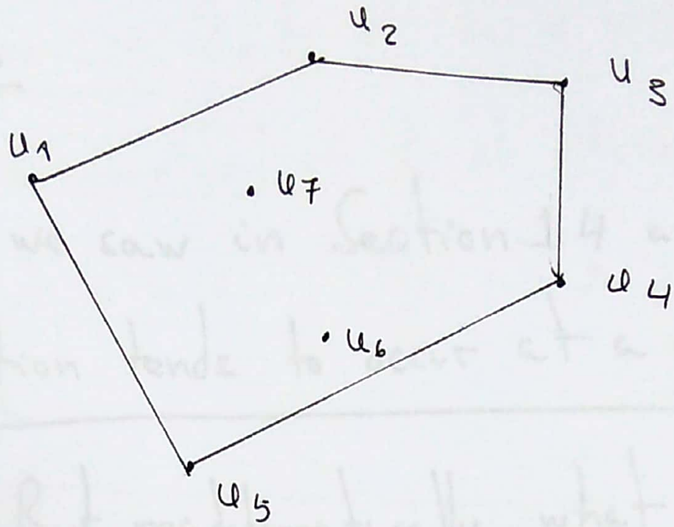
Definition 2.5 Let $u^1, \dots, u^k$ be vectors in $\mathbb{R}^n$ and let $\lambda_1, \dots, \lambda_k$ be nonnegative scalars whose sum is unity.

(a) The vector $\sum_{i=1}^{k} \lambda_i u^i$ is said to be a $\boxed{\text{convex combination}}$ of the vectors $u^1, \dots, u^k$

(b) The $\boxed{\text{convex hull}}$ of the vectors $u^1, \dots, u^k$ is the set of all convex combinations of these vectors.

Notes: In Portuguese: Envoltória Convexa

Q: How can this be visualized?



Theorem 2.1 (Proof is in the book)

(a) The intersection of convex sets is convex

(b) Every polyhedron is a convex set

(c) A convex combination of a finite number of elements of a convex set also belongs to that set

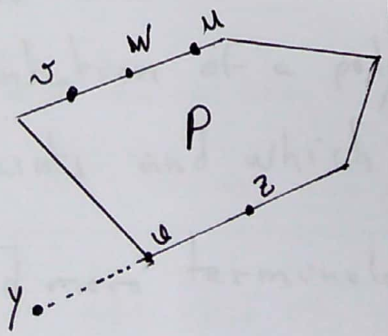
(d) The convex hull of a finite number of vectors is a convex set

2.2. Extreme points, vertices and basic feasible solutions

- As we saw in Section 1.4 an optimal solution tends to occur at a corner
- $\boxed{\text{Q:}}$ But mathematically, what is a corner?

The book suggests three different ways
- Extreme point
- Vertex
- Basic feasible solution

Definition 2.6 Let P be a polyhedron. A vector $u \in P$ is an $\boxed{\text{extreme point}}$ of P if we cannot find two vectors $y, z \in P$, both different from u, and a scalar $\lambda \in [0, 1]$ such that $u = \lambda y + (1-\lambda) z$

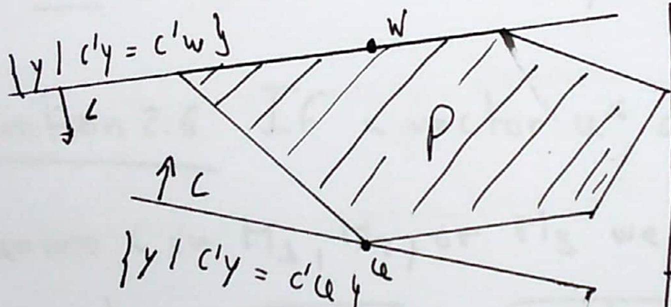


Notes:
- Vector w is not an extreme point because it is a convex combination of v and u
- Vector u is an extreme point since if $u = \lambda y + (1-\lambda) z$ with $\lambda \in [0,1]$, then either $y \notin P$, or $z \in P$, or $u = y$ or $u = z$

Alternative geometric definition:

Definition 2.7 Let P be a polyhedron. A vector $u \in P$ is a $\boxed{\text{vertex}}$ of P if there exists some $\boxed{c}$ (cost vector) such that $c'u < c'y$ for all y satisfying $y \in P$ and $y \neq u$

$\boxed{Q:}$ How can we visualize this?



