SOME COMMENTS AND A BIBLIOGRAPHY ON THE LAGUERRE-SAMUELSON INEQUALITY WITH EXTENSIONS AND APPLICATIONS IN STATISTICS AND MATRIX THEORY

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Abstract. We examine an 1880 theorem of Laguerre [49] concerning polynomials with all real roots and a 1968 inequality of Samuelson [103] for the maximum and minimum deviation from the mean, and establish their equivalence. The bounds provided by Laguerre's Theorem involve the first three coefficients of an n-th degree polynomial while Samuelson's Inequality is in terms of the standard deviation (and the mean) of a set of n real numbers (observations). We present eight proofs of this Laguerre-Samuelson inequality and survey the literature; we also give various extensions and applications in statistics and matrix theory. We include some historical and biographical information and present an extensive bibliography with over 100 entries.

1. Introduction and mise-en-scène

1.1. The Laguerre-Samuelson Inequality. Throughout this paper $x_1, x_2, \ldots, x_n$ will denote n real numbers with (arithmetic) mean

(1.1) $$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

and standard deviation (with divisor n):

(1.2) $$s = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2} = \sqrt{\frac{1}{n}\left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right)}.$$

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Then

$$(1.3)\qquad \bar{x} - s\sqrt{n-1} \le x_j \le \bar{x} + s\sqrt{n-1} \qquad \text{for all } j = 1, 2, \ldots, n$$

or equivalently

$$(1.4)\qquad (x_j - \bar{x})^2 \le (n-1)s^2 \qquad \text{for all } j = 1, 2, \ldots, n.$$

Equality holds in (1.4) if and only if all the x_i other than x_j are equal and so then x_j is either the largest or the smallest of the x_i; equality holds on the left (right) of (1.3) if and only if the $n-1$ largest (smallest) x_i are all equal.

We see, therefore, that given the mean and standard deviation of a set of real numbers, their minimum is bounded below and their maximum bounded above. These bounds are often referred to as "Samuelson's Inequality" in the statistical literature[1] in view of the inequalities established in 1968 by the American economist and Nobel laureate Paul Anthony Samuelson[2] (b. 1915) in the *Journal of the American Statistical Association* [103].

The inequalities (1.3) were (almost certainly first) established in 1880 by the well-known French mathematician Edmond Nicolas Laguerre[3] (1834–1886) in the *Nouvelles Annales de Mathématiques (Paris)* [49]. Laguerre's results were obtained in a completely different notation and context[4].

Laguerre's interest focused on n-th degree polynomials with all roots real. Let $x_1, x_2, \ldots, x_n$ denote the roots, all of which we will assume to be real, of the n-th degree polynomial equation with $n \ge 2$:

$$(1.5)\qquad f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0.$$

Since we will assume that this polynomial has degree n we will now suppose, without loss of generality, that

$$(1.6)\qquad a_0 = 1.$$

[1] Cf. e.g., Arnold [3], Borwein, Styan and Wolkowicz [16], Chaganty and Vaish [26], Farnum [31], Kabe [45], Mărgăritescu [57], Mathew and Nordström [59], Murty [76], Patel, Kapadia and Owen [92] (p. 263), Puntanen [99] (Example 6.16, pp. 275–276), and Wolkowicz and Styan [121].

[2] For an "autobiographical account of his career" see Samuelson [105].

[3] For a biographical account see Brezinski [20].

[4] While several authors in the mathematical literature refer to Laguerre (cf. e.g., Lupaş [52], Madhava Rao and Sastry [54], Mitrinović [71], pp. 210–211, Popoviciu [97], Sz.-Nagy [112], [113], [114], and Weber [120], pp. 364–371, the only author who we could find in the statistical literature to do so was Rodica-Cristina Vodă [118] in 1983 (in Romanian), who also references Mihăileanu [67].

Let

$$t_1 = \sum_{i=1}^{n} x_i \qquad \text{and} \qquad t_2 = \sum_{i=1}^{n} x_i^2. \tag{1.7}$$

Then

$$a_1 = -\sum_{i=1}^{n} x_i = -t_1 \qquad \text{and} \qquad a_2 = \sum_{i<j} x_i x_j = \tfrac{1}{2}(t_1^2 - t_2). \tag{1.8}$$

Laguerre [49] proved that

$$-\frac{a_1}{n} - b\sqrt{n-1} \le x_j \le -\frac{a_1}{n} + b\sqrt{n-1} \qquad \text{for all } j = 1, 2, \ldots, n, \tag{1.9}$$

where

$$b = \sqrt{\frac{(n-1)a_1^2}{n^2} - \frac{2a_2}{n}} = \frac{\sqrt{nt_2 - t_1^2}}{n} \tag{1.10}$$

using (1.8). It follows at once that

$$-\frac{a_1}{n} = \bar{x} \qquad \text{and} \qquad b = s, \tag{1.11}$$

respectively the mean and the standard deviation defined in (1.1) and (1.2) above, and so the inequalities (1.9) coincide with (1.3).

Laguerre [49], however, did not observe that $-a_1/n$ and b were in fact the mean and standard deviation[5] of the roots x_i; his interest was in obtaining bounds for the roots, whenever they are all real, of an n-th degree polynomial given the first three coefficients—in our formulation the first of these: $a_0 = 1$, cf. (1.6)[6].

In this paper we will, therefore, refer to the inequalities (1.3) or (1.4) as the "Laguerre-Samuelson Inequality".

While "Samuelson's Inequality" is certainly the most popular name for (1.3), the name "Extreme Deviations Inequality" is also used in the (relatively recent) statistical literature[7]; in 1974 Arnold used "extreme deviance" in

[5] The term "standard deviation" was introduced in 1893 (by Karl Pearson (1857–1936) "in a lecture to the Royal Society", cf. Hart [37], p. 626; Stigler [110], p. 328, "although the idea was by then nearly a century old", cf. Abbott [1], p. 105.

[6] Laguerre [49] did not assume that $a_0 = 1$ and so his results involve a_1/a_0 and a_2/a_0 instead of our a_1 and a_2.

[7] Cf. Dwass (1975) [30], O'Reilly (1976) [91], and Quesenberry (1974) [100].

the title of his paper [3], while "How deviant can you be?" is the title of the seminal paper by Samuelson (1968) [103]; the 1992 survey paper by Olkin [89] is entitled "A matrix formulation on how deviant an observation can be". Much earlier, however, the term "extreme deviate" appears in the title of the 1948 paper by Nair [79] and "extreme observation" in the titles of the papers by Hartley and David (1954) [38] and McKay (1935) [60]. In the hydrology journal *Water Resources Research*, Kirby (1974) [47] uses "standardized maximum deviate".

Wolkowicz and Styan (1988) call (1.3) the "Samuelson-Nair Inequality" in their *Encyclopedia of Statistical Sciences* entry [125], while Arnold and Balakrishnan in their 1989 monograph *Relations, Bounds and Approximations for Order Statistics* [4] present many inequalities related to and including the Laguerre-Samuelson Inequality in their Section 3.2 entitled "Variations on the Samuelson-Scott theme"[8].

