Chapter 9 - Number Systems

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Motivation

Lets start the semester with an easy subject:

In everyday life how do you count numbers? Any ideas?

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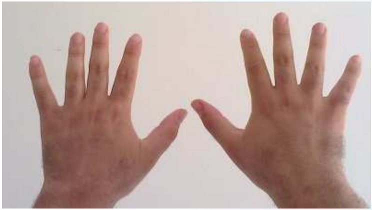

- 1 Fingers;
- 2 Fingers;
- 3 Fingers;
- . . .
- 10 Fingers;

;)

Decimal system is used to represent numbers:

- Digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9;

Consider the number 83:

What does this mean using the decimal system? Any ideas?

Example

Consider the number 83:

What does this mean using the decimal system? Any ideas?

- Number 10 was counted 8 times:
  - $8 \times 10 = 8 \times 10^1$
- Number 1 was counted 3 times:
  - $3 = 3 \times 10^0$
- Combining these elements:
  - *I.e.*: $83 = 8 \times 10 + 3 = 8 \times 10^1 + 3 \times 10^0$

Example

Consider the number 4728:

What does this mean using the decimal system? Any ideas?

Exercise

Consider the number 4728:

What does this mean using the decimal system? Any ideas?

- Number 1000 was counted X times:
  -
- Number 100 was counted Y times:
  -
- Number 10 was counted Z times:
  -
- Number 1 was counted D times:
  -
- Combining these elements:
  -

Exercise

Consider the number 4728:

What does this mean using the decimal system? Any ideas?

- Number 1000 was counted 4 times:
  - $4 \times 1000 = 4 \times 10^3$
- Number 100 was counted 7 times:
  - $7 \times 100 = 7 \times 10^2$
- Number 10 was counted 2 times:
  - $2 \times 10 = 2 \times 10^1$
- Number 1 was counted 8 times:
  - $8 = 8 \times 10^0$
- Combining these elements:
  - *I.e.*: $4728 = 4 \times 10^3 + 7 \times 10^2 + 2 \times 10^1 + 8 \times 10^0$

But what if we have decimal fractions? Any ideas?

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- *E.g.:* how do we represent the number 0.256 using the decimal system?

Example

Consider the number 0.256:

What does this mean using the decimal system? Any ideas?

- Number 0.1 was counted 2 times:
  - $2 \times 0.1 = 2 \times 10^{-1}$
- Number 0.01 was counted 5 times:
  - $5 \times 0.01 = 5 \times 10^{-2}$
- Number 0.001 was counted 6 times:
  - $6 \times 10^{-3} = 6 \times 10^{-3}$
- Combining these elements:
  - *I.e.:* $0.256 = 2 \times 10^{-1} + 5 \times 10^{-2} + 6 \times 10^{-3}$

But wait:

What if we have an **integer** part and a **fractional** part? Any ideas?

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What if we have an **integer** part and a **fractional** part? Any ideas?

- *E.g.:* how do we represent the number 442.256 using the decimal system?

Example

Consider the number 442.256:

> What does this mean using the decimal system? Any ideas?

Exercise

Consider the number 442.256:

What does this mean using the decimal system? Any ideas?

- Number 100 was counted X times:
    -
- Number 10 was counted Y times:
    -
- Number 1 was counted Z times:
    -
- Number 0.1 was counted Q times:
    -
- Number 0.01 was counted W times:
    -
- Number 0.001 was counted E times:
    -

Combining these elements:

- *I.e.:* $442.256 =$

Exercise

Consider the number 442.256:

> What does this mean using the decimal system? Any ideas?

- Number 100 was counted 4 times:
  - $4 \times 100 = 4 \times 10^2$
- Number 10 was counted 4 times:
  - $4 \times 10 = 4 \times 10^1$
- Number 1 was counted 2 times:
  - $2 \times 1 = 2 \times 10^0$
- Number 0.1 was counted 2 times:
  - $2 \times 0.1 = 2 \times 10^{-1}$
- Number 0.01 was counted 5 times:
  - $5 \times 0.01 = 5 \times 10^{-2}$
- Number 0.001 was counted 6 times:
  - $6 \times 0.001 = 6 \times 10^{-3}$

Combining these elements:

- *I.e.:* $442.256 = 4 \times 10^2 + 4 \times 10^1 + 2 \times 10^0 + 2 \times 10^{-1} + 5 \times 10^{-2} + 6 \times 10^{-3}$

Some important observations:

