Topics on "Operational Research"
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LP: algebraic vs. tableaux

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A parallel is presented for the resolution of the LP in the algebraic form versus the tabular form. The revised (matrix) form is also shown.

Key words: linear programming; simplex method; algebraic form; tableaux; revised

0 The model

Original form of the model ("s.t.", subject to):

$$\begin{aligned} &[\max]\ z = 3x_1 + 5x_2 \\ &\text{s.t.} \\ &x_1 \le 4 \\ &2x_2 \le 12 \\ &3x_1 + 2x_2 \le 18 \end{aligned} \qquad \{1\}$$

and $\mathbf{x} \ge 0$.

Augmented form of the model:

$$\begin{aligned} &[\max]\ z = 3x_1 + 5x_2 + 0x_3 + 0x_4 + 0x_5 \\ &\text{s. to} \\ &x_1 + x_3 = 4 \\ &2x_2 + x_4 = 12 \\ &3x_1 + 2x_2 + x_5 = 18 \end{aligned} \qquad \{2\}$$

and $\mathbf{x} \ge 0$ ($x_i \ge 0$, $i = 1..5$). Variables x_i, $i = 3..5$, are the "slack variables". Better,

Maximize Z

subject to

$$\begin{aligned} (0)\quad & Z = 3x_1 + 5x_2 + 0x_3 + 0x_4 + 0x_5 \\ (1)\quad & x_1 + x_3 = 4 \\ (2)\quad & 2x_2 + x_4 = 12 \\ (3)\quad & 3x_1 + 2x_2 + x_5 = 18 \end{aligned} \qquad \{3\}$$

1 Algebraic form of the Simplex Method

Initialization

$$\text{Basic variables:}\quad \{x_3 \quad x_4 \quad x_5\} = \{4 \quad 12 \quad 18\}$$

$$Z = 0$$

$$\text{Non-basic variables:}\quad \{x_1 \quad x_2\} = \mathbf{0}$$

Optimality test

The rates of improvement are *positive*. Therefore, this solution is not optimal.

Iteration 1

STEP 1) Determining the direction of movement

The choice of which nonbasic variable is increased is as follows:

{4} $$Z = 3x_1 + 5x_2$$

Entering: x_2

Consequences ?

STEP 2) Determining where to stop

(Keep nonbasic variables null.) All the variables must be *nonnegative*.

How far can the **entering** variable be increased ?

Minimum ratio test.

{5}
$$\begin{aligned} x_3 &= 4 - 0x_2 \ge 0 \\ x_4 &= 12 - 2x_2 \ge 0 \\ x_5 &= 18 - 2x_2 \ge 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_2 &\le 4/0 = \infty \\ x_2 &\le 12/2 = 6 \leftarrow \min \\ x_2 &\le 18/2 = 9 \end{aligned}$$

From Eq. (2), x_2 has pushed x_4 to 0, so

Leaving: x_4

x_2 replaces x_4

Normalize (to *one*) the coefficient of the entering variable (x_2) in its equation [(2)],

{6} (2') $$x_2 + \frac{1}{2}x_4 = 6$$

and replace x_2 (the "new" basic variable) in all the other Equations.

STEP 3) Solving for the new BF solution

{7}
$$\begin{array}{llllllll} (0) & Z = 30 & +3x_1 & +0x_2 & +0x_3 & -\frac{5}{2}x_4 & +0x_5 & \triangleright (0)-5(2') \\ (1) & & x_1 & & +\underline{x_3} & & = 4 & \triangleright (1)-0(2') \\ (2) & & & \underline{x_2} & & +\frac{1}{2}x_4 & = 6 & \triangleright (2') \\ (3) & & 3x_1 & & & -x_4 & +\underline{x_5} = 6 & \triangleright (3)-2(2') \end{array}$$

Basic variables: $\{x_3 \quad x_2 \quad x_5\} = \{4 \quad 6 \quad 6\}$

$Z = \quad 30$

Non-basic variables: $\{x_1 \quad x_4\} = \mathbf{0}$

Optimality test

Some rates of improvement are *positive*; therefore, the solution is not optimal.

Iteration 2

STEP 1) Determining the direction of movement

The choice of which nonbasic variable is increased is as follows:

{8} $$Z = 30 + 3x_1 - \frac{5}{2}x_4$$

Entering: x_1

Consequences ?

STEP 2) Determining where to stop

(Keep nonbasic variables null.) All the variables must be *nonnegative*.

How far can the **entering** variable be increased ?

Minimum ratio test.