The Indian statistician Keshavan Raghavan Nair[9] (b. 1910) established the Laguerre-Samuelson Inequality (1.3) in his 1947 Ph.D. thesis [77], publishing his proof a year later in 1948 in the *Journal of the Indian Society of Agricultural Statistics (Delhi)* [78], cf. also Nair [82], [83]. J. M. C. Scott[10] established several inequalities (see §1.4 below) on ordered absolute deviations $|x_j - \bar{x}|$ in the Appendix to the 1936 paper [94] by Egon Sharpe Pearson (1895–1980), assisted by C. Chandra Sekar in *Biometrika (London)*; as noted by Arnold and Balakrishnan [4] (Theorem 3.2, p. 44) the Laguerre-Samuelson Inequality is a special case of one of Scott's inequalities.

1.2. The Brunk Inequalities. Now let us arrange the x_i's in nondecreasing order:

$$x_{\min} = x_{(n)} \leq x_{(n-1)} \leq \cdots \leq x_{(2)} \leq x_{(1)} = x_{\max} \tag{1.12}$$

so that $x_{(j)}$ is the j-th largest. Then:

$$\bar{x} + \frac{s}{\sqrt{n-1}} \leq x_{\max} = x_{(1)} \leq \bar{x} + s\sqrt{n-1} \tag{1.13}$$

and

$$\bar{x} - s\sqrt{n-1} \leq x_{\min} = x_{(n)} \leq \bar{x} - \frac{s}{\sqrt{n-1}}. \tag{1.14}$$

[8] Cf. [4], Theorem 3.3, pp. 45–46, for six proofs of the Laguerre-Samuelson Inequality; a further proof using the arithmetic-geometric mean inequality is proposed in [4] as Exercise 7, p. 62.

[9] For an "autobiographical article" see Nair [83].

[10] We believe J. M. C. Scott was at the Cavendish Laboratory, Cambridge, England, in the mid-50s, but have no further biographical information.

The right-hand inequality in (1.13) and the left-hand inequality in (1.14) are the Laguerre-Samuelson inequality (1.3). The left-hand inequality in (1.13) and the right-hand inequality in (1.14) were established (possibly for the first time) in 1959 by Hugh Daniel Brunk (b. 1919), also in the *Journal of the Indian Society of Agricultural Statistics* [22], and so we will refer to them as the "Brunk Inequalities". Unaware of Brunk's results these inequalities were established again by Boyd (1971) [18], Hawkins (1971) [40] and Wolkowicz and Styan (1979) [121].

Equality holds on the left of (1.13) if and only if equality holds on the left of (1.14) if and only if:

$$x_{(1)} = \cdots = x_{(n-1)} = \bar{x} + \frac{s}{\sqrt{n-1}} \qquad \text{and} \qquad x_{(n)} = \bar{x} - s\sqrt{n-1}\,;$$

equality holds on the right of (1.13) if and only if equality holds on the right of (1.14) if and only if:

$$x_{(1)} = \bar{x} + s\sqrt{n-1} \qquad \text{and} \qquad x_{(2)} = \cdots = x_{(n)} = \bar{x} - \frac{s}{\sqrt{n-1}}\,.$$

1.3. The Boyd-Hawkins Inequalities. For the k-th largest observation or "order statistic" $x_{(k)}$ we have the following inequalities

$$\bar{x} - s\sqrt{\tfrac{k-1}{n-k+1}} \le x_{(k)} \le \bar{x} + s\sqrt{\tfrac{n-k}{k}} \qquad \text{for } k = 2, \ldots, n-1; \tag{1.15}$$

equality holds on the left of (1.15) if and only if

$$x_{(1)} = \cdots = x_{(k-1)} = \bar{x} + s\sqrt{\tfrac{n-k+1}{k-1}} \quad \text{and} \quad x_{(k)} = \cdots = x_{(n)} = \bar{x} - s\sqrt{\tfrac{k-1}{n-k+1}}$$

and on the right of (1.15) if and only if

$$x_{(1)} = \cdots = x_{(k)} = \bar{x} + s\sqrt{\tfrac{n-k}{k}} \quad \text{and} \quad x_{(k+1)} = \cdots = x_{(n)} = \bar{x} - s\sqrt{\tfrac{k}{n-k}}.$$

If we put $k = 1$ in (1.15) then we obtain the same upper bound for $x_{\max} = x_{(1)}$ as in (1.13) but a weaker lower bound. Similarly, if we put $k = n$ in (1.15) then we obtain the same lower bound for $x_{\min} = x_{(n)}$ as in (1.14) but a weaker upper bound.

The inequalities (1.15) were established (possibly for the first time[11]) in 1971 by A. V. Boyd [18] in the *Publikacije Elektrotehničkog Fakulteta Univerziteta u Beogradu, Serija Matematika i Fizika (Belgrade)*[12] (in English)

[11] Rodica-Cristina Vodă [118], p. 547, comments (in Romanian) that (1.15) "este şi el inclus parţial in rezultatul lui Laguerre" (p. 547) or (in English) "can be partially derived from an old inequality due to Laguerre" (p. 548): no further details are given.

[12] The masthead of this journal also carries the French subtitle: *Publications de la Faculté d'Électrotechnique de l'Université à Belgrade, Série Mathématiques et Physique.*

and, also in 1971, by Douglas M. Hawkins [40] in the *Journal of the American Statistical Association*; see also Wolkowicz and Styan [121], [122], [123]. As observed by Arnold and Balakrishnan [4] (p. 49) and Wolkowicz and Styan [121], the inequalities (1.15) are "implicit" in the papers by Mallows and Richter (1969) [55] and Arnold and Groeneveld (1979) [7], while Scott (1936) [108] gives (without proof) the inequality

$$x_{(2)} \leq \bar{x} + s\sqrt{\frac{n-2}{2}}, \tag{1.16}$$

the special case of the upper bound in (1.15) for $k = 2$.

We will call (1.15) the "Boyd-Hawkins Inequalities".

1.4. The Scott Inequalities. The first (explicit) proof of the Laguerre-Samuelson Inequality in the statistical literature was almost certainly that given in 1936 by J. M. C. Scott [108] in the Appendix to the paper by Pearson and Chandra Sekar [94]; the Laguerre-Samuelson Inequality appears there as a special case of (1.19a), the first of three inequalities below, cf. Arnold and Balakrishnan [4], Theorem 3.2, p. 44, where it is observed that "Scott's ingenious constructive proof is apparently the only proof available in the literature."

Let us define the absolute deviations:

$$\delta_i = |x_i - \bar{x}|; \qquad i = 1, \ldots, n, \tag{1.17}$$

and let $\delta_{(i)}$ denote the i-th largest absolute deviation so that

$$\delta_{(n)} \leq \delta_{(n-1)} \leq \cdots \leq \delta_{(1)}. \tag{1.18}$$

Of course the i-th largest absolute deviation $\delta_{(i)}$ will not, in general, be equal to $|x_{(i)} - \bar{x}|$.