Notes:
- Line at the bottom touches P at a single point and u is a vertex
- W is not a vertex because there is no hyperplane that meets P only at w

- The two geometric definitions are not easy to work with from an algorithmic point of view.
- We would like to have a definition that relies on a representation of a polyhedron in terms of linear constraints and which reduces to an algebraic test.
- We need more terminology

Consider a polyhedron $P \subset \mathbb{R}^n$ defined in terms of the linear equality and inequality constraints:

$$a_i' u \geq b_i \quad , \quad i \in M_1$$

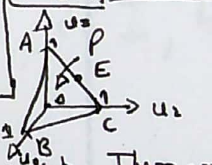
$$a_i' u \leq b_i \quad , \quad i \in M_2$$

$$a_i' u = b_i \quad , \quad i \in M_3$$

where M_1, M_2 and M_3 are finite set index sets, each $\underline{a_i}$ is a vector in $\mathbb{R}^n$, and each $\underline{b_i}$ is a scalar

Definition 2.6 If a vector u^* satisfies $a_i' u^* = b_i$ for some i in M_1, M_2, or M_3 we say that the correspond constraint is $\boxed{\text{active}}$ or $\boxed{\text{binding}}$ at u^*

Notes:

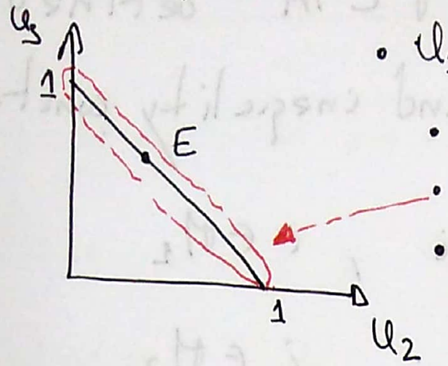


Let $P = \{(u_1, u_2, u_3) \mid u_1 + u_2 + u_3 = 1,\ u_1, u_2, u_3 \geq 0\}$. There are three constraints that are active at each one of the $\underline{A}$, $\underline{B}$, $\underline{C}$ and $\underline{D}$ points. There are only two constraints that are active at point E, namely $u_1 + u_2 + u_3 = 1$ and $u_1 = 0$ Why? Turn the page

- If there are $\underline{n}$ constraints that are active at a vector $\underline{u^*}$, then $\underline{u^*}$ satisfies a certain system of $\underline{n}$ linear equations in $\underline{n}$ unknowns
- This system has a unique solution if and only if these $\underline{n}$ equations are "linearly independent"

$\boxed{\text{Q:}}$ What does this statement mean?

Notes: **On point E:**



- $u_1 + u_2 + u_3 = 1$
  - $u_1 = 0 \Rightarrow u_2 = 1 - u_3$
- This is the line visually represented
- Note that points on this line may vary in their values of u_2 and u_3, but $u_1 = 0$

Theorem 2.2 - Let u^* be an element of $\mathbb{R}^n$ and let $I = \{ i \mid a_i' u^* = b_i \}$ be the set of indices of constraints that are active at u^*.

Then, the following are equivalent:

(a) There exist n vectors in the set $\{a_i \mid i \in I\}$ which are linearly independent.

(b) The span of the vectors a_i, $i \in I$, is all of $\mathbb{R}^n$, that is every element of $\mathbb{R}^n$ can be expressed as a linear combination of the vectors a_i, $i \in I$

(c) The system of equations $a_i' u = b_i$, $i \in I$, has a unique solution.

Proofs:

- (a) and (b) are well know results from linear algebra (see page 49)
- Regarding (c):
  - If the system of equations $a_i' u = b_i$, $i \in I$ has multiple solutions say u^1 and u^2 then:

$$\begin{cases} a_i' u^1 = b_i \\ a_i' u^2 = b_i \end{cases} \Rightarrow a_i' u^1 = a_i' u^2 \Leftrightarrow a_i' \underbrace{(u^1 - u^2)}_{d} = 0 \quad \forall i \in I$$

- Since $\underline{d}$ is orthogonal to every vector $\underline{a_i}, i \in I$, $\underline{d}$ is not a linear combination of these vectors

|Notes:|

Linear Combination: [source: wiki]

- Let $\vec{v_1}, \vec{v_2}, \vec{v_3}, \dots, \vec{v_n}$ be vectors in $\mathbb{R}^n$
- Let $c_1, c_2, c_3, \dots, c_n$ be scalars
- Then:

  $\vec{b} = c_1 \vec{v_1} + c_2 \vec{v_2} + \dots + c_n \vec{v_n}$ is a linear combination of $\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}$

- Therefore the vectors $a_i, i \in I$ do not span $\mathbb{R}^n$
- If $\underline{u}$ satisfies $a_i' u = b \;\; \forall i \in I$ then we also have

$$a_i'(u + d) = b_i \quad , \text{ but why?}$$

$$\Leftrightarrow a_i' u + \underbrace{a_i' d}_{0} = b_i \quad \forall i \in I$$

0, since $\underline{d}$ is orthogonal to every vector $a_i, i \in I$

$\therefore_1$ This means that multiple solutions exist

$\therefore_2$ Therefore, if a_i $i \in S$ span all of $\mathbb{R}^n$, a linear combination must exist and therefor no multiple solutions exist. I.e a single solution exist ▨

