Characteristic Polynomial

♣ **Preleminary Results**. Let $A = (a_{ij})$ be an $n \times n$ matrix. If $Au = \lambda u$, then λ and u are called the *eigenvalue* and *eigenvector* of A, respectively. The eigenvalues of A are the roots of the *characteristic polynomial*

$$K_A(\lambda) = \det(\lambda I_n - A).$$

The eigenvectors are the solutions to the *Homogeneous system*

$$(\lambda I_n - A)\, X = \theta.$$

Note that $K_A(\lambda)$ is a *monic polynomial* (i.e., the leading coefficient is one).

Cayley-Hamilton Theorem. *If* $K_A(\lambda) = \lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n$ *is the characteristic polynomial of the* $n \times n$ *matrix* A*, then*

$$K_A(A) = A^n + p_1 A^{n-1} + \cdots + p_{n-1}A + p_n I_n = Z_n,$$

where Z_n *is the* $n \times n$ *zero matrix.*

Corollary. *Let* $K_A(\lambda) = \lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n$ *be the characteristic polynomial of the* $n \times n$ *invertible matrix* A*. Then*

$$A^{-1} = \frac{1}{-p_n}\left[A^{n-1} + p_1A^{n-2} + \cdots + p_{n-2}A + p_{n-1}I_n\right].$$

Proof. According to the Cayley Hamilton's theorem we have

$$A\left[A^{n-1} + p_1A^{n-2} + \cdots + p_{n-1}I_n\right] = -p_nI_n,$$

Since A is nonsingular, $p_n = (-1)^n \det(A) \neq 0$; thus the result follows. ■

Newton's Identity. *Let* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *be the roots of the polynomial*

$$P(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots \ \cdots + c_{n-1}\lambda + c_n.$$

If $s_k = \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k$*, then*

$$c_k = -\frac{1}{k}\left(s_k + s_{k-1}c_1 + s_{k-2}c_2 + \cdots + s_2c_{k-2}c_1 + s_1c_{k-1}\right).$$

Proof. From

$$P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots\ \ldots(\lambda - \lambda_{n-1})(\lambda - \lambda_n)$$

and the use of logarithmic differentiation, we obtain

$$\frac{P'(\lambda)}{P(\lambda)} = \frac{n\lambda^{n-1} + (n-1)c_1\lambda^{n-2} + \cdots + 2c_{n-2}\lambda + c_{n-1}}{\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n} = \sum_{i=1}^{n} \frac{1}{(\lambda - \lambda_i)}\ .$$

By using the geometric series for $\dfrac{1}{(\lambda - \lambda_i)}$ and choosing $|\lambda| > \max_{1\le i\le n} |\lambda_i|$, we obtain

$$\sum_{i=1}^{n} \frac{1}{(\lambda - \lambda_i)} = \frac{n}{\lambda} + \frac{s_1}{\lambda^2} + \frac{s_2}{\lambda^3} + \cdots\cdots$$

Hence

$$n\lambda^{n-1} + (n-1)c_1\lambda^{n-2} + \cdots + c_{n-1} = \left(\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n\right)\left(\frac{n}{\lambda} + \frac{s_1}{\lambda^2} + \frac{s_2}{\lambda^3} + \cdots\right).$$

By equating both sides of the above equality we may obtain the Newton's identities. ■

♣ **The Method of Direct Expansion**. The characteristic polynomial of an $n \times n$ matrix $A = (a_{ij})$ is defined as:

$$K_A(\lambda) = \det(\lambda I_n - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \dots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \dots & \lambda - a_{nn} \end{vmatrix} = \lambda^n - \sigma_1\lambda^{n-1} + \sigma_2\lambda^{n-2} - \dots + (-1)^n\sigma_n,$$

where

$$\sigma_1 = \sum_{i=1}^{n} a_{ii} = trace(A)$$

is the sum of all first-order diagonal minors of A,

$$\sigma_2 = \sum_{i<j} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}$$

is the sum of all second-order diagonal minors of A,

$$\sigma_3 = \sum_{i<j<k} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix}$$

is the sum of all third-order diagonal minors of A, and so forth. Finally,

$$\sigma_n = det(A)$$

There are $\binom{n}{k}$ diagonal minors of order k in A. From this we find that the direct computation of the coefficients of the characteristic polynomial of an $n \times n$ matrix is equivalent to computing

$$\binom{n}{1} + \binom{n}{2} + \cdots \binom{n}{n} = 2^n - 1$$

determinants of various orders, which, generally speaking, is a major task. This has given rise to special methods for expanding characteristic polynomial. We shall explain some of these methods.

Example. *Compute the characteristic polynomial of* $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -4 \\ 1 & 0 & 2 \end{pmatrix}$.