Then

$$\delta_{(j)} \leq s\sqrt{\tfrac{n(n-j)}{j(n-j)+1}} \quad \text{for } j \text{ odd and } j \neq n, \tag{1.19a}$$

$$\delta_{(n)} \leq s\sqrt{\tfrac{n-1}{n(n+1)}} \quad \text{for } n \text{ odd}, \tag{1.19b}$$

$$\delta_{(j)} \leq s\sqrt{\tfrac{1}{j}} \quad \text{for } j \text{ even}. \tag{1.19c}$$

We note that $j = 1$ in (1.19a) corresponds to the Laguerre-Samuelson Inequality (1.4). The inequality (1.19b) is, of course, quite different to the

Brunk Inequality, cf. (1.14):

$$x_{\min} \leq \bar{x} - \frac{s}{\sqrt{n-1}}. \tag{1.20}$$

Indeed, we obtain equality in (1.19b) when $(n-1)/2$ of the x_i are equal to b and all other x_i are equal to $-1/b$, where

$$b = \sqrt{\frac{n+1}{n-1}}. \tag{1.21}$$

On the other hand equality holds in (1.20) if and only if the largest $n-1$ of the x_i are equal.

1.5. Purpose and Overview. Our main purpose in this paper is to survey the literature associated with the Laguerre-Samuelson, Brunk, and Boyd-Hawkins Inequalities, and to give several proofs. As observed by Arnold and Balakrishnan [4] (in their introduction to Chapter 3) the publication by Samuelson [103] "... spawned a torrent of generalizations, several of which referred to bounds on order statistics. It also spawned a flurry of rediscoveries of earlier notes on these topics. Ultimate priority seems hard to pin down ..."

In Section 2 we present eight different proofs of the "Laguerre-Samuelson Inequality" (1.3):

- 2.1. Laguerre (1880), Madhava Rao & Sastry (1940), Mitrinović (1970)
- 2.2. Thompson (1935)
- 2.3. Nair (1947, 1948), Kempthorne (1973), Arnold & Balakrishnan (1989)
- 2.4. Arnold (1974), Dwass (1975), Arnold & Balakrishnan (1989)
- 2.5. Arnold (1974), O'Reilly (1975, 1976), Arnold & Balakrishnan (1989), Murty (1990)
- 2.6. Wolkowicz and Styan (1979, 1980)
- 2.7. Smith(1980), Arnold & Balakrishnan (1989)
- 2.8. Olkin (1992).

Arnold and Balakrishnan [4], pp. 45–46, present six proofs, of which five (all but their first) are included in four (§2.3–2.5, 2.7) of our eight. As Arnold and

Balakrishnan [4] point out (p. 45): "It is instructive to ... consider several alternative proofs. The alternative proofs often suggest different possible extensions ... The Schwarz inequality[13] may be perceived to be lurking in the background of many of the proofs."

We also present several related inequalities and some applications in statistics and matrix theory. We include some historical and biographical information and present an extensive bibliography of over 100 entries from both the mathematical and the statistical literature. References to *Jahrbuch für die Fortschritte der Mathematik* are denoted by JFM (for reviews published in 1868–1930), *Mathematical Reviews* by MR (for reviews published since 1940), and to *Zentralblatt für Mathematik* [126] by Zbl (for reviews published since 1931).

2. The Laguerre-Samuelson Inequality: Eight Proofs

2.1. Laguerre (1880), Madhava Rao & Sastry (1940), Mitrinović (1970). Our first proof is that given in 1880 by Edmond Nicolas Laguerre [49], cf. also Madhava Rao and Sastry [54] and Mitrinović [71], pp. 210–211.

For any real scalar u, we have the sum of squares expansion:

$$\sum_{i=1}^{n}(u-x_i)^2 = nu^2 - 2t_1u + t_2 \geq (u-x_j)^2 = u^2 - 2x_ju + x_j^2 \tag{2.1}$$

for any particular x_j, since a sum of squared terms is always greater than or equal to any one of its summands. Here t_1 and t_2 are as in (1.7).

Rearranging (2.1), we see that for any real u,

$$(n-1)u^2 + 2(x_j - t_1)u + (t_2 - x_j^2) \geq 0. \tag{2.2}$$

Since this quadratic function in u is nonnegative, its discriminant must be non-positive:

$$4(x_j - t_1)^2 - 4(n-1)(t_2 - x_j^2) \leq 0. \tag{2.3}$$

Rearranging and simplifying (2.3) as a quadratic in x_j yields:

$$nx_j^2 - 2t_1x_j + t_1^2 - (n-1)t_2 \leq 0 \tag{2.4}$$

[13]Named after [Karl] Hermann Amandus Schwarz (1843–1921) for the inequality he established in 1888 in [107], pp. 343–345; the inequality was established, however, already in 1821 by [Baron] Augustin-Louis Cauchy (1789–1857) in [23], pp. 373–374, and in 1859 by Viktor Yakovlevich Bouniakowsky [Buniakovski, Bunyakovsky] (1804–1899) in [17], pp. 3–4. In this paper we will call it the Cauchy-Schwarz Inequality, cf. (2.14) below.

and so x_j must lie in the closed interval $[\alpha_1,\ \alpha_2]$, where $\alpha_1,\ \alpha_2$ are the roots of

$$nx_j^2 - 2t_1x_j + t_1^2 - (n-1)t_2 = 0. \tag{2.5}$$

These roots $\alpha_1,\ \alpha_2$ are:

$$\frac{2t_1 \pm \sqrt{4t_1^2 - 4n(t_1^2 - (n-1)t_2)}}{2n} = \frac{-a_1}{n} \pm b\sqrt{n-1} \tag{2.6}$$

using (1.10) and so (1.9) is established. □

We may arrive at the inequality (2.4) more easily, however, cf. Madhava Rao and Sastry [54], since

$$\begin{aligned} -\{nx_j^2 - 2t_1x_j + t_1^2 - (n-1)t_2\} &= (n-1)(t_2 - x_j^2) - (t_1 - x_j)^2 \\ &= (n-1)\sum_{i\neq j} x_i^2 - (\sum_{i\neq j} x_i)^2 \\ &= (n-1)\sum_{i\neq j}(x_i - \hat{x})^2 \geq 0, \end{aligned}$$

cf. (1.2), where

$$\hat{x} = \frac{1}{n-1}\sum_{i\neq j} x_i \tag{2.7}$$

is the "reduced" mean of the $n-1$ roots $x_1, ..., x_n$ excluding x_j. □

2.2. Thompson (1935). Almost certainly the first proof in a statistical context is the following proof which is implicit in the 1935 paper of William R. Thompson[116].

Let $\hat{x}$ denote the "reduced" mean of the $n-1$ real numbers $x_1, ..., x_n$ excluding x_j, cf. (2.7), and let $\bar{x}$ and s denote the mean and standard deviation, respectively, of all n observations, cf. (1.1) and (1.2). Then

$$\bar{x} - \hat{x} = \frac{1}{n}(x_j - \hat{x}) = \frac{1}{n-1}(x_j - \bar{x}) \tag{2.8}$$

and so

$$
\begin{aligned}
ns^2 &= \sum_{i=1}^{n}(x_i - \hat{x} + \hat{x} - \bar{x})^2 \\
&= \sum_{i \neq j}(x_i - \hat{x})^2 + (x_j - \hat{x})^2 - n(\hat{x} - \bar{x})^2 \\
&= \sum_{i \neq j}(x_i - \hat{x})^2 + n(n-1)(\hat{x} - \bar{x})^2 \\
&= \sum_{i \neq j}(x_i - \hat{x})^2 + \frac{n}{n-1}(x_j - \bar{x})^2 \qquad (2.9) \\
&\geq \frac{n}{n-1}(x_j - \bar{x})^2, \qquad (2.10)
\end{aligned}
$$

using (2.8). The inequality (1.4) follows at once.