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Definition 2.9 Consider a polyhedron $\underline{P}$ defined by linear equality and inequality constraints, and let $\underline{u^*}$ be an element of $\mathbb{R}^n$

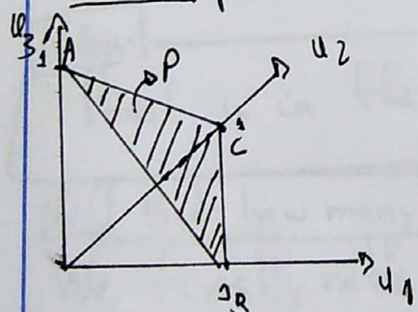
(a) The vector $\underline{u^*}$ is a $\boxed{\text{basic solution}}$ if

(i) All equality constraints are active or

(ii) Out of the constraints that are active at $\underline{u^*}$ there are $\underline{n}$ of them that are linearly independent

(b) If $\underline{u^*}$ is a basic solution that satisfies all of the constraints, we say that it is a basic ... feasible solution

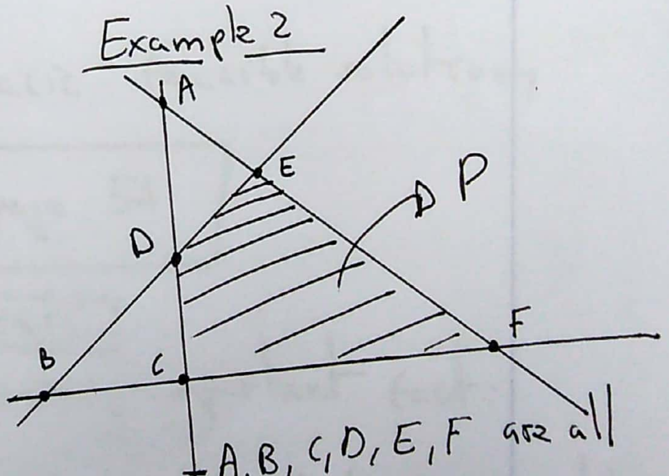
Example 1



$P: \{(u_1, u_2, u_3) \mid u_1 + u_2 + u_3 = 1,\ u_1, u_2, u_3 \geq 0\}$

A, B, C are basic feasible solutions
(All of the constraints are satisfied)

Example 2



- A, B, C, D, E, F are all basic solutions, because there are two linearly independent constraints that are active (Definition 2.9.a.(ii))
- C, D, E, F are basic feasible solutions (all constraints are satisfied)

We have given three different definitions:

Extreme Point } These are geometric ("easy")
Vertex

Basic Feasible solution } This is algebraic ("hard")

All three definitions are equivalent.

Theorem 2.3 Let P be a nonempty polyhedron and let $u^* \in P$. Then the following are equivalent

(a) u^* is a vertex

(b) u^* is an extreme point

(c) u^* is a basic feasible solution

Notes: Proof is in the book page 57

Q: But how many solutions exist?

We finally note the following important fact:

Corollary 2.1 Given a finite number of linear inequality constraints, there can only be a finite number of basic or basic feasible solutions

Proof:

- Consider a system of $\underline{m}$ linear inequality constraints imposed on a vector $u \in \mathbb{R}^n$
- At any <u>basic solution</u> (by definition 2.9)

  There are $\underline{n}$ linearly independent active constraints
- Since any $\underline{n}$ linearly independent active constraints define a unique point:

  $\neq$ basic solutions correspond to $\neq$ sets of $\underline{n}$ linearly independent active constraints.
- <u>Therefore:</u>

  Number of basic solutions is <u>bounded above</u> by the number of ways that we can choose $\underline{n}$ <u>constraints</u> out of a total of $\underline{m}$, which is finite

$\boxed{\text{Q: But what does all of this mean?}}$

Example:

- Consider the <u>unit cube</u> $\{u \in \mathbb{R}^n \mid 0 \leq u_i \leq 1,\ i = 1, \dots, n\}$
- Note that $0 \leq u_i \leq 1 \Rightarrow \left\{\begin{matrix} 0 \leq u_i \\ u_i \leq 1 \end{matrix}\right\}$ I.e. two constraints
- Therefore a total of $\underline{2 \times n}$ constraints exist
- This implies that a total of:

$$\underbrace{2}_{u_1} \times \underbrace{2}_{u_2} \times \underbrace{2}_{u_3} \times \dots \times \underbrace{2}_{u_n} = 2^n \text{ basic feasible solutions}$$

2.3 Polyhedra in standard form

- Lets now focus on polyhedra in standard form.
  (This will be essential for the simplex method later on)

- **Recall:**
  - $P = \{u \in \mathbb{R}^n \mid Au = b,\ u \geq 0\}$ be a polyhedron in standard form
  - A has dimension $m \times n$, where m is the number of equality constraints
  - Assume the m rows are linearly independent

- We will see that linearly dependent rows of A correspond to redundant assumptions that can be discarded.