We have:

$$\sigma_1 = 1 + 1 + 2 = 4, \quad \sigma_2 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = (-3) + (2) + (-1) = -2,$$

and

$$\sigma_3 = \det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & -4 \\ 1 & 0 & 2 \end{vmatrix} = -17.$$

Thus

$$K_A(\lambda) = \det(\lambda I_3 - A) = \lambda^3 - \sigma_1\lambda^2 + \sigma_2\lambda - \sigma_3 = \lambda^3 - 4\lambda - 2\lambda + 17.$$

♣ **Leverrier's Algorithm**. This method allows us to find the characteristic polynomial of any $n \times n$ matrix A using the trace of the matrix A^k, where $k = 1, 2, \cdots n$. Let

$$\sigma(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$$

be the set of all eigenvalues of A which is also called the *spectrum* of A. Note that

$$s_k = trace(A^k) = \sum_{i=1}^{n} \lambda_i^k \ , \ \text{for all } k = 1, 2, \cdots, n.$$

Let

$$K_A(\lambda) = \det(\lambda I_n - A) = \lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n$$

be the characteristic polynomial of the matrix A, then for $k \leq n$, the *Newton's identities* hold true:

$$p_k = -\frac{1}{k}\left[s_k + p_1 s_{k-1} + \cdots + p_{k-1}s_1\right] \quad (k = 1, 2, \cdots, n)$$

Example. *Let* $A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 1 & 0 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ 4 & -5 & 0 & 4 \end{pmatrix}$. *Then*

$$A^2 = \begin{pmatrix} 1 & 8 & 4 & 0 \\ 9 & -1 & -1 & 9 \\ 13 & -12 & 5 & 8 \\ 15 & -12 & -6 & 7 \end{pmatrix} \quad A^3 = \begin{pmatrix} 17 & 6 & 13 & 19 \\ 42 & -28 & 8 & 23 \\ 43 & -9 & -16 & 22 \\ 19 & -11 & -3 & -17 \end{pmatrix} \quad A^4 = \begin{pmatrix} 125 & -48 & 16 & 104 \\ 122 & -23 & -22 & 46 \\ 90 & -40 & 41 & -12 \\ -66 & 120 & 0 & -107 \end{pmatrix}.$$

So $s_1 = 4$, $s_2 = 12$, $s_3 = -44$, *and* $s_4 = 36$. *Hence*

$$\begin{cases} p_1 = -s_1 = -4, \\ p_2 = -\frac{1}{2}(s_2 + p_1 s_1) = -\frac{1}{2}(12 + (-4)4) = 2, \\ p_3 = -\frac{1}{3}(s_3 + p_1 s_2 + p_2 s_1) = -\frac{1}{3}(-44 + (-4)12 + 2(4)) = 28, \\ p_4 = -\frac{1}{4}(s_4 + p_1 s_3 + p_2 s_2 + p_3 s_1) = -\frac{1}{4}(36 + (-4)(-44) + 2(12) + 28(4)) = -87. \end{cases}$$

Therefore

$$K_A(\lambda) = \lambda^4 - 4\lambda^3 + 2\lambda^2 + 28\lambda - 87$$

and

$$A^{-1} = \frac{1}{87}\left[A^3 - 4A^2 + 2A + 28I_4\right] =$$

$$\frac{1}{87}\left[\begin{pmatrix} 17 & 6 & 13 & 19 \\ 42 & -28 & 8 & 23 \\ 43 & -9 & -16 & 22 \\ 19 & -11 & -3 & -17 \end{pmatrix} - 4\begin{pmatrix} 1 & 8 & 4 & 0 \\ 9 & -1 & -1 & 9 \\ 13 & -12 & 5 & 8 \\ 15 & -12 & -6 & 7 \end{pmatrix} + 2\begin{pmatrix} 1 & 2 & 1 & -1 \\ 1 & 0 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ 4 & -5 & 0 & 4 \end{pmatrix} + 28\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]$$

$$A^{-1} = \frac{1}{87}\begin{pmatrix} 43 & -22 & -1 & 17 \\ 8 & 4 & 16 & -11 \\ -5 & 41 & -10 & -4 \\ -33 & 27 & 21 & -9 \end{pmatrix}. \qquad \blacksquare$$

♣ **The Method of Souriau (or Fadeev and Frame)**. This is an elegant modification of the Leverrier's method.

Let A be an $n \times n$ matrix, then define

$$\begin{array}{lll} A_1 = A, & q_1 = -trace(A_1), & B_1 = A_1 + q_1 I_n \\ A_2 = AB_1, & q_2 = -\frac{1}{2} trace(A_2), & B_2 = A_2 + q_2 I_n \\ \vdots \; \vdots \; \vdots & \vdots \; \vdots \; \vdots \; \vdots & \vdots \; \vdots \; \vdots \\ A_n = AB_{n-1}, & q_n = -\frac{1}{n} trace(A_n), & B_n = A_n + q_n I_n \end{array}$$

Theorem. *$B_n = Z_n$, and*

$$K_A(\lambda) = \det(\lambda I_n - A) = \lambda^n + q_1 \lambda^{n-1} + \cdots + q_{n-1}\lambda + q_n \,.$$

If A is nonsingular, then

$$A^{-1} = -\frac{1}{q_n} B_{n-1} \,.$$

Proof. Suppose the characteristic polynomial of A is

$$K_A(\lambda) = \det(\lambda I_n - A) = \lambda^n + p_1 \lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n \,,$$

where $p_k's$ are defined in the Leverrier's method.