This proof also shows that equality holds in (1.4) if and only if equality holds in (2.10) and this is so if and only if $x_i = \hat{x}$ for all $i \neq j$. Hence equality holds in (1.4) if and only if all the x_i other than x_j are equal. □

Thompson [116] obtains (2.9) explicitly—cf. his (6) on p. 215—but apparently does not obtain the inequality (2.10). Thompson's interest focused on the distribution of the "Studentized deviations" $(x_j - \bar{x})/s$ when the "'observations" $x_1, ..., x_n$ are independently and identically distributed as a normal random variable with unknown mean and variance.

2.3. Nair (1947, 1948), Kempthorne (1973), Arnold & Balakrishnan (1989). We consider the $n \times n$ orthogonal matrix $\mathbf{E} =$

$$
\begin{pmatrix}
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} \\
\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \frac{-(n-2)}{\sqrt{(n-1)(n-2)}} & 0 \\
\frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \cdots & \frac{1}{\sqrt{n(n-1)}} & \frac{-(n-1)}{\sqrt{n(n-1)}}
\end{pmatrix},
$$

the so-called Helmert matrix[14] and let $\mathbf{x} = \{x_i\}$ and $\mathbf{y} = \mathbf{E}\mathbf{x} = \{y_i\}$. Then

$$\sum_{i=1}^{n} x_i^2 = \mathbf{x}'\mathbf{x} = \mathbf{x}'\mathbf{E}'\mathbf{E}\mathbf{x} = \mathbf{y}'\mathbf{y} = \sum_{i=1}^{n} y_i^2 \geq y_1^2 + y_n^2. \tag{2.11}$$

Since

$$y_1 = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \qquad \text{and} \qquad y_n = \sqrt{\frac{n}{n-1}}(\bar{x} - x_n) \tag{2.12}$$

it follows at once from (2.11) that

$$\sum_{i=1}^{n} x_i^2 \geq \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2 + \frac{n}{n-1}(\bar{x} - x_n)^2. \tag{2.13}$$

If we rearrange the components of the vector $\mathbf{x}$ so that x_j is in the n-th position then, with x_n replaced by x_j, (2.13) becomes (1.4).

Equality holds in (2.13) if and only if equality holds in (2.11) and this is so if and only if $y_2 = \cdots = y_{n-1} = 0$, i.e., all the x_i are equal except for x_n (which we now choose to be x_j). □

This is the third proof given by Arnold and Balakrishnan [4], p. 45, and follows that given by K. R. Nair in "a small section of the third part" of his 1947 Ph.D. thesis [77] and published in 1948 [78], and by Oscar Kempthorne in a 1973 "Personal communication" [46] to Barry C. Arnold[15].

2.4. Arnold (1974), Dwass (1975), Arnold & Balakrishnan (1989). Barry C. Arnold [3] and Meyer Dwass [30] proved (1.4) using the Cauchy-Schwarz inequality:

$$(\mathbf{a}'\mathbf{b})^2 \leq \mathbf{a}'\mathbf{a} \cdot \mathbf{b}'\mathbf{b} \tag{2.14}$$

for any $n \times 1$ real vectors $\mathbf{a}$ and $\mathbf{b}$. This is the second proof given by Arnold and Balakrishnan [4], p.45.

[14] Named after Friedrich Robert Helmert (1843–1919) for the matrix he introduced in 1876 [41], cf. also Harville [39], pp. 85–86, Lancaster [50], Read [102], and Stuart and Ord [111], Example 11.3.

[15] Cf. Arnold and Balakrishnan [4], pp. 45 & 158, and Arnold [3] where, in an acknowledgement, it is observed that: "Upon seeing an earlier draft of this note, Oscar Kempthorne supplied me with three of several alternative proofs that he derived for Samuelson's inequality".

Since $\sum_{i=1}^{n}(x_i - \bar{x}) = 0$, it follows that

$$x_j - \bar{x} = -\sum_{i \neq j}(x_i - \bar{x}) \tag{2.15}$$

and so

$$\begin{aligned} (x_j - \bar{x})^2 &= \left(\textstyle\sum_{i \neq j}(x_i - \bar{x})\right)^2 \\ &\leq (n-1) \sum_{i \neq j}(x_i - \bar{x})^2 \\ &= (n-1) \textstyle\sum_{i=1}^{n}(x_i - \bar{x})^2 - (n-1)(x_j - \bar{x})^2 \end{aligned}$$

from (2.14) with the vectors $\mathbf{a} = \{x_i - \bar{x}\}_{i \neq j}$ and $\mathbf{b} = (1, 1, \ldots, 1)'$ both $(n-1) \times 1$. Hence

$$(x_j - \bar{x})^2 \leq \frac{n-1}{n} \sum_{i=1}^{n}(x_i - \bar{x})^2 = (n-1)s^2$$

from (2.14), and so (1.4) follows immediately. Equality holds if and only if the vectors $\mathbf{a}$ and $\mathbf{b}$ are proportional, i.e., all the x_i except for x_j are equal. □

2.5. Arnold (1974), O'Reilly (1975, 1976), Arnold & Balakrishnan (1989), Murty (1990). Barry C. Arnold (1974) gave a second proof in [3] which used the "hat" matrix from linear regression analysis; see also O'Reilly [90], [91], and Murty [76].

In the usual full-rank Gauss-Markov linear statistical model

$$\mathsf{E}\mathbf{y} = \mathbf{X}\boldsymbol{\beta}, \tag{2.16}$$

where E denotes (mathematical) expectation and the "model" or "design" matrix $\mathbf{X}$ is $n \times p$ with rank $p < n$. Then it is well known that the $n \times n$ "hat matrix"

$$\mathbf{H} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' \tag{2.17}$$

is symmetric and idempotent, and hence nonnegative definite, as is the residual matrix $\mathbf{M} = \mathbf{I} - \mathbf{H}$.

We now let $p = 2$ and $\mathbf{X} = (\mathbf{e} : \mathbf{Cx})$ as in (centered) simple linear regression; here the $n \times 1$ sum vector

$$\mathbf{e} = (1, 1, \ldots, 1)', \tag{2.18}$$

while the $n \times n$ centering matrix

$$\mathbf{C} = \mathbf{I}_n - \frac{1}{n}\mathbf{ee}' \tag{2.19}$$

is symmetric and idempotent. Hence

$$\mathbf{H} = \frac{1}{n}\mathbf{ee}' + \frac{1}{\mathbf{x}'\mathbf{Cx}}\mathbf{Cxx}'\mathbf{C} = \frac{1}{n}\mathbf{ee}' + \frac{n}{s^2}\mathbf{Cxx}'\mathbf{C} \tag{2.20}$$

and so the j-th diagonal element of $\mathbf{M} = \mathbf{I} - \mathbf{H}$:

$$m_{jj} = 1 - \frac{1}{n} - \frac{(x_j - \bar{x})^2}{ns^2} \geq 0, \tag{2.21}$$

since $\mathbf{M}$ is nonnegative definite; the Laguerre-Samuelson Inequality (1.4) follow at once.