Example: Consider the following constraints:

$$\begin{cases} u_1 + u_2 + u_3 = 3 \\ 2u_1 + 2u_2 + 2u_3 = 6 \end{cases} \times 2$$

This is the same constraint but just multiplied by 2, therefore it is redundant: linearly dependent

Example:

~~$3u_1 + 4u_2 + 5u_3$~~

$$\begin{cases} 3u_1 + 4u_2 = 5 \\ 2u_1 + 9u_2 = 2 \\ 4u_1 + 18u_2 = 4 \end{cases}$$

This is the same constraint multiplied by 2: linearly dependent

Example:

$$\begin{cases} 2u_1 + 3u_2 + u_3 = 0 \\ -u_2 = -2 \\ u_3 = 1 \end{cases}$$

None of the rows are a combination of the others: Linearly independence. All constraints are important

Recall that:

- At any basic solution: there must be n linearly independent constraints that are active
- At any basic solution: $Au = b$ (definition 2.9. a) which provides us with m active constraints. These are linearly independent, given our assumption.
- In order to obtain n active constraints (definition 2.9.b) we need to choose $n-m$ of the variable u_i and set them to zero
- However, for the resulting set of n constraints to be linearly independent the choice of these $n-m$ variables is not entirely arbitrary.

Theorem 2.4 Consider the constraints $Au = b$ and $u \geq 0$ and assume that the $m \times n$ matrix A has linearly independent rows. A vector $u \in \mathbb{R}^n$ is a basic solution if and only if we have $Au = b$ and there exist indices $B(1), \dots, B(m)$ such that:

(a) The columns $A_{B(1)}, \dots, A_{B(m)}$ are linearly independent

(b) If $i \neq B(1), \dots, B(m)$, then $u_i = 0$

Proof: [Full details can be found on page 53]

- Consider some $u \in \mathbb{R}^n$ and suppose that there are indices $B(1), \dots, B(m)$ that satisfy (a) and (b). The active constraints $u_i = 0$, $i \neq B(1), \dots, B(m)$ and $Au = b$ imply that

$$\sum_{i=1}^{m} A_{B(i)} u_{B(i)} = \sum_{i=1}^{n} A_i u_i = Au = b$$

Since the columns $A_{B(i)}$ are linearly independent (2.4.a) $u_{B(1)}, \dots, u_{B(m)}$ are unique. Thus the system of equations has a unique solution. By Theorem 2.2 there are n L.I active constraints and since $m \leq n$ then u is a basic solution

Procedure for constructing basic solutions

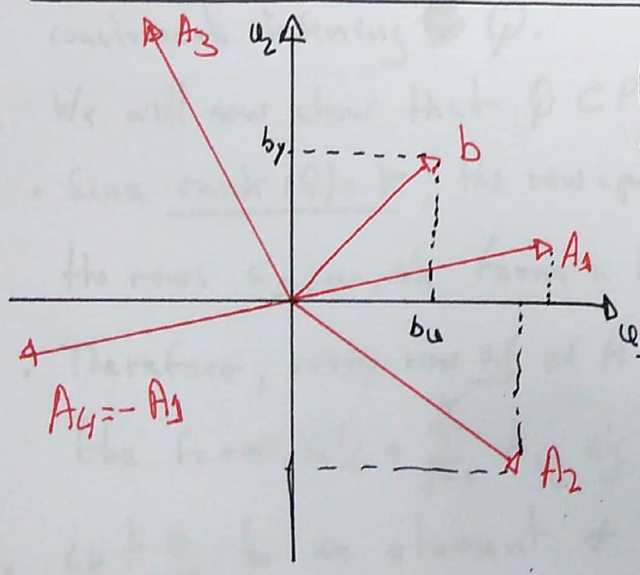
1) Choose $\underline{m}$ linearly independent columns
$A_{B(1)}, \dots, A_{B(m)}$

2) Let $u_i = 0$ for all $i \neq B(1), \dots, B(m)$

3) Solve the system of $\underline{m}$ equations $\underline{Au = b}$ for the unknowns $u_{B(1)}, \dots, u_{B(m)}$

Since every basic feasible solution is a basic solution it can be obtained from this procedure

$\boxed{Q}$: How can this be visualized?