{9} $$\begin{aligned} x_3 &= 4 - x_1 \geq 0 \\ x_2 &= 6 - 0x_1 \geq 0 \\ x_5 &= 6 - 3x_1 \geq 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &\leq 4/1 = 4 \\ x_1 &\leq 6/0 = \infty \\ x_1 &\leq 6/3 = 2 \leftarrow \text{min} \end{aligned}$$

From Eq. (3), x_1 has pushed x_5 to 0, so

Leaving: x_5

x_1 replaces x_5

Normalize (to *one*) the coefficient of the entering variable (x_1) in its equation [(3)],

{10} (3') $$x_1 + \frac{1}{3}x_5 = 2$$

and replace x_1 (the "new" basic variable) in all the other Equations.

STEP 3) Solving for the new BF solution

{11}
$$\begin{array}{llllllll}
(0) & Z = 36 & +0x_1 & +0x_2 & +0x_3 & -\frac{3}{2}x_4 & -x_5 & \\
(1) & & & & +x_3 & +\frac{1}{3}x_4 & -\frac{1}{3}x_5 & = 2 \\
(2) & & & x_2 & & +\frac{1}{2}x_4 & & = 6 \\
(3) & & x_1 & & & -\frac{1}{3}x_4 & +\frac{1}{3}x_5 & = 2
\end{array}$$

Basic variables: $\{x_3 \quad x_2 \quad x_1\} = \{2 \quad 6 \quad 2\}$

$Z = \quad 36$

Non-basic variables: $\{x_5 \quad x_4\} = \mathbf{0}$

Optimality test

The rates of improvement are all *negative*; therefore, this solution is **optimal**.

The optimal solution is thus (the values of the ***structural*** variables are **emphasized**)

{12} $$\{x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5\} = \{\mathbf{2} \quad \mathbf{6} \quad 2 \quad 0 \quad 0\}$$

2 Tabular form of the Simplex Method

The tabular form of the simplex method records only the essential information: (1) the coefficients of the variables, (2) the constants on the right-hand sides of the equations, and (3) the basic variable appearing in each equation.

Notice that Eq. (0), for *Z*, in the tableau is written as $Z - \mathbf{c}^{\mathrm{T}}\mathbf{x} = z_{\mathrm{RHS}}$, so the coefficients of *x* have their signs reversed.

Compare the following with Eq. {3}.

Table 1 Simplex tableaux for the Wyndor Glass Co. problem

Basic variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right side	Ratio
Z	(0)	1	–3	**–5**	0	0	0	0	
x_3	(1)	0	1	0	1	0	0	4	∞
x_4	(2)	0	0	**2**	0	1	0	12	6
x_5	(3)	0	3	2	0	0	1	18	9

Compare the following with Eq. {7}.

Table 2 Simplex tableaux for the Wyndor Glass Co. problem

Basic variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right side	Ratio
Z	(0)	1	**–3**	0	0	5/2	0	30	
x_3	(1)	0	1	0	1	0	0	4	4
x_2	(2)	0	0	1	0	1/2	0	6	∞
x_5	(3)	0	**3**	0	0	–1	1	6	2

Compare the following with Eq. {11}.

Table 3 Simplex tableaux for the Wyndor Glass Co. problem

Basic variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right side	Ratio
Z	(0)	1	0	0	0	3/2	1	36	
x_3	(1)	0	0	0	1	1/3	−1/3	2	
x_2	(2)	0	0	1	0	1/2	0	6	
x_1	(3)	0	1	0	0	−1/3	1/3	2	

The coefficients [line (0)] of the basic variables —which have their signs reversed— are all *negative*; therefore, this solution is **optimal**.

3 Revised (matrix form) Simplex Method

The “revised simplex method” —a *matrix* form of the simplex method that is totally equivalent to the previous two— records only the necessary information: *(1)* the coefficients of the variables, *(2)* the constants on the right-hand sides of the equations, and *(3)* the basic variable appearing in each equation. (Notation is partly altered for coherence of some available software: *z* for *Z*, *p* for *c*, etc..) Vectors are systematically considered here column matrices. For minimization, the changes are obvious.