Clearly $p_1 = -trace(A) = -trace(A_1) = q_1$, and now suppose that we have proved that

$$q_1 = p_1, q_2 = p_2, \; \ldots \; , q_{k-1} = p_{k-1} \,.$$

Then by the hypothesis we have

$$\begin{aligned} A_k = AB_{k-1} = A(A_{k-1} + q_{k-1} I_n) &= AA_{k-1} + q_{k-1} A \\ &= A[A(A_{k-2} + q_{k-2} I_n)] + q_{k-1} A \\ &= A^2 A_{k-1} + q_{k-2} A^2 + q_{k-1} A \\ &= \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\ &= A^k + q_1 A^{k-1} + \; \cdots \; + q_{k-1} A \,. \end{aligned}$$

Let $s_i = trace(A^i)$ $\big(i = 1, 2, \ldots, k\big)$, then by *Newton's identities*

$$\begin{aligned} -kq_k = trace(A_k) &= trace(A^k) + q_1 \; trace(A^{k-1}) + \cdots + q_{k-1} \; trace(A) \\ &= s_k + q_1 s_{k-1} + \cdots + q_{k-1} s_1 \\ &= s_k + p_1 s_{k-1} + \cdots + p_{k-1} s_1 \\ &= -kp_k \,. \end{aligned}$$

showing that $p_k = q_k$. Hence this relation holds for all k.

By the Cayley-Hamilton theorem,

$$B_n = A^n + q_1 A^{n-1} + \cdots + q_{n-1} A + q_n I_n = Z_n \,.$$

and so

$$B_n = A_n + q_n I_n = Z_n; \qquad A_n = AB_{n-1} = -q_n I_n.$$

If A is nonsingular, then $\det(A) = (-1)^n K_A(0) = (-1)^n q_n \neq 0$, and thus

$$A^{-1} = -\frac{1}{q_n} B_{n-1} \,.$$

■

Example. *Find the characteristic polynomial and if possible the inverse of the matrix*

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 1 & 0 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ 4 & -5 & 0 & 4 \end{pmatrix}.$$

For $k = 1, 2, 3, 4$, compute

$$A_k = AB_{k-1} \qquad q_k = \frac{-1}{k} trace(A_k), \qquad B_k = A_k + q_k I_4 .$$

$$A_1 = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 1 & 0 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ 4 & -5 & 0 & 4 \end{pmatrix}, \qquad q_1 = -4, \qquad B_1 = \begin{pmatrix} -3 & 2 & 1 & -1 \\ 1 & -4 & 2 & 1 \\ 2 & 1 & -5 & 3 \\ 4 & -5 & 0 & 0 \end{pmatrix};$$

$$A_2 = \begin{pmatrix} -3 & 0 & 0 & 4 \\ 5 & -1 & -9 & 5 \\ 5 & -16 & 9 & -4 \\ -1 & 8 & -6 & -9 \end{pmatrix}, \qquad q_2 = 2, \qquad B_2 = \begin{pmatrix} -1 & 0 & 0 & 4 \\ 5 & 1 & -9 & 5 \\ 5 & -16 & 11 & -4 \\ -1 & 8 & -6 & -7 \end{pmatrix};$$

$$A_3 = \begin{pmatrix} 15 & -22 & -1 & 17 \\ 8 & -24 & 16 & -11 \\ -5 & 41 & -38 & -4 \\ -33 & 27 & 21 & -37 \end{pmatrix}, \qquad q_3 = 28, \qquad B_3 = \begin{pmatrix} 43 & -22 & -1 & 17 \\ 8 & 4 & 16 & -11 \\ -5 & 41 & -10 & -4 \\ -33 & 27 & 21 & -9 \end{pmatrix};$$

$$A_4 = \begin{pmatrix} 87 & 0 & 0 & 0 \\ 0 & 87 & 0 & 0 \\ 0 & 0 & 87 & 0 \\ 0 & 0 & 0 & 87 \end{pmatrix}, \qquad q_4 = -87, \qquad B_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore the characteristic polynomial of A is:

$$K_A(\lambda) = \lambda^4 - 4\lambda^3 + 2\lambda^2 + 28\lambda - 87.$$

Note that A_4 is a diagonal matrix, so we only need to multiply the first row of A by the first column of B_3 to obtain 87. Since $q_4 = -87$, the matrix A has an inverse.

$$A^{-1} = \frac{-1}{q_4} B_3 = \frac{1}{87} \begin{pmatrix} 43 & -22 & -1 & 17 \\ 8 & 4 & 16 & -11 \\ -5 & 41 & -10 & -4 \\ -33 & 27 & 21 & -9 \end{pmatrix}.$$