Notes:

- $Au = b \iff \sum_{i=1}^{n} A_i . u_i = b$

$A_1 . u_1 + A_2 . u_2 = b \iff$

$A_1 . b_u + A_2 . b_y = b \Rightarrow \begin{cases} b_u \geq 0 \\ b_y < 0 \end{cases}$

Since $u_i \geq 0$ the solution is infeasible

- Standard form with $\underline{n = 4}$ and $\underline{m = 2}$:
  - A_1 and A_2 form a basis, but the corresponding solution is <u>infeasible</u> because a negative value of u_2 is needed to synthesize $\underline{b}$ from A_1 and A_2
  - A_1 and A_3 form a basis, corresponding solution is <u>feasible</u>
  - A_1 and A_4 do not form a basis: <u>linear dependent</u>

Theorem 2.5 Let $P = \{u \mid Au = b, u \geq 0\}$ be a nonempty polyhedron where $\underline{A}$ is a matrix of dimension $\underline{m \times n}$ with rows $a'_1, \dots, a'_m$. Suppose that ~~rank(A)~~ $\text{rank}(A) = k < m$ and that the rows $a'_{i_1}, \dots, a'_{i_k}$ I.e. the rows that are linearly independent are linearly independent. Consider the polyhedron:

$$\mathbb{Q} = \{u \mid a'_{i_1} u = b_{i_1}, \dots, a'_{i_k} u = b_{i_k}, u \geq 0\}$$

Then $\mathbb{Q} = P$

Proof:

- Proof for $i_1 = 1, \dots, i_k = k$, i.e. first $\underline{k}$ rows are l.i.
- General case can be reduced to this by rearranging the rows of $\underline{A}$
- Clearly $P \subset \mathbb{Q}$ since any element of $\underline{P}$ automatically satisfies the constraints defining $\mathbb{Q}$.
- We will now show that $\mathbb{Q} \subset P$
  - Since $\underline{\text{rank}(A) = k}$, the row space of $\underline{A}$ has dimension $\underline{k}$ and the rows $a'_1, \dots, a'_k$ form a basis of the row space
  - Therefore, every row $\underline{a'_i}$ of A can be expressed in the form $a'_i = \sum_{j=1}^{k} \lambda_{ij} a'_j$ for some scalars λ_{ij}.
  - Let $\underline{u}$ be an element of P and note that

$$b_i = a'_i u = \sum_{j=1}^{k} \lambda_{ij} a'_j u = \sum_{j=1}^{k} \lambda_{ij} b_j, \quad i = 1, \dots, m$$

Similar to this equality

• Consider now an element $\underline{y}$ of $\underline{Q}$. We will show that it belongs to $\underline{P}$. Indeed for any $\underline{i}$

$$a_i'y = \sum_{j=1}^{k} \lambda_{ij} a_j' y = \sum_{j=1}^{k} \lambda_{ij} \boxed{b_j} = b_i$$

Therefore $y \in P$, since the equality is observed ⇓ and also establishes that $Q \subset P$, since the restriction is observed in Q and P. ▨ (end of proof)

$\therefore_1$ Polyhedron Q in Theorem 2.5 is in standard form namely, $Q = \{u \mid Du = f, u \geq 0\}$ where $\underline{D}$ is a $k \times n$ submatrix of A with rank equal to $\underline{k}$ and $\underline{f}$ is a k-dimensional subvector of $\underline{b}$

$\boxed{Q}$: Ok... But what does this all mean?

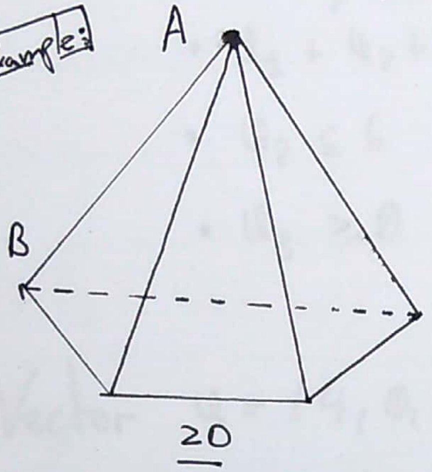
$\therefore_2$ Linearly dependent rows of $\underline{A}$ correspond to redundant constraints that can be discarded ⇓

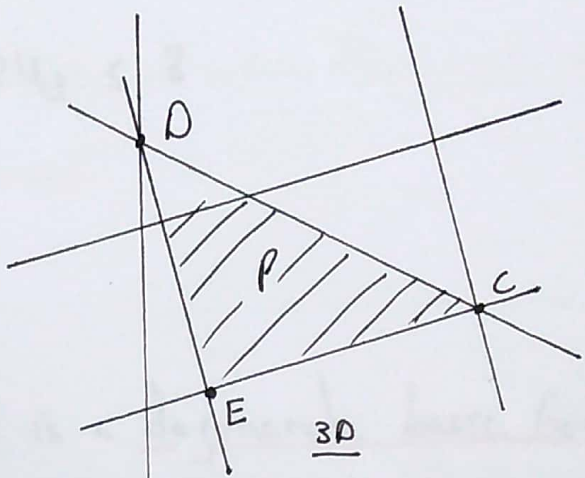
2.4 Degeneracy

Notes: Degeneração in Portuguese

Definition 2.10 A basic solution $u \in \mathbb{R}^n$ is said to be $\boxed{\text{degenerate}}$ if more than n of the constraints are active at u.