Equality holds in (1.4) if and only if equality holds throughout (2.21) and this is so if and only if all the elements in the j-th row (and column) of $\mathbf{M}$ are zero, i.e., all the x_i except for x_j are equal. □

The proof given by O'Reilly [90], [91], is similar but uses the model matrix $\mathbf{X} = (\mathbf{e} : \mathbf{x})$ as in uncentered simple linear regession. This O'Reilly proof is the fifth proof of the Laguerre-Samuelson Inequality given by Arnold and Balakrishnan [4], p. 46, while the Arnold-Murty proof is their fourth.

2.6. Wolkowicz & Styan (1979, 1980). The proof given by Henry Wolkowicz and George P. H. Styan (1979, 1980) [121], [123], cf. also Bancroft [10], Chaganty [24], Chaganty and Vaish [25], [26], Neudecker and Liu [84], Puntanen [99] (Example 6.16, pp. 275–276), and Trenkler [117], essentially uses the following result (Lemma 2.1 in [123], p. 475):

Lemma 2.6.1. *Let* $\mathbf{w}$ *and* $\mathbf{x}$ *be real nonnull* $n \times 1$ *vectors and let* $\bar{x}$ *and* s *be defined as in* (1.1) *and* (1.2) *above, so that* $\bar{x} = \mathbf{x}'\mathbf{e}/n$ *and* $s^2 = \mathbf{x}'\mathbf{Cx}/n$*, where the centering matrix* $\mathbf{C} = \mathbf{I} - \mathbf{ee}'/n$ *as in* (2.19)*, with* $\mathbf{e}$ *the* $n \times 1$ *vector of ones. Then*

$$-s\sqrt{n\,\mathbf{w}'\mathbf{Cw}} \leq \mathbf{w}'\mathbf{Cx} \leq s\sqrt{n\,\mathbf{w}'\mathbf{Cw}}\,. \tag{2.22}$$

Equality holds on the left (right) of (2.22) *if and only if*

$$\mathbf{x} = c\mathbf{w} + d\mathbf{e} \tag{2.23}$$

for some scalars c *and* d *with* $c < 0$ $(c > 0)$.

Proof. The inequality string (2.22) follows at once from the Cauchy-Schwarz Inequality (2.14) with $\mathbf{a} = \mathbf{Cw}$ and $\mathbf{b} = \mathbf{Cx}$. □

If in (2.22) we now substitute

$$\mathbf{w} = \mathbf{e}_j - \mathbf{e}/n = \mathbf{h}_j, \tag{2.24}$$

say, where

$$\mathbf{e}_j = (0, \dots, 0, 1, 0, \dots, 0)' \tag{2.25}$$

with 1 in the j-th position, then (2.22) becomes (1.3). The equality condition $\mathbf{x} = c\mathbf{w} + d\mathbf{e} = c\mathbf{e}_j + d\mathbf{e}$ shows that equality holds in (1.4) if and only if all the x_i are equal except for x_j. □

2.7. Smith (1980), Arnold & Balakrishnan (1989). Arnold and Balakrishnan [4], p. 46, give the following proof credited to William P. Smith [109], as their sixth (and last) proof of the Laguerre-Samuelson Inequality. This proof is based on the Cantelli Inequality[16], cf. e.g., Patel, Kapadia and Owen [92], p. 51.

Let X denote a random variable with mean 0 and variance 1. Then

$$\text{Prob}(X \le u) \le \frac{1}{1+u^2} \qquad \text{if } u \le 0 \tag{2.26}$$

$$\text{Prob}(X \ge u) \le \frac{1}{1+u^2} \qquad \text{if } u \ge 0. \tag{2.27}$$

We now suppose that X is a discrete uniform random variable with

$$\text{Prob}\left(X = \frac{x_i - \bar{x}}{s}\right) = \frac{1}{n} \qquad \text{for all } i = 1, \dots, n. \tag{2.28}$$

Then X has expectation $\mathsf{E}X = 0$ and variance $\mathsf{var}X = 1$.

[16]Named after Francesco Paolo Cantelli (1875–1966); for a biographical account see Benzi [15].

If we substitute $u = (x_{\min} - \bar{x})/s < 0$ in (2.26) then it becomes

$$\frac{1}{n} \leq 1 \Bigg/ \left\{ 1 + \left(\frac{x_{\min} - \bar{x}}{s} \right)^2 \right\}$$

and so

$$\left(\frac{x_{\min} - \bar{x}}{s} \right)^2 \leq n - 1. \tag{2.29}$$

Substituting $u = (x_{\max} - \bar{x})/s > 0$ in (2.27) gives

$$\frac{1}{n} \leq 1 \Bigg/ \left\{ 1 + \left(\frac{x_{\max} - \bar{x}}{s} \right)^2 \right\}$$

and so

$$\left(\frac{x_{\max} - \bar{x}}{s} \right)^2 \leq n - 1. \tag{2.30}$$

Combining (2.29) and (2.30) yields the Laguerre-Samuelson Inequality (1.4).
□

2.8. Olkin (1992). Ingram Olkin, in his 1992 survey paper [89], used the following result:

(2.31)

$$c(x_j - \bar{x})^2 \leq \sum_{i=1}^{n} (x_i - \bar{x})^2 \text{ for all } j = 1, \ldots, n \quad \Longleftrightarrow \quad 0 \leq c \leq \frac{n}{n-1}.$$

To prove (2.31) we express both sides of its right-hand side as quadratic forms. Let $\mathbf{x} = (x_1, \ldots, x_n)'$, $\mathbf{e} = (1, \ldots, 1)'$ and where, cf. (2.25), $\mathbf{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0)'$ with 1 in the j-th position—all $n \times 1$. We may write

$$x_j - \bar{x} = \mathbf{x}'\mathbf{h}_j \quad \text{with} \quad \mathbf{h}_j = \mathbf{e}_j - \frac{1}{n}\mathbf{e}, \tag{2.32}$$

cf. (2.24) above, and so the right-hand side of (2.31) becomes

$$c(x_j - \bar{x})^2 = c\mathbf{x}'\mathbf{h}_j\mathbf{h}_j{}'\mathbf{x} \leq \mathbf{x}'\mathbf{C}\mathbf{x} = \sum_{i=1}^{n} (x_i - \bar{x})^2, \tag{2.33}$$

where the centering matrix $\mathbf{C}$ is defined as in (2.19). Then (2.33) holds if and only if

$$\mathbf{C} - c\mathbf{h}_j\mathbf{h}_j{}' = \mathbf{I}_n - \frac{1}{n}\mathbf{e}\mathbf{e}' - c\mathbf{h}_j\mathbf{h}_j{}' = \mathbf{I}_n - \mathbf{A}\mathbf{A}' \tag{2.34}$$

is nonnegative definite; here $\mathbf{A} = (\mathbf{e}/\sqrt{n} : \sqrt{c}\,\mathbf{h}_j)$. Since the nonzero eigenvalues of the matrices $\mathbf{A}\mathbf{A}'$ and $\mathbf{A}'\mathbf{A}$ coincide, it follows at once that $\mathbf{C} - c\mathbf{h}_j\mathbf{h}_j{}'$ is nonnegative definite whenever

$$\mathbf{I}_2 - \mathbf{A}'\mathbf{A} = \mathbf{I}_2 - \begin{pmatrix} \mathbf{e}'/\sqrt{n} \\ \sqrt{c}\,\mathbf{h}_j{}' \end{pmatrix} (\mathbf{e}/\sqrt{n} : \sqrt{c}\,\mathbf{h}_j) = \begin{pmatrix} 0 & 0 \\ 0 & 1 - c(n-1)/n \end{pmatrix}$$

is nonnegative definite. The result (2.31) follows at once.