{13}
$$\begin{aligned}[\max]z &= \mathbf{p}^{\mathrm{T}}\mathbf{x} \\ \text{s. to} \quad \mathbf{A}\,\mathbf{x} &\le \mathbf{b} \\ \text{with} \quad \mathbf{x} &\ge \mathbf{0}\end{aligned}$$

The augmented form (to be used, as always), keeping the nonnegativity, is

{14}
$$\begin{aligned}[\max]z &= \left[\mathbf{p}_D \;\vert\; \mathbf{p}_I\right]^{\mathrm{T}}\begin{bmatrix}\mathbf{x}_D \\ -- \\ \mathbf{x}_I\end{bmatrix} \\ \text{s. to} \quad \left[\mathbf{A}_D \;\vert\; \mathbf{A}_I\right]\begin{bmatrix}\mathbf{x}_D \\ -- \\ \mathbf{x}_I\end{bmatrix} &= \mathbf{b}\end{aligned}$$

The *basic* variables will be here called ***dependent variables*** (which they are), hence subscript "D"; and the *non-basic* variables will be called ***independent variables*** ("independently" made zero), hence subscript "I". As the non-basic variables, $\mathbf{x}_\mathrm{I}$, are set equal to 0, it is, successively:

{15}
$$\mathbf{A}_D\mathbf{x}_D = \mathbf{b}$$

Therefore, it is

{16}
$$\begin{aligned}\mathbf{x}_{D0} &= \mathbf{A}_D^{-1}\mathbf{b} \\ &\text{and} \\ z_0 &= \mathbf{p}_D^{\mathrm{T}}\mathbf{x}_{D0}\end{aligned}$$

To express the objective function in terms of the non-basic variables [Tavares, 1996, p 53][1] and, thus, to annul the coefficients of the basic variables, we have to premultiply the constraint (following 2.nd equation) by $-\mathbf{p}_D^{\mathrm{T}}\mathbf{A}_D^{-1}$ and add it to the objective function, that is, successively:

{17}
$$\begin{aligned}z &= \mathbf{p}_D^{\mathrm{T}}\mathbf{x}_D + \mathbf{p}_I^{\mathrm{T}}\mathbf{x}_I \\ \mathbf{b} &= \mathbf{A}_D\mathbf{x}_D + \mathbf{A}_I\mathbf{x}_I \qquad \times\left(-\mathbf{p}_D^{\mathrm{T}}\mathbf{A}_D^{-1}\right)\end{aligned}$$

{18}
$$\begin{aligned}z - \mathbf{p}_D^{\mathrm{T}}\mathbf{A}_D^{-1}\mathbf{b} &= \left\{\mathbf{p}_D^{\mathrm{T}}\mathbf{x}_D - \mathbf{p}_D^{\mathrm{T}}\mathbf{A}_D^{-1}\mathbf{A}_D\mathbf{x}_D\right\} + \left\{\mathbf{p}_I^{\mathrm{T}}\mathbf{x}_I - \mathbf{p}_D^{\mathrm{T}}\mathbf{A}_D^{-1}\mathbf{A}_I\mathbf{x}_I\right\} = \\ &= 0 + \mathbf{p}_I^{\mathrm{T}}\mathbf{x}_I - \mathbf{p}_D^{\mathrm{T}}\mathbf{A}_D^{-1}\mathbf{A}_I\mathbf{x}_I = \left(\mathbf{p}_I^{\mathrm{T}} - \mathbf{p}_D^{\mathrm{T}}\mathbf{A}_D^{-1}\mathbf{A}_I\right)\mathbf{x}_I = \\ &= \left[\mathbf{p}_I - \left(\mathbf{A}_D^{-1}\mathbf{A}_I\right)^{\mathrm{T}}\mathbf{p}_D\right]^{\mathrm{T}}\mathbf{x}_I\end{aligned}$$

The following vector, d, is usually called the *reduced cost vector*

{19}
$$\mathrm{d} = \mathbf{p}_I - \left(\mathbf{A}_D^{-1}\mathbf{A}_I\right)^{\mathrm{T}}\mathbf{p}_D$$

or, introducing $\mathbf{K}$ and $\overline{\mathbf{p}}$ (as in some software),

{20}
$$\begin{aligned}\mathbf{K} &= \mathbf{A}_D^{-1}\mathbf{A}_I \\ \overline{\mathbf{p}} &= \mathbf{K}^{\mathrm{T}}\mathbf{p}_D\end{aligned} \qquad \mathrm{d} = \mathbf{p}_I - \overline{\mathbf{p}}$$

[1] See Bibliography on the course website.

It represents the constrained derivatives $\partial z/\partial x_I$, the nonbasic (independent) variables being now the decision variables. The ratios, as a criterion for the leaving variable, will be called q, with the (nonnegative) minimum ratio giving the leaving variable:

{21} $$Q = \frac{\mathbf{x}_D}{\mathbf{K}_{i\,e}}$$

Following are copies of the resolution of the prototype example: ***(a)*** with one of the course website resolutions; and ***(b)*** of an Excel resolution (just for this illustrative purpose).

```
================================ BASIS :  0 ================================

XD  3   4.00000      AD inv =    1.000       0.000       0.000
    4   12.0000                  0.000       1.000       0.000
    5   18.0000                  0.000       0.000       1.000

  z =  0.000000

    K =  1.000       0.000
         0.000       2.000
         3.000       2.000

P bar =  0.000       0.000

Delta =  3.000       5.000

EntVa =      2  (rank)   Del =  5.0000

         4.000       12.00       18.00
Theta  ----------- ----------- -----------
         0.000       2.000       2.000

i. e.,  -1.000       6.000       9.000

LeaVa =      2        Thet^ =  6.0000                  Next z = 30.0000

================================ BASIS :  1 ================================

XD  3   4.00000      AD inv =    1.000       0.000       0.000
    2   6.00000                  0.000      0.5000       0.000
    5   6.00000                  0.000      -1.000       1.000

  z =  30.00000

    K =  1.000       0.000
         0.000      0.5000
         3.000      -1.000

P bar =  0.000       2.500

Delta =  3.000      -2.500

EntVa =      1  (rank)   Del =  3.0000

         4.000       6.000       6.000
Theta  ----------- ----------- -----------
         1.000       0.000       3.000

i. e.,   4.000      -1.000       2.000

LeaVa =      3        Thet^ =  2.0000                  Next z = 36.0000
```

```
================================ BASIS :  2 ==================================

XD  3   2.00000       AD inv =   1.000      0.3333     -0.3333
    2   6.00000                  0.000      0.5000      0.000
    1   2.00000                  0.000     -0.3333      0.3333

  z =  36.00000

    K = 0.3333     -0.3333
        0.5000      0.000
       -0.3333      0.3333

P bar =  1.500      1.000

Delta = -1.500     -1.000

EntVa =      2  (rank)   Del = -1.0000

Output data  ::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::::

     Optimal objective funct.    36.0000      (MAXIMUM)      Last basis:    2

                              Var. #   Value
                                  1    2.00000
                                  2    6.00000
                                  3    2.00000
                                  4    0.00000
                                  5    0.00000
End of LINEAR PROGRAM.
```

Feb-2007 **The revised simplex (matrix form)**

Wyndor Glass Co.

[max] z =

x_1	x_2	x_3	x_4	x_5
3	5	0	0	0

A =

1	0	1	0	0
0	2	0	1	0
3	2	0	0	1

B =

4
12
18

1 x

X_D =

3
4
5

A_D =

1	0	0
0	1	0
0	0	1

A_D^{-1} =

1	0	0
0	1	0
0	0	1

$X_{D0} = A_D^{-1} B$ =

4
12
18

P_D =

0
0
0

P_D^T =

0	0	0

$z = P_D^T X_{D0}$ = 0

A_1 =

1	0
0	2
3	2

$K = A_D^{-1} A_1$ =

1	0
0	2
3	2
	↑

K^T =

1	0	3
0	2	2

$P^- = K^T P_D$ =

0
0

P_1 =

3
5

$\Delta = P_1 - P^-$ =

3
5

Enter **2**

Θ =

#DIV/0!
6
9

Leave **4**

$z' = z + \theta\,\delta$ = 30

2 x

X_D =

3
2
5

A_D =

1	0	0
0	2	0
0	2	1

A_D^{-1} =

1	0	0
0	0,5	0
0	-1	1

$X_{D0} = A_D^{-1} B$ =

4
6
6

P_D =

0
5
0

P_D^T =

0	5	0

$z = P_D^T X_{D0}$ = 30

A_1 =

1	0
0	1
3	0

$K = A_D^{-1} A_I =$

1	0
0	0,5
3	-1

↑

$K^T =$

1	0	3
0	0,5	-1

$P^- = K^T P_D =$

0
2,5

$P_I =$

3
0

$\Delta = P_I - P^- =$

3
-2,5

Enter 1

$\Theta =$

4
#DIV/0!
2

Leave 5

$z' = z + \theta\,\delta =$ 36

3 x

$X_D =$

3
2
1

$A_D =$

1	0	1
0	2	0
0	2	3

$A_D^{-1} =$

1	0,333333	-0,333333
0	0,5	0
0	-0,333333	0,333333

$X_{D0} = A_D^{-1} B =$

2
6
2

$P_D =$

0
5
3

$P_D^T =$

0	5	3

$z = P_D^T X_{D0} =$ **36**

$A_I =$

0	0
1	0
0	1

$K = A_D^{-1} A_I =$

0,333333	-0,333333
0,5	0
-0,333333	0,333333

↑

$K^T =$

0,333333	0,5	-0,333333
-0,333333	0	0,333333

$P^- = K^T P_D =$

1,5
1

$P_I =$

0
0

$\Delta = P_I - P^- =$

-1,5
-1

OPTIMUM

❖