Example:





Degenerate basic feasible solutions: A, C

Nondegenerate basic feasible solutions: B, E

Degenerate basic solution: D

Example 2.4 Consider the polyhedron P defined by the constraints:

$$u_1 + u_2 + 2u_3 \leq 8$$

$$u_2 + 6u_3 \leq 12$$

$$u_1 \leq 4$$

$$u_2 \leq 6$$

$$u_1, u_2, u_3 \geq 0$$

- Vector $u = (2, 6, 0)$ is a nondegenerate basic feasible solution

  Why?

- Vector $u = (4, 0, 2)$ is a degenerate basic feasible solution

  Why?

LD

Vector $u = (2, 6, 0)$ is a nondegenerate basic feasible solution:

- There are exactly three active and linearly independent active constraints
- Namely:
  - $u_1 + u_2 + 2u_3 \leq 8$
  - $u_2 \leq 6$
  - $u_3 \geq 0$

Vector $u = (4, 0, 2)$ is a degenerate basic feasible solution:

- There are four active constraints, three of them linearly independent
- Namely:
  - $u_1 + u_2 + 2u_3 \leq 8$
  - $u_2 + 6u_3 \leq 12$
  - $u_1 \leq 4$
  - $u_2 \geq 0$

$\boxed{\text{Notes:}}$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\times(-1)+} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 6 \\ 0 & -1 & -2 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\times 1+,\ \times(-1)+} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 6 \\ 0 & 0 & 4 \\ 0 & 0 & -6 \end{bmatrix} \xrightarrow{\times\left(\frac{1}{4}\right)}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & -6 \end{bmatrix} \xrightarrow{\times 6+} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore_1$ Notice that one of the restrictions disappeared

$\therefore_2$ The number of leading 1's is three, therefore the rank is 3

$\therefore_3$ This means that only 3 constraints are linearly independent.

Degeneracy in standard form polyhedra

Definition 2.11 Consider the standard form polyhedron $P = \{u \in \mathbb{R}^n \mid Au = b,\ u \geq 0\}$ and let $\underline{u}$ be a basic solution. Let $\underline{m}$ be the number of rows of $\vec{A}$. The vector $\underline{u}$ is a degenerate basic solution if more than $\underline{n-m}$ of the components of $\underline{u}$ are zero.

Q: Why is this so?

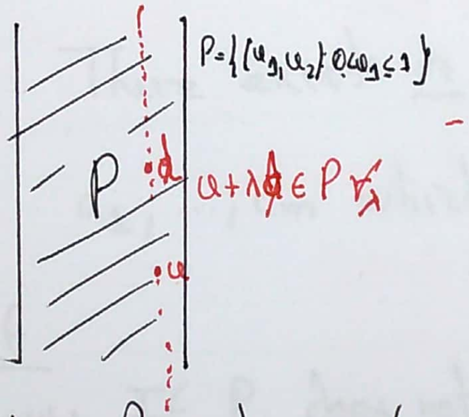
from $\underset{m \times n}{A}u = b$

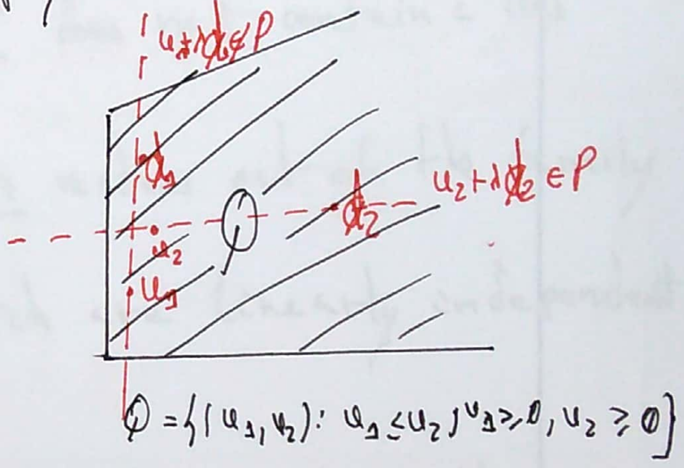
- At a basic solution of a polyhedron the $\underline{m}$ equality constraints are always active (by definition 2.5.a. i)
- Therefore we don't need $\underline{n}$ active constraints for degeneracy since $\underline{m}$ are already met
- I.e. we need $\underline{n-m}$

2.5 Existence of extreme points

Q: What are the necessary and sufficient conditions for a polyhedron to have at least one extreme point?

It turns out that the existence of an extreme point depends on whether a polyhedron contains an infinite line or not.