Substituting $c = n/(n-1)$ in the right-hand side of (2.31) gives the Laguerre-Samuelson Inequality (1.4). □

Some discussion of this proof is given in [10], [24], [25], [84], and [117]—see §4.2 below for additional commentary.

3. Proofs of Some Inequalities Closely Related to the Laguerre-Samuelson Inequality

3.1. The Brunk Inequalities. Let us arrange the n real numbers $x_1, \ldots, x_n$ in nondecreasing order as in (1.12):

$$x_{\min} = x_{(n)} \leq x_{(n-1)} \leq \cdots \leq x_{(2)} \leq x_{(1)} = x_{\max} \tag{3.1}$$

so that $x_{(j)}$ is the j-th largest. Then:

$$\bar{x} + \frac{s}{\sqrt{n-1}} \leq x_{\max} = x_{(1)} \leq \bar{x} + s\sqrt{n-1} \tag{3.2}$$

and

$$\bar{x} - s\sqrt{n-1} \leq x_{\min} = x_{(n)} \leq \bar{x} - \frac{s}{\sqrt{n-1}}. \tag{3.3}$$

The right-hand inequality in (3.2) and the left-hand inequality in (3.3) are the Laguerre-Samuelson inequality (1.3). As announced in §1.3 above, we will refer to the left-hand inequality in (3.2) and the right-hand inequality in (3.3) as the "Brunk inequalities" since we believe that they were established for the first time in 1959 by H. D. Brunk [22].

Equality holds on the left of (3.2) if and only if equality holds on the left of (3.3) if and only if:

$$x_{(1)} = \cdots = x_{(n-1)} = \bar{x} + \frac{s}{\sqrt{n-1}} \quad \text{and} \quad x_{(n)} = \bar{x} - s\sqrt{n-1}\,; \tag{3.4}$$

equality holds on the right of (3.2) if and only if equality holds on the right of (3.3) if and only if:

$$x_{(1)} = \bar{x} + s\sqrt{n-1} \quad \text{and} \quad x_{(2)} = \cdots = x_{(n)} = \bar{x} - \frac{s}{\sqrt{n-1}}\,. \tag{3.5}$$

3.1.1. Brunk (1959). To prove the "Brunk inequalities" Brunk used the following result ([22], Corollary 1), which we find to be interesting in its own right:

Lemma 3.1.1. *Let the random variable Z be distributed over the closed interval $[0,1]$ and let p be a nonnegative constant so that $p \le \text{Prob}(Z=1)$. Then*

$$p\,\mathsf{E}Z^2 \le (\mathsf{E}Z)^2, \tag{3.6}$$

with equality if and only if

$$\text{Prob}(Z=0) = 1-p \quad \text{and} \quad \text{Prob}(Z=1) = p. \tag{3.7}$$

Proof. Since $0 \le Z \le 1$ we have $Z^2 \le Z$ with probability one and so $\mathsf{E}Z^2 \le \mathsf{E}Z$ and $\text{Prob}(Z=1) \le \mathsf{E}Z$. Combining these two inequalities yields

$$p\,\mathsf{E}Z^2 \le \text{Prob}(Z=1)\cdot \mathsf{E}Z \le (\mathsf{E}Z)^2, \tag{3.8}$$

and (3.6) is established. Equality holds in (3.6) if and only if equality holds throughout (3.8) if and only if $p = \text{Prob}(Z=1)$ and $Z = Z^2$ with probability one, and so the equality condition (3.7) follows at once. □

To prove the "Brunk inequalities" we now let the random variable X assume each of the n values in (3.1) with probability $1/n$. Then the random variable $Z = (x_{\max} - X)/r$, where the range $r = x_{\max} - x_{\min}$, is distributed over $[0,1]$. The expectation $\mathsf{E}X = \bar{x}$ and the variance $\mathsf{var}X = s^2$. Hence

$$\mathsf{E}Z^2 = \mathsf{var}Z + (\mathsf{E}Z)^2 = \frac{s^2 + (x_{\max} - \bar{x})^2}{r^2} \tag{3.9}$$

and so from Lemma 3.1.1:

$$\frac{1}{n}\mathsf{E}Z^2 = \frac{s^2 + (x_{\max} - \bar{x})^2}{nr^2} \le \frac{(x_{\max} - \bar{x})^2}{r^2}, \tag{3.10}$$

which simplifies to

$$s^2 \le (n-1)(x_{\max} - \bar{x})^2, \tag{3.11}$$

from which the left-hand inequality in (3.2) follows at once. Equality holds in (3.10) if and only if (3.7) holds and here this becomes (3.4).

To establish the right-hand inequality in (3.3) we repeat the above argument with $Z = (X - x_{\min})/r$. □

3.1.2. Wolkowicz and Styan (1979). Wolkowicz and Styan [121] provided a completely algebraic (non-statistical) proof of the Brunk inequalities. Since $n(x_{\max} - \bar{x}) = \sum_{i=1}^{n}(x_{\max} - x_i)$ it follows that

$$\begin{aligned} n^2(x_{\max} - \bar{x})^2 &= \left\{\sum_{i=1}^{n}(x_{\max} - x_i)\right\}^2 \\ &= \sum_{i=1}^{n}(x_{\max} - x_i)^2 + \sum_{i \neq i'}(x_{\max} - x_i)(x_{\max} - x_{i'}) \\ &\ge \sum_{i=1}^{n}(x_{\max} - x_i)^2 \\ &= \sum_{i=1}^{n}(x_{\max} - \bar{x} + \bar{x} - x_i)^2 = n\{(x_{\max} - \bar{x})^2 + s^2\}, \end{aligned}$$

from which the left-hand inequality in (3.2) follows at once, with equality if and only if $x_{\max} = x_{(1)} = \ldots = x_{(n-1)}$ or (3.4) holds.