■

Matlab Program

```
A = input('Enter a square matrix : ')
m = size(A); n = m(1); q = zeros(1,n); B = A; AB = A; In = eye(n);
for k = 1 : n-1, q(k) = -(1/k) * trace(AB)B = AB + q(k) * In; AB = A * B; end
C = B; q(n) = -(1/n) * trace(AB); Q = [1 q];
disp('The Characteristic polynomial looks like : ')
disp( 'K_A(x) = x ∧ n + q(1)x ∧ (n-1) + ... + q(n-1)x + q(n)' ), disp(' '),
disp('The coefficients list c(k) is : '), disp(' '),
disp(Q), disp(' ')
if q(n) == 0, disp('The matrix is singular ');
else, disp('The matrix has an inverse. '), disp(' ')
    C = -(1/q(n)) * B;
    disp('The inverse of A is : '),disp(' '),
    disp(C)
end
```

■

♣ **The Method of Undetermined Coefficients**. If one has to expand large numbers of characteristic polynomials of the same order, then the method of undetermined coefficients may be used to produce characteristic polynomials of those matrices.

Let A be an $n \times n$ matrix and

$$K_A(\lambda) = \det(\lambda I_n - A) = \lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n$$

be its characteristic polynomial. In order to find the coefficients p_i's of $K_A(\lambda)$ we evaluate

$$D_j = K_A(j) = \det(jI_n - A) \qquad j = 0, 1, 2, \ldots, n-1$$

and obtain the following system of linear equations:

$$\begin{cases} p_n = D_0 \\ 1^n + p_1.1^{n-1} + \cdots\cdots\cdots + p_n = D_1 \\ 2^n + p_1.2^{n-1} + \cdots\cdots\cdots + p_n = D_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (n-1)^n + p_1.(n-1)^{n-1} + \cdots + p_n = D_{n-1} \end{cases}$$

Which can be changed into:

$$S_{n-1}P = \begin{bmatrix} 1^{n-1} & 1^{n-2} & \ldots & 1 \\ 2^{n-1} & 2^{n-2} & \ldots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ (n-1)^{n-1} & (n-1)^{n-2} & \ldots & n-1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n-1} \end{bmatrix} = \begin{bmatrix} D_1 - D_0 - 1^n \\ D_2 - D_0 - 2^n \\ \vdots \ \vdots \ \vdots \ \vdots \\ D_{n-1} - D_0 - (n-1)^n \end{bmatrix} = D\,.$$

The system may be solved as follows:

$$P = S_n^{-1}D.$$

Since the $(n-1) \times (n-1)$ matrix S_n depends only on the order of A, we may store R_n, the inverse of S_{n-1} beforehand and use it to find the coefficients of characteristic polynomial of various $n \times n$ matrices.

Examples. *Compute the characteristic polynomials of the* 4×4 *matrices*

$$A = \begin{pmatrix} 1 & 3 & 0 & 4 \\ 2 & -3 & 1 & 3 \\ 1 & 2 & 1 & 2 \\ -1 & 3 & 2 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 1 & 0 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ 4 & -5 & 0 & 4 \end{pmatrix}.$$

First we find

$$S = \begin{pmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ 27 & 9 & 3 \end{pmatrix} \qquad \text{and} \qquad R = S^{-1} = -\frac{1}{12}\begin{pmatrix} -6 & 6 & -2 \\ 30 & -24 & 6 \\ -36 & 18 & -4 \end{pmatrix}.$$

Then for the matrix A we obtain

$$D_0 = \det(-A) = -48, \quad D_1 = \det(I_4 - A) = -72,$$
$$D_2 = \det(2I_4 - A) = -128 \quad \text{and} \quad D_3 = \det(3I_4 - A) = -180$$

$$D = \begin{pmatrix} D_1 - D_0 - 1^4 \\ D_2 - D_0 - 2^4 \\ D_3 - D_0 - 3^4 \end{pmatrix} = \begin{pmatrix} -25 \\ -96 \\ -213 \end{pmatrix}.$$

Hence

$$P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = -\frac{1}{12} \begin{pmatrix} -6 & 6 & -2 \\ 30 & -24 & 6 \\ -36 & 18 & -4 \end{pmatrix} \begin{pmatrix} -25 \\ -96 \\ -213 \end{pmatrix} = \begin{pmatrix} 0 \\ -23 \\ -2 \end{pmatrix}.$$

Thus

$$K_A(\lambda) = \lambda^4 - 23\lambda^2 - 2\lambda - 48$$

For the matrix B we have

$$D_0 = \det(-B) = -87, \quad D_1 = \det(I_4 - B) = -60,$$
$$D_2 = \det(2I_4 - B) = -39 \quad \text{and} \quad D_3 = \det(3I_4 - B) = -12$$

$$D = \begin{pmatrix} D_1 - D_0 - 1^4 \\ D_2 - D_0 - 2^4 \\ D_3 - D_0 - 3^4 \end{pmatrix} = \begin{pmatrix} 26 \\ 32 \\ -6 \end{pmatrix}.$$