Polyhedron $\underline{P}$ contains a line and does not have an extreme point

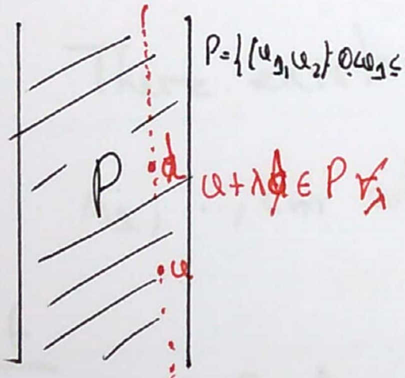
Polyhedron Q does not contain a line and has extreme points

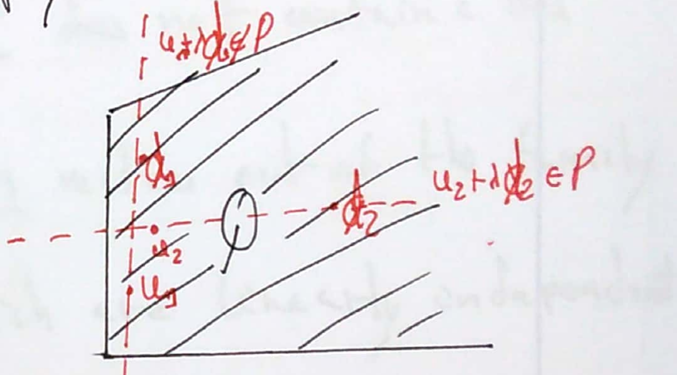
Definition 2.12 A polyhedron $P \subset \mathbb{R}^n$ $\boxed{\text{contains a line}}$ if there exists a vector $\underline{u \in P}$ and a nonzero vector $\underline{d} \in \mathbb{R}^n$ such that $u + \lambda d \in P$ for all scalars λ.

2.5 Existence of extreme points

> **Q:** What are the necessary and sufficient conditions for a polyhedron to have at least one extreme point?

It turns out that the existence of an extreme point depends on whether a polyhedron contains an infinite line or not.





$Q = \{(u_1,u_2): u_1 \le u_2, u_1 \ge 0, u_2 \ge 0\}$

Polyhedron P contains a line and does not have an extreme point

Polyhedron Q does not contain a line and has extreme points

Definition 2.12 A polyhedron $P \subset \mathbb{R}^n$ $\boxed{\text{contains a line}}$ if there exists a vector $u \in P$ and a nonzero vector $d \in \mathbb{R}^n$ such that $u + \lambda d \in P$ for all scalars λ.

Theorem 2.6 Suppose that the polyhedron $P = \{u \in \mathbb{R}^n \mid a_i' u \geq b_i,\ i = 1, \dots, m\}$ is nonempty. Then, the following are equivalent:

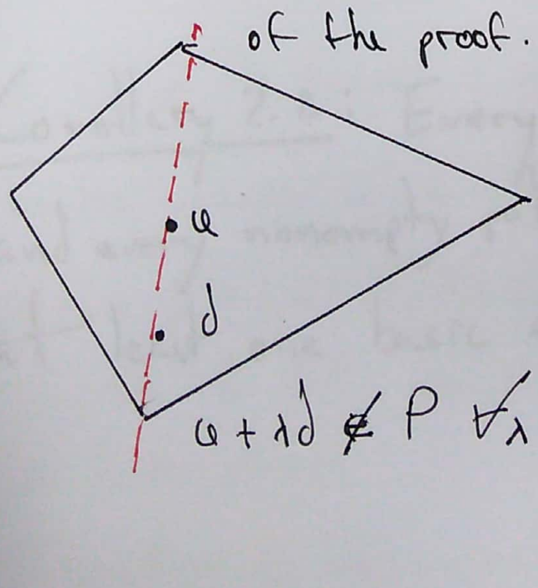
(a) The polyhedron $\underline{P}$ has at least one extreme point

(b) The polyhedron $\underline{P}$ does not contain a line

(c) There exists $\underline{n}$ vectors out of the family $a_1, \dots, a_m$ which are linearly independent

Proof

Focus: If $\underline{P}$ does not contain a line, then it has a basic feasible solution, and therefore an extreme point. We will see a geometric interpretation of the proof.



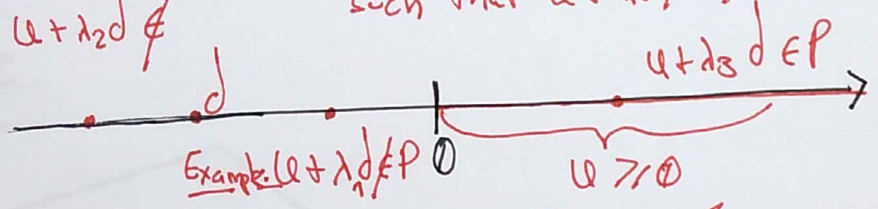
$u + \lambda d \notin P \ \forall \lambda$

$\therefore_1$ A <u>bounded</u> polyhedron does not contain a line

$\therefore_2$ Similarly, the positive orthant $\{u \mid u \geq 0\}$ does not contain a line

Q: Why $\therefore_2$?