- Decimal system is said to have a **base**, or **radix**, of 10;
- In any number:
    - Leftmost digit is referred to as the most significant digit (MSD);
    - Rightmost digit is called the least significant digit (LSD);

In conclusion:

4	7	2	2	5	6
100s	10s	1s	tenths	hundredths	thousandths
10^2	10^1	10^9	10^{-1}	10^{-2}	10^{-3}
position 2	position 1	position 0	position –1	position –2	position –3

Figure: Positional interpretation of decimal number: 472, 256 (Source: (Stallings, 2015))

TYPO:

- Position 0 of the table should read 10^0

In general, ÊX where:

- $X = \{\cdots d_2 d_1 d_0 . d_{-1} d_{-2} d_{-3} \cdots\}$
- $X = \sum_i (d_i \times 10^i)$

Positional Number Systems

Decimal system illustrates a positional number system (1/2):

- Each number is represented by a string of digits;
- Each digit position i has an associated weight r^i:
  - r is the radix / base of the system;
- General form of a number in such a system with radix r is:

$$(\cdots a_3 a_2 a_1 a_0 . a_{-1} a_{-2} a_{-3} \cdots)_r$$

  - Where $a_i : 0 \leq a_i < r$

Positional Number Systems

Decimal system illustrates a positional number system (2/2):

- Number is defined to have the value:

$$\cdots a_3 r^3 + a_2 r^2 + a_1 r^1 + a_0 r^0 + a_{-1} r^{-1} \cdots$$

The question is:

> Do we really need to use the decimal system? Any ideas?

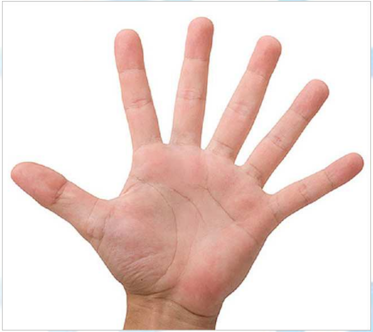
Decimal system:

- Radix 10;
- Digits in the range 0 through 9

> What if human beings had 12 fingers?

What if human beings had 12 fingers?

- Radix 12;
- Digits in the range 0 through 11;

Fun fact

You think 12 fingers is weird?

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You think 12 fingers is weird?

Have a look at Polydactyly:

Binary System

Binary system only uses two digits:

- Radix / base 2;
- Binary digits 1 and 0 have the same meaning as in decimal notation:

$$0_2 = 0_{10}$$
$$1_2 = 1_{10}$$

- Also a positional number system:
  - Each binary digit in a number has a value;

Exercise

$$10_2 = ?_{10}$$
$$11_2 = ?_{10}$$
$$100_2 = ?_{10}$$

Exercise

$$10_2 = (1 \times 2^1) + (0 \times 2^0) = 2_{10}$$
$$11_2 = (1 \times 2^1) + (1 \times 2^0) = 3_{10}$$
$$100_2 = (1 \times 2^2) + (0 \times 2^1) + (0 \times 2^0) = 4_{10}$$

But what if are trying to represent a binary number with a fractional part? Any ideas?

But what if are trying to represent a binary number with a fractional part? Any ideas?

- Binary number 1001.101_2 converts to what decimal number?

Exercise

$$1001.101_2 = ?_{10}$$

Exercise

$$
\begin{aligned}
1001.101 &= 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 1 \times 2^0 + 1 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3} \\
&= 9.625_{10}
\end{aligned}
$$

Remember this formula for the decimal system:

- $X = \{\cdots d_2 d_1 d_0 . d_{-1} d_{-2} d_{-3} \cdots\}$
- $X = \sum_i (d_i \times 10^i)$

What do you think would be the necessary changes for a **binary** system? Any ideas?

Remember this formula for the **decimal** system:

- $X = \{\cdots d_2 d_1 d_0 . d_{-1} d_{-2} d_{-3} \cdots\}$
- $X = \sum_i (d_i \times 10^i)$

What do you think would be the necessary changes for a **binary** system? Any ideas?

- Radix / base has value 2;
- *I.e.:* $X = \sum_i (d_i \times 2^i)$

Converting between Decimal and Binary

How can we convert between decimal and binary numbers? Any ideas?

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How can we convert between decimal and binary numbers? Any ideas?