Hence

$$P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = -\frac{1}{12} \begin{pmatrix} -6 & 6 & -2 \\ 30 & -24 & 6 \\ -36 & 18 & -4 \end{pmatrix} \begin{pmatrix} 26 \\ 32 \\ -6 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 28 \end{pmatrix}.$$

Thus

$$K_B(\lambda) = \lambda^4 - 4\lambda^3 + 2\lambda^2 + 28\lambda - 87$$

Matlab Program

```
N = input('Enter the size of your square matrix : ');
n = N - 1; In = eye(N); S = zeros(n); R = zeros(n); D = zeros(1,n);
DSP1 = [' For any ',int2str(N),'-square matrix, you need S =='];
DSP2 = ['Do you want to try with another ',int2str(N),'-square matrix? (Yes = 1/No = 0)'];
%DEFINING S
for i = 1 : n, for j = 1 : n, S(i,j) = i ^ (N - j); end; end;
disp(' '), disp(DSP1), disp(' '), disp(S),
R = inv(S);
ok = 1;
while ok == 1;
   A = input(['Enteran ',int2str(N),' x ',int2str(N),' matrix A : ');     disp(' ')
   D0 = det(A);
   for k = 1 : n; D(k) = det(k * In - A); end;
   for i = 1 : n; DD(i) = D(i) - D0 - i ^ N; end;
   P = R * DD';
   disp('The Characteristic polynomial looks like : ')
   disp( 'K_A(x) = x ^ n + p(1)x ^ (n - 1) + ... + p(n - 1)x + p(n)' ), disp(' '),
   disp('The coefficients list p(k) is : '), disp(' '),
   disp([1 P' D0]), disp(' '),
   disp(DSP2), disp(' '),
   ok = input(DSP2);
end
```

■

♣ **The Method of Danilevsky**. Consider an $n \times n$ matrix A and let

$$K_A(\lambda) = det(\lambda I_n - A) = \lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n$$

be its characteristic polynomial. Then the **companion matrix** of $K_A(\lambda)$

$$F[A] = \begin{pmatrix} -p_1 & -p_2 & -p_3 & \dots & -p_{n-1} & -p_n \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

is similar to A and is called the *Frobenius form* of A.

The method of Danilevsky (1937) applies the Gauss-Jordan method to obtain the Frobenius form of an $n \times n$ matrix. According to this method the transition from the matrix A to $F[A]$ is done by means of $n-1$ similarity transformations which successively transform the rows of A, beginning with the last, into corresponding rows of $F[A]$.

Let us illustrate the beginning of the process. Our purpose is to carry the nth row of

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2,n-1} & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3,n-1} & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{n,n-1} & a_{nn} \end{pmatrix}$$

into the row $(0 \quad 0 \quad \dots \quad 1 \quad 0)$. Assuming that $a_{n,n-1} \neq 0$, we replace the $(n-1)th$ row of the $n \times n$ identity matrix with the nth row of A and obtained the matrix

$$U_{n-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \dots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{n,n-1} & a_{nn} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

The inverse of U_{n-1} is

$$V_{n-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \dots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ v_{n-1,1} & v_{n-1,2} & v_{n-1,3} & \dots & v_{n-1,n-1} & v_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

where

$$v_{n-1,i} = -\frac{a_{ni}}{a_{n,n-1}} \qquad \text{for} \quad i \neq n-1$$

and

$$v_{n-1,n-1} = -\frac{1}{a_{n,n-1}} \; .$$

Multiplying the right side of A by V_{n-1}, we obtain

$$AV_{n-1} = B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1,n-1} & b_{1n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2,n-1} & b_{2n} \\ \vdots & \vdots & \ddots & \dots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & b_{n-1,n-1} \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

However the matrix $B = AM_{n-1}$ is not similar to A. To have a similarity transformation, it is necessary to multiply the left side of B by $U_{n-1} = V_{n-1}^{-1}$. Let $C = U_{n-1}AV_{n-1}$, then C is similar to A and is of the form

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & \dots & c_{1,n-1} & c_{1n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2,n-1} & b_{2n} \\ \vdots & \vdots & \ddots & \dots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_{n-1,1} & c_{n-1,2} & c_{n-1,3} & \dots & c_{n-1,n-1} & c_{n-1,n-1} \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

Now, if $c_{n-1,n-1} \neq 0$, then similar operations are performed on matrix C by taking its $(n-2)th$ row as the principal one. We then obtain the matrix

$$D = U_{n-2}CV_{n-2} = U_{n-2}U_{n-1}AV_{n-1}V_{n-2}$$

with two reduced rows. We continue the same way until we finally obtain the Frobenius form

$$F[A] = U_1U_2 \cdots U_{n-2}U_{n-1}AV_{n-1}V_{n-2} \cdots V_2V_1$$

if, of course, all the $n-1$ intermediate transformations are possible.