If $n^2(\bar{x} - x_{\min})^2$ is expanded similarily, then the right-hand inequality in (3.3) follows at once, with equality if and only if $x_{(2)} = \cdots = x_{(n)} = x_{\min}$ or (3.5) holds. □

3.2. The Boyd-Hawkins Inequalities. As observed above in §1.3 the k-th largest observation or "order statistic" $x_{(k)}$ satisfies the following "'Boyd-Hawkins inequalities":

$$\bar{x} - s\sqrt{\tfrac{k-1}{n-k+1}} \le x_{(k)} \le \bar{x} + s\sqrt{\tfrac{n-k}{k}} \qquad \text{for } k = 2, \ldots, n-1; \tag{3.12}$$

equality holds on the left of (3.12) if and only if

$$x_{(1)} = \cdots = x_{(k-1)} = \bar{x} + s\sqrt{\tfrac{n-k+1}{k-1}} \quad \text{and} \quad x_{(k)} = \cdots = x_{(n)} = \bar{x} - s\sqrt{\tfrac{k-1}{n-k+1}}$$

and on the right of (3.12) if and only if

$$x_{(1)} = \cdots = x_{(k)} = \bar{x} + s\sqrt{\tfrac{n-k}{k}} \quad \text{and} \quad x_{(k+1)} = \cdots = x_{(n)} = \bar{x} - s\sqrt{\tfrac{k}{n-k}}.$$

3.2.1. Wolkowicz & Styan (1979). Possibly the simplest proof of (3.12) is that presented in 1979 by Wolkowicz and Styan [121]. We use our Lemma 2.6.1 above, a version of the Cauchy-Schwarz inequality given by Wolkowicz and Styan (Lemma 2.1 in [123], p. 475):

$$-s\sqrt{n\,\mathbf{w}'\mathbf{C}\mathbf{w}} \leq \mathbf{w}'\mathbf{C}\mathbf{x} \leq s\sqrt{n\,\mathbf{w}'\mathbf{C}\mathbf{w}}\,, \tag{3.13}$$

where $\mathbf{w}$ and $\mathbf{x}$ are real nonnull $n \times 1$ vectors and the centering matrix $\mathbf{C} = \mathbf{I} - \mathbf{e}\mathbf{e}'/n$ as in (2.19), with $\mathbf{e}$ the $n \times 1$ vector of ones. Equality holds on the left (right) of (2.22) if and only if

$$\mathbf{x} = c\mathbf{w} + d\mathbf{e} \tag{3.14}$$

for some scalars c and d with $c < 0$ ($c > 0$).

Now let $\mathbf{w} = \sum_{i=k}^{l} \mathbf{e}_i/(l-k+1)$ and $\mathbf{x} = \{x_{(i)}\}$, where $\mathbf{e}_i$ is defined as in (2.25) above and

$$x_{\min} = x_{(n)} \leq x_{(n-1)} \leq \cdots \leq x_{(2)} \leq x_{(1)} = x_{\max}\,. \tag{3.15}$$

Then $\mathbf{w}'\mathbf{C}\mathbf{x} = \bar{x}_{(k,l)} - \bar{x}$, where the "subsample mean"

$$\bar{x}_{(k,l)} = \sum_{i=k}^{l} x_{(i)}/(l-k+1) \qquad \text{for} \quad 1 \leq l \leq k \leq n. \tag{3.16}$$

Moreover, $\mathbf{w}'\mathbf{C}\mathbf{w} = (l-k+1)^{-1} - n^{-1}$. Hence (3.13) implies

$$\bar{x} - s\sqrt{\tfrac{k-1}{n-k+1}} \leq \bar{x}_{(k,n)} \leq \bar{x}_{(k,l)} \leq \bar{x}_{(1,l)} \leq \bar{x} + s\sqrt{\tfrac{n-l}{l}} \tag{3.17}$$

which, when $l = k$, reduces to

$$\bar{x} - s\sqrt{\tfrac{k-1}{n-k+1}} \leq x_{(k)} \leq \bar{x} + s\sqrt{\tfrac{n-k}{k}} \tag{3.18}$$

as in (3.12). From (3.14) we note that equality holds in (3.12) if and only if $\mathbf{x} = c\mathbf{w} + d\mathbf{e}$ for some scalars c and d. The equality conditions for (3.12) follow at once. □

4. Some Matrix-theoretic Extensions Related to the Cauchy-Schwarz and Laguerre-Samuelson Inequalities

4.1. Bounds for Eigenvalues. When the real $n \times n$ matrix $\mathbf{A}$ has all its eigenvalues real, e.g., when $\mathbf{A}$ is symmetric, then the Laguerre-Samuelson, Brunk and Boyd-Hawkins inequalities provide bounds for the eigenvalues of $\mathbf{A}$ as observed by Wolkowicz and Styan [123], [124]; see also, e.g., Merikoski [63], Merikoski, Styan and Wolkowicz [64], Merikoski and Virtanen [65], Merikoski and Wolkowicz [66], and Tarazaga [115].

As Mirsky [69] and Brauer and Mewborn [19] pointed out, the mean and variance of the eigenvalues λ_i may be expressed in terms of the trace of $\mathbf{A}$ and the trace of $\mathbf{A}^2$:

$$m = \frac{1}{n}\sum_{i=1}^{n}\lambda_i = \frac{1}{n}\mathrm{tr}\mathbf{A} \tag{4.1}$$

and

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}\lambda_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n}\lambda_i\right)^2 = \frac{1}{n}\mathrm{tr}\mathbf{A}^2 - \left(\frac{1}{n}\mathrm{tr}\mathbf{A}\right)^2. \tag{4.2}$$

Then from (1.13) and (1.14) we obtain:

$$m + \frac{s}{\sqrt{n-1}} \leq \lambda_{\max} = \lambda_1 \leq m + s\sqrt{n-1} \tag{4.3}$$

and

$$m - s\sqrt{n-1} \leq \lambda_{\min} = \lambda_n \leq m - \frac{s}{\sqrt{n-1}}, \tag{4.4}$$

while from (1.15):

$$m - s\sqrt{\tfrac{k-1}{n-k+1}} \leq \lambda_k \leq m + s\sqrt{\tfrac{n-k}{k}} \qquad \text{for } k = 2, \ldots, n-1, \tag{4.5}$$

where λ_k is the k-th largest eigenvalue of $\mathbf{A}, k = 2, \ldots, n-1$.

4.2. Some Matrix Inequalities Related to the Cauchy-Schwarz and Laguerre-Samuelson Inequalities. Two of our eight proofs of the Laguerre-Samuelson inequality were based explicitly on the Cauchy-Schwarz inequality which, as we noted at the end of Section 1, "may be perceived to be lurking in the background of many of the proofs"[17] of the Laguerre-Samuelson inequality, cf. §2.4, §2.6, and Lemma 2.6.1. Moreover, the discussion in [10], [24], [25], [84], and [117] of the proof given in §2.8 is all centered around the Cauchy-Schwarz inequality.

In their 1996 paper Pečarić, Puntanen and Styan [96] presented the following matrix-theoretic extension of the Cauchy-Schwarz inequality; here a g-inverse (generalized inverse) $\mathbf{X}^-$ is any matrix $\mathbf{X}^-$ such that $\mathbf{X}\mathbf{X}^-\mathbf{X} = \mathbf{X}$.

