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http://people.brunel.ac.uk/~mastjjb/jeb/or/moreip.html

The main areas in which IP is used in practice include:

- imposition of logical conditions in LP problems (such as the either/or condition)
- blending with a limited number of ingredients
- depot location
- job shop scheduling
- assembly line balancing
- airline crew scheduling
- timetabling

Blending problem

Consider the example of a manufacturer of animal feed who is producing feed mix for dairy cattle. In our simple example the feed mix contains two active ingredients and a filler to provide bulk. One kg of feed mix must contain a minimum quantity of each of four nutrients as below:

Nutrient	A	B	C	D
gram	90	50	20	2

The ingredients have the following nutrient values and cost

	A	B	C	D	Cost/kg
Ingredient 1 (gram/kg)	100	80	40	10	40
Ingredient 2 (gram/kg)	200	150	20	-	60

What should be the amounts of active ingredients and filler in one kg of feed mix?

Solution

Variables

In order to solve this problem it is best to think in terms of 1 kg of feed mix. That kilogram is made up of three parts; ingredient 1, ingredient 2 and filler. So: let

x_1 = amount (kg) of ingredient 1 in 1 kg of feed mix

x_2 = amount (kg) of ingredient 2 in 1 kg of feed mix

x_3 = amount (kg) of filler in 1 kg of feed mix

where $x_1, x_2, x_3 >= 0$.

Constraints

- balancing constraint (*implicit* constraint due to the definition of the variables)

$x_1 + x_2 + x_3 = 1$

- nutrient constraints

$100x_1 + 200x_2 >= 90$ (nutrient A)
$80x_1 + 150x_2 >= 50$ (nutrient B)
$40x_1 + 20x_2 >= 20$ (nutrient C)
$10x_1 >= 2$ (nutrient D)

Note the use of an inequality rather than an equality in these constraints, following the rule we put forward in the Two Mines example, where we assume that the nutrient levels we want are lower limits on the amount of nutrient in 1 kg of feed mix.

Objective

Presumably to minimise cost, i.e.

minimise $40x_1 + 60x_2$

which gives us our complete LP model for the blending problem:

$$\begin{array}{rrrrrr} [\max]z = & 40x_1 & +60x_2 & & & \\ \text{s. to} & x_1 & +x_2 & +x_3 & = & 1 \\ & 100x_1 & +200x_2 & & \geq & 90 \\ & 80x_1 & +150x_2 & & \geq & 50 \\ & 40x_1 & +20x_2 & & \geq & 20 \\ & 10x_1 & & & \geq & 2 \end{array}$$

In Lindo:

```
! Beasley "Animal feed"
! http://people.brunel.ac.uk/~mastjjb/jeb/or/moreip.html
! "Linear Programming" X=0.3667 0.2667 0.3667, z*=30.67
min 40x1 + 60x2 + 0x3
st
 x1 + x2 + x3 = 1
100 x1 +200 x2 > 90
 80 x1 +150 x2 > 50
 40 x1 + 20 x2 > 20
 10 x2 > 2
end
```

Suppose now we have the additional conditions:

- if we use any of ingredient 2 we incur a fixed cost of 15
- we need not satisfy all 4 nutrient constraints but need only satisfy 3 of them (i.e. whereas before the optimal solution required all 4 nutrient constraints to be satisfied now the optimal solution could (if it is worthwhile to do so) only have 3 (any 3) of these nutrient constraints satisfied and the 4.th violated.

Give the complete MIP formulation of the problem with these two new conditions added.

Solution

To cope with the condition that if x_2>=0 we have a fixed cost of 15 incurred we have the standard trick of introducing a zero-one variable y defined by

```
y = 1 if x2>=0
  = 0 otherwise
```

and

- add a term +15y to the objective function

and add the additional constraint

- x_2 <= [largest value x_2 can take]y

In this case it is easy to see that x_2 can never be greater than one and hence our additional constraint is x_2<= y.

To cope with condition that need only satisfy 3 of the 4 nutrient constraints, we introduce 4 zero-one variables z_i (i =1..4) where

z_i = 1 if nutrient constraint i (i=1,2,3,4) is satisfied
   = 0 otherwise

and add the constraint

- $z_1+z_2+z_3+z_4$ >= 3 (at least 3 constraints satisfied)

and alter the nutrient constraints to be

- $100x_1 + 200x_2$ >= $90z_1$
- $80x_1 + 150x_2$ >= $50z_2$
- $40x_1 + 20x_2$ >= $20z_3$
- $10x_1$ >= $2z_4$

The logic behind this change is that if a z_i=1 then the constraint becomes the original nutrient constraint which needs to be satisfied. However if a z_i=0 then the original nutrient constraint becomes

- same left-hand side >= zero

which (for the 4 left-hand sides dealt with above) is always true and so can be neglected — meaning the original nutrient constraint need not be satisfied. Hence the complete (MIP) formulation of the problem is given by

minimise $40x_1 + 60x_2 + 15y$
subject to
$x_1 + x_2 + x_3 = 1$
$100x_1 + 200x_2$ >= $90z_1$
$80x_1 + 150x_2$ >= $50z_2$
$40x_1 + 20x_2$ >= $20z_3$
$10x_1$ >= $2z_4$
$z_1 + z_2 + z_3 + z_4$ >= 3
x_2 <= y
z_i = 0 or 1   i=1,2,3,4
y = 0 or 1
x_i >= 0   i=1,2,3

$$
\begin{array}{lcccccccccc}
[\min]z = & 40x_1 & +60x_2 & +0x_3 & +15y & +0z_1 & +0z_2 & +0z_3 & +0z_4 & & \\
\text{s. to} & x_1 & +x_2 & +x_3 & & & & & & = & 1 \\
 & 100x_1 & +200x_2 & & & -90z_1 & & & & \geq & 0 \\
 & 80x_1 & +150x_2 & & & & -50z_2 & & & \geq & 0 \\
 & 40x_1 & +20x_2 & & & & & -20z_3 & & \geq & 0 \\
 & 10x_1 & & & & & & & -2z_4 & \geq & 0 \\
 & & x_2 & & -1 & & & & & \leq & 0 \\
 & & & & & z_1 & +z_2 & +z_3 & +z_4 & \geq & 3
\end{array}
$$

with y, z binary.

In Lindo:

```
! Beasley "Animal feed"
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! Beasley "Animal feed"
! http://people.brunel.ac.uk/~mastjjb/jeb/or/moreip.html
! "Integer Programming" X=0.9 0 0.1 0 1 1 1 0, z*=36
min 40x1 + 60x2 + 0x3 + 15y
st
 x1 + x2 + x3 = 1
100 x1 +200 x2 - 90 z1 > 0
 80 x1 +150 x2 - 50 z2 > 0
 40 x1 + 20 x2 - 20 z3 > 0
 10 x2 - 2 z4 > 0
 x2 - y < 0
 z1 + z2 + z3 + z4 > 3
end
int y
int z1
int z2
int z3
int z4
```