Exceptional case in the Danilevsky method. Suppose that in the transformation of the matrix A into its Frobenius form $F[A]$ we arrived, after a few steps, at a matrix of the form

$$R = \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1k} & \dots & r_{1,n-1} & r_{1n} \\ r_{21} & r_{22} & \dots & r_{2k} & \dots & r_{2,n-1} & r_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ r_{k1} & r_{k2} & \dots & r_{kk} & \dots & r_{k,n-1} & r_{kn} \\ 0 & 0 & \dots & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \ddots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots & 1 & 0 \end{pmatrix}$$

and it was found that $r_{k,k-1} = 0$ or $|r_{k,k-1}|$ is very small. It is then possible to continue the transformation by the Danilevsky method.

Two cases are possible here.

Case 1. Suppose for some $j = 1, 2, \dots, k-2$, $r_{kj} \neq 0$. Then by permuting the jth row and $(k-1)th$ row and the jth column and $(k-1)th$ column of R we obtain a matrix $R' = (r'_{ij})$ similar to R with $r'_{k,k-1} \neq 0$.

Case 2. Suppose now that $r_{kj} = 0$ for all $j = 1, 2, \ldots, k-2$. Then R is in the form

$$R = \begin{bmatrix} R_1 & R_2 \\ 0 & R_3 \end{bmatrix} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1,k-1} & r_{1,k} & r_{1,k+1} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2,k-1} & r_{2,k} & r_{2,k+1} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{k-1,1} & r_{k-1,2} & \cdots & r_{k-1,k-1} & r_{k-1,k} & r_{k-1,k+1} & \cdots & r_{k-1,n} \\ \hline 0 & 0 & \cdots & 0 & r_{kk} & r_{k,k+1} & \cdots & r_{kn} \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

In this case the characteristic polynomial of R breaks up into two determinants:

$$\det(\lambda I_n - R) = \det(\lambda I_{k-1} - R_1)\det(\lambda I_{n-k+1} - R_3).$$

Here, the matrix R_3 is already reduced to the Frobenius form. It remains to apply the Danilevsky's method to the matrix R_1.

Note. *Since $U_k A_{k-1}$ only changes the kth row of A_{k-1}, it is more efficient to multiply first A_{k-1} by its $(k+1)th$ row and then multiply on the right side the resulting matrix by V_k.*

The next result shows that once we transform A into its Frobenius form ; we may obtain the eigenvectors with the help of the matrices $V_i's$.

Theorem. *Let A be an $n \times n$ matrix and let $F[A]$ be its Frobenius form. If λ is an eigenvalue of A, then*

$$v = \begin{pmatrix} \lambda^{n-1} \\ \lambda^{n-2} \\ \vdots \\ \lambda \\ 1 \end{pmatrix} \qquad \text{and} \qquad w = V_{n-1}V_{n-2}\cdots V_2V_1v$$

are the eigenvectors of $F[A]$ and A respectively.

Proof. Since

$$\det(\lambda I_n - A) = \det(\lambda I_n - F[A]\,) = \lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n\,,$$

we have

$$(\lambda I_n - F[A])v = \begin{pmatrix} \lambda - p_1 & -p_2 & -p_3 & \cdots & -p_{n-1} & -p_n \\ 1 & \lambda & 0 & \cdots & 0 & 0 \\ \lambda & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \lambda \end{pmatrix} \begin{pmatrix} \lambda^{n-1} \\ \lambda^{n-2} \\ \lambda^{n-3} \\ \vdots \\ \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Since $F[A] = V_1^{-1}V_2^{-1}\cdots V_{n-2}^{-1}V_{n-1}^{-1}AV_{n-1}V_{n-2}\cdots V_2V_1$ and $F[A]v = \lambda v$, we conclude that

$$\lambda w = V_{n-2}\cdots V_2V_1(\lambda v) = (V_{n-2}\cdots V_2V_1)\,F[A]v = A\,(V_{n-1}V_{n-2}\cdots V_2V_1v) = Aw \qquad \blacksquare$$

Note. *For expanding characteristic polynomials of matrices of order higher than fifth, the method of Danilevsky requires less multiplications and additions than other methods.*

Example. *Reduce the matrix*

$$A = \begin{pmatrix} 1 & 1 & 3 & 4 \\ 2 & 0 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

to its Frobenius form.