Theorem 4.1. *Let* $\mathbf{A}$ *be an* $n \times n$ *symmetric and nonnegative definite matrix with* $\mathbf{A}^{\{p\}}$ *defined as*

$$\begin{aligned} \mathbf{A}^{\{p\}} &= \mathbf{A}^p; \quad p = 1, 2, \ldots, \\ &= \mathbf{P}_{\mathbf{A}} = \mathbf{A}(\mathbf{A}'\mathbf{A})^- \mathbf{A}'; \qquad p = 0, \\ &= (\mathbf{A}^+)^{|p|}; \qquad p = -1, -2, \ldots, \end{aligned}$$

where $\mathbf{A}^+$ *is the (unique) Moore-Penrose inverse of* $\mathbf{A}$*, and* $\mathbf{P}_{\mathbf{A}}$ *denotes the orthogonal projector onto the column space* $\mathcal{C}(\mathbf{A})$ *of* $\mathbf{A}$*. Let* $\mathbf{t}$ *and* $\mathbf{u}$ *be* $n \times 1$ *vectors, and let* h *and* k *be integers. Then*

$$(\mathbf{t}'\mathbf{A}^{\{(h+k)/2\}}\mathbf{u})^2 \leq \mathbf{t}'\mathbf{A}^{\{h\}}\mathbf{t} \cdot \mathbf{u}'\mathbf{A}^{\{k\}}\mathbf{u} \tag{4.6}$$

for $h, k = \ldots, -1, 0, 1, 2, \ldots,$ *with equality if and only if*

$$\mathbf{A}\mathbf{t} \propto \mathbf{A}^{\{1+(k-h)/2\}}\mathbf{u}. \tag{4.7}$$

Several extensions of the Theorem 5.1.1 and some statistical applications are also given in Pečarić, Puntanen and Styan [96].

When $h = 1$ and $k = -1$, then the inequality (4.6) becomes

$$(\mathbf{t}'\mathbf{P}_{\mathbf{A}}\mathbf{u})^2 \leq \mathbf{t}'\mathbf{A}\mathbf{t} \cdot \mathbf{u}'\mathbf{A}^+\mathbf{u}, \tag{4.8}$$

cf. Bancroft [10].

Equality holds in (4.8) if and only if

$$\mathbf{A}\mathbf{t} \propto \mathbf{P}_{\mathbf{A}}\mathbf{u}. \tag{4.9}$$

When $\mathbf{t} = \mathbf{w}$, $\mathbf{u} = \mathbf{x}$ and $\mathbf{A} = \mathbf{C}$, the centering matrix $\mathbf{I}_n - n^{-1}\mathbf{e}\mathbf{e}'$ as in (2.19), then $\mathbf{A}^+ = \mathbf{P}_{\mathbf{A}} = \mathbf{C}$ and (4.8) becomes

$$(\mathbf{w}'\mathbf{C}\mathbf{x})^2 \leq \mathbf{w}'\mathbf{C}\mathbf{w} \cdot \mathbf{x}'\mathbf{C}\mathbf{x}, \tag{4.10}$$

[17] Arnold and Balakrishnan [4], p. 45.

which is equivalent to (2.22) in Lemma 2.6.1, and the equality condition (4.9) becomes

$$\mathbf{Cw} \propto \mathbf{Cx}, \tag{4.11}$$

which is equivalent to (2.23) in Lemma 2.6.1[18].

We may also express (4.8) as

$$(\mathbf{t}'\mathbf{u})^2 \leq \mathbf{t}'\mathbf{At} \cdot \mathbf{u}'\mathbf{A}^-\mathbf{u} \qquad \text{for all} \quad \mathbf{u} \in \mathcal{C}(\mathbf{A}) \tag{4.12}$$

and for any, and hence for every g-inverse $\mathbf{A}^-$, cf. Neudecker and Liu [84]. The quadratic form $\mathbf{u}'\mathbf{A}^-\mathbf{u}$ in (4.12) is invariant with respect to the choice of g-inverse $\mathbf{A}^-$ when $\mathbf{u} \in \mathcal{C}(\mathbf{A})$, since then $\mathbf{u} = \mathbf{Av}$ for some $\mathbf{v}$ and so $\mathbf{u}'\mathbf{A}^-\mathbf{u} = \mathbf{v}'\mathbf{AA}^-\mathbf{Av} = \mathbf{v}'\mathbf{Av} = \mathbf{v}'\mathbf{AA}^{\sim}\mathbf{Av}$ for any g-inverse $\mathbf{A}^{\sim}$. Equality holds in (4.12) if and only if

$$\mathbf{At} \propto \mathbf{u}. \tag{4.13}$$

Chaganty [24] presents (4.12) with the Moore-Penrose inverse $\mathbf{A}^+$ instead of a g-inverse $\mathbf{A}^-$ and observes that equality holds in (4.12) when $\mathbf{t} = \mathbf{A}^+\mathbf{u}$ which, since $\mathbf{u} \in \mathcal{C}(\mathbf{A})$, implies $\mathbf{At} = \mathbf{u}$, cf. (4.13).

Trenkler [117] observes that Baksalary and Kala [9] showed that

$$(\mathbf{t}'\mathbf{u})^2 \leq \alpha\mathbf{t}'\mathbf{At} \qquad \text{for all} \quad \mathbf{u} \in \mathcal{C}(\mathbf{A}) \tag{4.14}$$

provided that then $\mathbf{u}'\mathbf{A}^-\mathbf{u} \leq \alpha$ for any, and hence for every g-inverse $\mathbf{A}^-$.

If we now let $\mathbf{u} = \mathbf{t} \in \mathcal{C}(\mathbf{A})$, then (4.12) becomes

$$(\mathbf{t}'\mathbf{t})^2 \leq \mathbf{t}'\mathbf{At} \cdot \mathbf{t}'\mathbf{A}^-\mathbf{t} \qquad \text{for all} \quad \mathbf{t} \in \mathcal{C}(\mathbf{A}) \tag{4.15}$$

for any, and hence for every g-inverse $\mathbf{A}^-$; when $\mathbf{t} \neq \mathbf{0}$ then equality holds in (4.15) if and only if $\mathbf{t}$ is an eigenvector of $\mathbf{A}$, cf. Lemma 2.1 of Dey and Gupta [29].

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[18] Since $\mathbf{Cw} = k\mathbf{Cx}$ is equivalent to $\mathbf{x} = (1/k)\mathbf{w} + (k\bar{x} - \bar{w})\mathbf{e}$, where $\bar{w} = \mathbf{w}'\mathbf{e}/n$.

and this was followed by an ongoing collaboration with Jorma Kaarlo Merikoski, and with Henry Wolkowicz. Our thanks go also to Josip E. Pečarić for drawing our attention to several references, and to Rajendra Bhatia, Chandler Davis, S. W. Drury, Simo Puntanen, Hans Joachim Werner, and Keith J. Worsley for helpful discussions. Much of the biographical information was obtained by visiting the excellent O'Connor-Robertson Internet website [88], while web access to the databases MathSciNet (for *Mathematical Reviews*) and MATH Database (for *Zentralblatt für Mathematik*) has been of great help in compiling our bibliography. This research was supported in part by a research grant from the Natural Sciences and Engineering Research Council of Canada (to the second author).

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