Volume in sphere: inverse function

Volume, V, of a liquid in a spherical reservoir as a function of the height of liquid, h.

Volume, V, as a function of h

$$V = \frac{1}{3}\pi h^2 (3R - h)$$

For $h = 0$, $V = 0$; for $h = 2R$, $V = \frac{1}{3}\pi (4R^2)(3R - 2R) = \frac{4}{3}\pi R^3$.

Now, h as a function of V

$$V = \frac{\pi}{3} h^2 (3R - h)$$

$$\frac{3}{\pi} V = 3Rh^2 - h^3$$

$$h^3 - (3R)h^2 + \frac{3}{\pi} V = 0$$

Cubic:

$$\boxed{a = 1 \qquad b = -3R \qquad c = 0 \qquad d = \frac{3}{\pi} V}$$

See http://web.tecnico.ulisboa.pt/~mcasquilho/text/utilities/cubic.html

$$p = \frac{c}{3a} - \left(\frac{b}{3a}\right)^2 = 0 - \left(\frac{-3R}{3}\right)^2 = -R^2$$

$$2q = \frac{d}{a} - \frac{bc}{3a^2} + 2\left(\frac{b}{3a}\right)^3 = \frac{3V}{\pi} - 0 + 2\left(\frac{-3R}{3}\right)^3 = \frac{3V}{\pi} - 2R^3$$

$$\boxed{\begin{aligned} p &= -R^2 \\ q &= \frac{3V}{2\pi} - R^3 \end{aligned}}$$

$$D = p^3 + q^2 = -R^6 + \left(\frac{3V}{2\pi} - R^3\right)^2 =$$

$$= -R^6 + \frac{9V^2}{4\pi^2} - \frac{3V}{\pi} R^3 + R^6 = \frac{9V^2}{4\pi^2} - \frac{3V}{\pi} R^3$$

$$\frac{4\pi^2}{9V} D = V - \frac{4\pi^2}{9V} \frac{3V}{\pi} R^3 = V - \frac{4\pi}{3} R^3 \le 0$$

$$\boxed{D \le 0}$$

This is case 7:

$$h_1 = 2\sqrt{|p|} \cos \frac{1}{3} \arccos \frac{-q}{\sqrt{|p|^3}} - \frac{b}{3a} = 2\sqrt{R^2} \cos\left(\frac{1}{3} \arccos \frac{R^3 - \frac{3V}{2\pi}}{R^3}\right) + R$$

$$h_1 = 2R\cos\left(\frac{1}{3}\arccos\left(1-\frac{3V}{2\pi R^3}\right)\right)+R$$

$$h_1 = 2R\cos\left(\frac{1}{3}\arccos\left(1-\frac{2}{2}\frac{3}{2\pi}\frac{V}{R^3}\right)\right)+R =$$

$$= 2R\cos\left(\frac{1}{3}\arccos\left(1-2\frac{V}{(4\pi/3)R^3}\right)\right)+R$$

Let $V_T = \frac{4}{3}\pi R^3$

$$h_1 = 2R\cos\left(\frac{1}{3}\arccos\left(1-2\frac{V}{V_T}\right)\right)+R$$

For $V = 0$, $h_1 = 3R$; and for $V = V_T$, $h_1 = 2R\cos(\pi/3) + R = 2R$ This is not the adequate root. Other roots (verify dimensional homogeneity):

$$B = \frac{1}{2}\left(\frac{b}{a}+h_1\right) = \frac{1}{2}\left(\frac{-3R}{1}+h_1\right) = \frac{1}{2}(h_1-3R)$$

$$C = \left(\frac{b}{a}+h_1\right)h_1 + \frac{c}{a} = (h_1-3R)h_1$$

$$B^2 - C = \frac{1}{4}(h_1-3R)^2 - (h_1-3R)h_1 =$$

$$= (h_1-3R)\left(\frac{1}{4}h_1 - \frac{3}{4}R - h_1\right) = (h_1-3R)\left(-\frac{3}{4}h_1 - \frac{3}{4}R\right)$$

$$\Delta = B^2 - C = \frac{3}{4}(3R-h_1)(R+h_1)$$

$$h_{2,3} = -B \pm \sqrt{\Delta} = \frac{1}{2}(3R-h_1) \pm \sqrt{\frac{3}{4}(3R-h_1)(R+h_1)}$$

$$h_{2,3} = \frac{1}{2}(3R-h_1) \pm \frac{1}{2}\sqrt{3(3R-h_1)(R+h_1)}$$

Negative, for $h_1 = 3R$,

$$h_2 = \frac{1}{2}(3R-3R) - \frac{1}{2}\sqrt{3(3R-3R)(R+3R)} = 0-0 = 0$$

For $h_1 = 2R$,

$$h_3 = \frac{1}{2}(3R-2R) - \frac{1}{2}\sqrt{3(3R-2R)(R+2R)} =$$

$$= \frac{1}{2}R - \frac{1}{2}\sqrt{3\times R\times 3R} = \frac{1}{2}R - \frac{3}{2}R = -R$$

Positive, for $h_1 = 3R$,

$$h_2 = \frac{1}{2}(3R - 3R) + \frac{1}{2}\sqrt{3(3R - 3R)(R + 3R)} =$$
$$= 0 + 0 = 0$$

For $h_1 = 2R$,

$$h_3 = \frac{1}{2}(3R - 2R) + \frac{1}{2}\sqrt{3(3R - 2R)(R + 2R)} =$$
$$= \frac{1}{2}R + \frac{1}{2}\sqrt{3 \times R \times 3R} = \frac{1}{2}R + \frac{3}{2}R = 2R$$

So, h_3 is the solution:

$$\boxed{h_{(3)} = \frac{1}{2}(3R - h_1) + \frac{1}{2}\sqrt{3(3R - h_1)(R + h_1)}}$$

Verify: $h_1 + h_2 + h_3 = -\frac{b}{a} = 3R$

$$h_1 + h_2 + h_3 = h_1 + 2 \times \left[\frac{1}{2}(3R - h_1)\right] = h_1 + 3R - h_1 = 3R$$

Verify: $h_1\ h_2\ h_3 = -\frac{d}{a} = -\frac{3V}{\pi}$

$$h_1 h_2 h_3 = h_1\left\{\left[\frac{1}{2}(3R - h_1)\right]^2 - \left[\frac{1}{2}\sqrt{3(3R - h_1)(R + h_1)}\right]^2\right\}$$

$$h_1 h_2 h_3 = h_1\left[\frac{1}{4}(9R^2 - 6Rh_1 + h_1^2) - \frac{1}{4}3(3R - h_1)(R + h_1)\right] =$$
$$= \frac{1}{4}h_1\left(-6Rh_1 + h_1^2 - 9Rh_1 + 3Rh_1 + 3h_1^2\right) =$$
$$= \frac{1}{4}h_1\left(4h_1^2 - 12Rh_1\right)$$

$$h_1 h_2 h_3 = h_1^2(h_1 - 3R)$$

Correct, by comparison with V in the beginning.

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