The matrix $B = A_3 = U_3AV_3$ is as follows:

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 & 4 \\ 2 & 0 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -3 & 1 \\ 2 & 0 & -2 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since $b_{32} = 0$, we need the permutation matrix $J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$; thus

$$C = JBJ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -3 & 1 \\ 2 & 0 & -2 & -1 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -2 & -1 \\ 1 & 1 & -3 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Next we obtain the matrix $D = A_2 = U_2CV_2$

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & -2 & -1 \\ 1 & 1 & -3 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & -2 & -3 \\ -1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Finally the Frobenius form $F[A] = A_1 = U_1DV_1$,

$$F[A] = \begin{pmatrix} -1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -2 & -2 & -3 \\ -1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus the Characteristic polynomial of A is :

$$K_A(\lambda) = x^4 - x^3 - 4x^2 - 2x - 3$$

■

Matlab Program

```
A = input('Enter the square matrix A : ');
m = size(A); N = m(1); b = [1]; B = zeros(N); i = 1;
while i < N,
  J = eye(N); h = A(N - i + 1, N - i)
  while h == 0;
   c = A(N - i + 1, 1 : N - i); z = norm(c, inf);
   if z  = 0;
    k = 1; r = 0;
    while r == 0 & k < N - i;
       r = r + c(N - i - k); k = k + 1;
       J(N - i, N - i) = 0; J(N - i, N - i - k + 1) = 1;
       J(N - i - k + 1, N - i - k + 1) = 0; J(N - i - k + 1, N - i) = 1,
```

```
        A = J * A * J;  k = k + 1;
      end
    else
      b = conv(b, [1 - A(N - i + 1, N - i + 1 : N)]);
      B = A(1 : N - i, 1 : N - i); A = B, N = N - i;
    end
     h = A(N - i + 1, N - i);
end
U = eye(N);  V = eye(N);
U(N - i, :) = A(N - i + 1, :);
V(N - i, :) = -A(N - i + 1, :)/A(N - i + 1, N - i);  V(N - i, N - i) = 1/A(N - i + 1, N - i);
A = U * A * V;
i = i + 1;
end;
b = conv(b,  [1 - A(N - i + 1, N - i + 1 : N)]);  disp(' '),
disp('The Characteristic polynomial looks like  :  '),  disp(' '),
disp(['K_A(x) = x ∧ n + c(1)x ∧ (n - 1), +... + c(n - 1)x + c(n)']), disp(' '),
disp('The coefficients list c(k) is  :  '), disp(' '),
disp(b),  disp(' ')
```

■

♣ **The Method of Krylov**. Let A be an $n \times n$ matrix. For any n-dimensional nonzero column vector v we associate its successive transforms

$$v_k = A^k v \qquad (k = 0, 1, 2, \ldots),$$

this sequence of vectors is called the *Krylov sequence* associated to the matrix A and the vector v.

At most n vectors of the sequence $v_0, v_1, v_2, \ldots$ will be linearly independent. Suppose for some $r = r(v) \leq n$, the vectors $v_0, v_1, v_2, \ldots, v_r$ are linearly independent and the vector v_{r+1} is a linear combination of the preceding ones. Hence there exists a monic polynomial

$$\phi(\lambda) = c_0 + c_1\lambda + c_2\lambda^2 + \cdots + c_{r-1}\lambda^{r-1} + \lambda^r$$

such that

$$\phi(A)v = (c_0 I_n + c_1 A + \cdots + c_{r-1}A^{r-1} + A^r)v = c_0 v_0 + c_1 v_1 + \cdots + c_{r-1}v_r + c_r v_{r+1} = \theta\,.$$

The polynomial $\phi(\lambda)$ is said to *annihilate* v and to be *minimal* for v. If $\omega(\lambda)$ is another monic polynomial which annihilates v,

$$\omega(A)v = 0\,,$$

then $\phi(\lambda)$ divides $\omega(\lambda)$. To show that; suppose

$$\omega(\lambda) = \gamma(\lambda)\phi(\lambda) + \rho(\lambda),$$

where $\rho(\lambda)$ is the remainder after dividing ω by ϕ, hence of degree strictly less than r, it follows that

$$\rho(A)v = 0.$$

But $\phi(\lambda)$ is minimal for v, hence $\rho(\lambda) = 0$.

Now of all vectors v there is at least one vector for which the degree v is maximal, since for any vector v, $r(v) \leq n$. We call such vector a *maximal vector*.

A monic polynomial $\mu_A(\lambda)$ is said to be the *minimal polynomial* for A, if $\mu_A(\lambda)$ is monic and of minimum degree satisfying

$$\phi(A) = Z_n.$$

Theorem 1. *Let A be an $n \times n$ matrix and let $\phi(\lambda)$ be a minimal polynomial for a maximal vector v. Then $\phi(\lambda)$ is the minimal polynomial for A.*

Proof. Consider any vector u such that u and v are linearly independent. Let $\psi(\lambda)$ be its minimal polynomial. If ω is the lowest common multiple of ϕ and ψ, then ω annihilates every vector in the plane of u and v, since

$$\omega(A)(\alpha u + \beta v) = \alpha\omega(A)u + \beta\omega(A)v = \theta.$$

Hence ω contains as a divisor the minimal polynomial of every vector in the plane. But ω is of degree $2n$ at most, hence has only finitely many divisors. Since there are infinitely many pairs of linearly independent vectors in the plane and finitely many divisors of ω, there is a pair of linearly independent vectors x and y in this plane with the same minimal polynomial. This polynomial also annihilates v since v is on this plane. Therefore ϕ is minimal for every vectors in the plane of u and v, and since u was any vector whatever, other than v, ϕ annihilates every n-dimensional vector.