Suppose we need to convert N from decimal into binary form (1/3):

- If we divide N by 2 we obtain a quotient N_1 and a remainder R_0;
- We then may write:
  - $N = 2 \times N_1 + R_0$
  - $R_0 = 0$ or 1

Suppose we need to convert N from decimal into binary form (2/3):

- N_1 can also be divided by 2, then:
  - $N_1 = 2 \times N_2 + R_1$
  - $R_1 = 0$ or 1

Suppose we need to convert N from decimal into binary form (3/3):

- N_2 can also be divided by 2, then:
  - $N_2 = 2 \times N_3 + R_2$
  - $R_2 = 0$ or 1

Continuing this sequence will eventually produce:

- a quotient $N_{m-1} = 1$
- a remainder R_{m-2} which is 0 or 1;

We are now able to obtain the **binary form**:

$$N = (R_{m-1} \times 2^{m-1}) + (R_{m-2} \times 2^{m-2}) + \cdots + (R_2 \times 2^2) + (R_1 \times 2^1) + R_0$$

Example

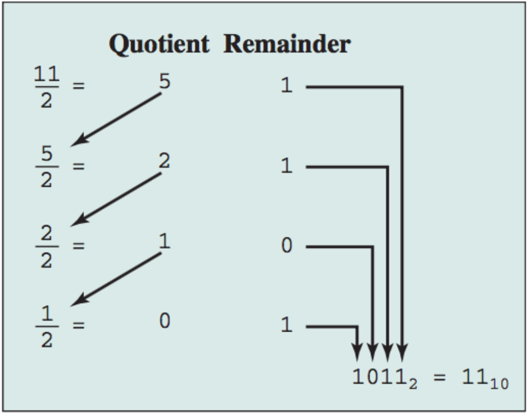

Figure: (Source: (Stallings, 2015))

Example

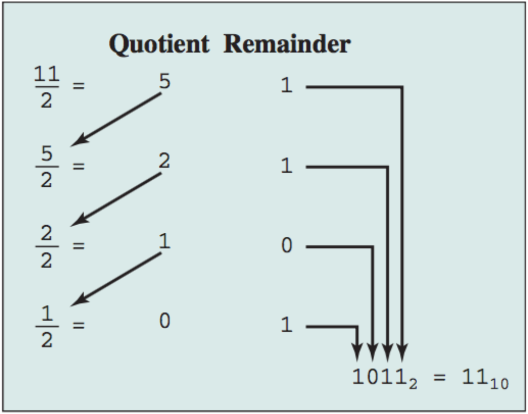

Figure: (Source: (Stallings, 2015))

Exercise

Converting the following decimal numbers into binary:

- $8_{10} = ?_2$
- $9_{10} = ?_2$
- $1024_{10} = ?_2$
- $1025_{10} = ?_2$
- $1027_{10} = ?_2$
- $2049_{10} = ?_2$
- $4096_{10} = ?_2$
- $4106_{10} = ?_2$

Hexadecimal Notation

In computation:

- All forms of data is represented in a binary fashion;
- Very cumbersome for human beings =(
- Most computer professionals prefer a more compact notation:
  - Hexadecimal notation FTW! =)
    - Binary digits are grouped into sets of four bits (nibble);
    - Each possible combination of four binary digits is given a symbol;

This is the hexadecimal table:

$0000 = 0$	$0100 = 4$	$1000 = 8$	$1100 = C$
$0001 = 1$	$0101 = 5$	$1001 = 9$	$1101 = D$
$0010 = 2$	$0110 = 6$	$1010 = A$	$1110 = E$
$0011 = 3$	$0111 = 7$	$1011 = B$	$1111 = F$

Figure: (Source: (Stallings, 2015))

In general:

$$Z = \sum_i (h_i \times 16^i)$$

- Radix / base has value 16;
- Each hexadecimal digit h_i is in the decimal range $0 \leq h_i < 15$;

Decimal (base 10)	Binary (base 2)	Hexadecimal (base 16)
0	0000	0
1	0001	1
2	0010	2
3	0011	3
4	0100	4
5	0101	5
6	0110	6
7	0111	7
8	1000	8
9	1001	9
10	1010	A
11	1011	B
12	1100	C
13	1101	D
14	1110	E
15	1111	F
16	0001 0000	10
17	0001 0001	11
18	0001 0010	12
31	0001 1111	1F
100	0110 0100	64
255	1111 1111	FF
256	0001 0000 0000	100

Figure: (Source: (Stallings, 2015))

Example

$$2C_{16} = ?_{10}$$

Example

$$
\begin{aligned}
2C_{16} &= (2_{16} \times 16^1) + (C_{16} \times 16^0) \\
&= (2_{10} \times 16^1) + (12_{10} \times 16^0) \\
&= 44
\end{aligned}
$$

This concludes this lession:

- Thank you for your time =)

References I

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