$\therefore$ There exist at least nonzero vector d such that $u + \lambda d \notin P$, i.e. does not contain a line



$$u + \lambda d \notin P \quad \forall \lambda$$

$\therefore_3$ Since a polyhedron in standard form $A.u = b$, $u \geq 0$, is contained in the positive orthant, it <u>does not</u> contain a line. Therefor P has an extreme point

<u>Corollary 2.2</u>: Every nonempty bounded polyhedron and every nonempty polyhedron in standard form has at least one basic solution

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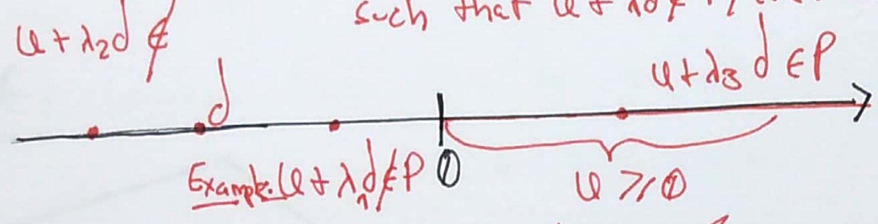


$$u + \lambda d \notin P \quad \forall \lambda$$

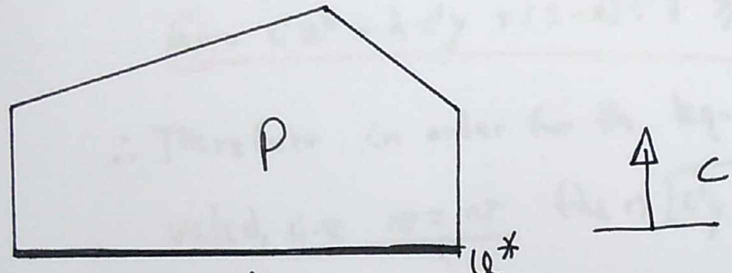
$\therefore_3$ Since a polyhedron in standard form $A.u = b,\ u \geq 0$, is contained in the positive orthant, it <u>does not</u> contain a line. Therefore $\underline{P}$ has an extreme point

<u>Corollary 2.2</u>: Every nonempty bounded polyhedron and every nonempty polyhedron in standard form has at least one basic solution

2.6 - Optimality of extreme points

Theorem 2.7 Consider the linear programming problem of minimizing $c'u$ over a polyhedron P. Suppose that P has at least one extreme point and that there exists an optimal solution. Then there exists an optimal solution which is an extreme point of P.

Proof:



Let:
- Q is the set of optimal solutions
- Extreme point u^* of Q is also an extreme point of P
- P be of the form $P = \{u \in \mathbb{R}^n \mid Au \geq b\}$ and let v be the optimal value of the cost $c'u$
- Then $Q = \{u \in \mathbb{R} \mid Au \geq b, \ c'u = v\}$ is also a polyhedron
- Since $Q \subset P$ and since P contains no lines, Q contains no lines either
- Therefore Q has an extreme point

↳

- Let $\underline{u}^*$ be an extreme point of $\underline{P}$, then $\underline{u}^*$ is also an extreme point of $\underline{P}$:

- Suppose that $\underline{u}^*$ is not an extreme point of P
- Then, there exist $y \in P$, $z \in P$ such that $\underline{y \neq u^*}$, $\underline{z \neq u^*}$ and some $\lambda \in [0,1]$ such that $u^* = \lambda y + (1-\lambda) z$ (by definition 2.6)
- Therefore: $v = c'u^* = \lambda c'y + (1-\lambda) c'z$
- Furthermore, since $\underline{v}$ is the optimal cost: $c'y \geq v$ and $c'z \geq v$
- This implies that $c'y = c'z = v$. Why?

$$v = c'u^* = \lambda c'y + (1-\lambda) c'z \geq \lambda v + (1-\lambda) v = \lambda v + v - \lambda v = v$$

∴ Therefore in order for the equation $v \geq v$ to remain valid, i.e. $\underline{v = v}$ then $\boxed{c'y = c'z = v}$

- And therefore $\underline{z \in P}$ and $\underline{y \in P}$.
- But this $\boxed{\text{contradicts}}$ the fact that $\boxed{u^* \text{ is an}}$ extreme point of $\underline{P}$, since we would be able to express it as $u^* = \lambda y + (1-\lambda) z$ with $y \neq u^*$ and $z \neq u^*$ for $y \in P$ and $\underline{z \in P}$
- The contradiction establishes that $\underline{u^*}$ is an $\underline{\text{extreme point}}$ of P.
- In addition, since $\underline{u^*}$ belongs to P it is optimal