Since ϕ annihilates every vector, it annihilates in particular every vector e_i, hence

$$\phi(A)I = \phi(A) = Z_n.$$

Thus $\phi(\lambda) = \mu_A(\lambda)$ is the minimal polynomial for A. ■

If the minimal polynomial and the characteristic polynomial of a matrix are equal, then they may be found by the use of Krylov's sequence. To produce the characteristic polynomial of A by Krylov method, first choose an arbitrary n-dimensional nonzero column vector v such as e_1, then use the Krylov sequence to define the matrix

$$V = [v, Av, A^2v, \ldots, An-2v, A^{n-1}v] = [v_0, v_1, v-2, \ldots v_{n-2}, v_{n-1}].$$

If the matrix V has rank n, then the system $Vc = -v_n$ has a unique solution

$$c^t = (c_0, c_1, c_2, \cdots, c_{n-1}).$$

The monic polynomial

$$\phi(\lambda) = c_0 + c_1\lambda + c_2\lambda^2 + \cdots + c_{n-1}\lambda^{n-1} + \lambda^n$$

which annihilates v is the characteristic polynomial of A. If the system $Vc = -v_n$ does not have a unique solution, then change the initial vector and try for example with e_2. ■

Examples. *Compute the characteristic polynomials of the following matrices:*

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 1 & 0 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ 4 & -5 & 0 & 4 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 1 & 2 & -3 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & -3 & -1 & 3 \\ 1 & 0 & 1 & -2 \end{pmatrix}.$$

Choosing the initial vector $v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ for both matrices, we obtain

$$V_A = [v, Av, A^2v, A^3v] = \begin{pmatrix} 1 & 1 & 1 & 17 \\ 0 & 1 & 9 & 42 \\ 0 & 2 & 13 & 43 \\ 0 & 4 & 15 & 19 \end{pmatrix} \quad \text{and} \quad V_B = [v, Bv, B^2v, B^3v] = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

The matrix V_A is nonsingular, hence $c_A = -V_A^{-1}A^4v = \begin{pmatrix} -87 \\ 28 \\ 2 \\ -4 \end{pmatrix}$. From c_A we obtain the characteristic polynomial of A which is

$$K_A(\lambda) = -87 + 28\lambda + 2\lambda^2 - 4\lambda^3 + \lambda^4.$$

The matrix V_B is singular, so we need another initial vector such as $v = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. The new matrix $V_B = \begin{pmatrix} 0 & 2 & 11 & 11 \\ 1 & 0 & 8 & 0 \\ 0 & -3 & 5 & -21 \\ 0 & 0 & -1 & 18 \end{pmatrix}$ is invertible, so $c_B = -V_B^{-1}A^4v = \begin{pmatrix} 9 \\ -2 \\ -10 \\ 2 \end{pmatrix}$. From the solution c_B we obtain the characteristic polynomial of B which is

$$K_B(\lambda) = 9 - 2\lambda - 10\lambda^2 + 2\lambda^3 + \lambda^4.$$

Remark. *The minimal polynomial and the characteristic polynomial of the matrix*

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

are

$$m_A(\lambda) = \lambda^3 - 3\lambda^2 - 7\lambda \quad \text{and} \quad K_A(\lambda) = \lambda^4 - 3\lambda^3 - 7\lambda^2,$$

respectively. Therefore by choosing any initial vector v, the matrix $V_A = [v, Av, A^2v, A^3v]$ will always be singular. This means that the Krylov sequence will never produce the characteristic polynomial $K_A(\lambda)$.

Matlab Program

```
A = input('Enter a square matrix A : ');
m = size(A); n = m(1); V = zeros(n,n);
DL1 = ['Enter an initial ', int2str(n) , ' - dimensional row vector v0 = '];
v0 = input(DL1);
z = 0; k = 1;
while z == 0 & k < 5
     w = v0; V(:,1) = w;
     for i = 2 : n,   w = A * w;   V(:,i) = w;   end,
     if det(V) ~= 0;   k = 8;   c = -inv(V) * A * w;
     else
       while k < 5
         v0 = input('The matrix V is singular, please enter another initial row vector v0 : ');
         k = k + 1;
       end;
     end;
    z = det(V);
end;
if k == 5;
  disp('Sorry, the Krylov method is not suited for this matrix. '), disp(' '),
else;
  disp('The Characteristic polynomial looks like : '), disp(' '),
  disp(['K_A(x) = c(0) + C(1)x + c(2)x ∧ 2 + ··· + c(n − 1)x ∧ (n − 1) + x ∧ n']), disp(' '),
  disp('The coefficients list c(k) is : '), disp(' '),
  disp([c', 1])
end;
```

■