Linear Programming, an introduction

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Linear Programming is presented at an introductory level, mainly from the book by Hillier and Lieberman [2005], abridged and adapted to suit the objectives of the "Operational Research" course. It begins with segments of its third chapter.

Key words: linear programming; simplex method.

I. Fundamentals and scope

Based on a prototype example, Linear Programming is presented, as well as the simplex method of resolution. This method was first presented by G. B. Dantzig in 1947 [MacTutor, 2007]. The text is based on the book by Hillier and Lieberman [2005], and begins with segments of the third chapter of the book.

II. Explanation of the simplex method

3 Introduction to Linear Programming

(H&L 25)

The development of linear programming has been ranked among the most important scientific advances in the mid-20.th century, and we must agree with this assessment. Its impact since just 1950 has been extraordinary. Today it is a standard tool that has saved many thousands or millions of dollars for most companies or businesses of even moderate size in the various industrialized countries of the world; and its use in other sectors of society has been spreading rapidly.

3.1 Prototype example

(H&L 26)

Table 1 Data for the Wyndor Glass Co. problem

	Production time per batch (h) Product		Production time available per week (h)
Plant	*1*	*2*	
A	1	0	4
B	0	2	12
C	3	2	18
Profit per batch	3 000	5 000	

x_1 = number of batches of product 1 produced per week
x_2 = number of batches of product 2 produced per week
Z = total profit per week (in \$1000) from producing these two
    products

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4 Solving Linear Programming problems: the simplex method
(H&L 103)

4.2 Setting up the Simplex Method
(H&L 108)

Original form of the model ("s.t.", subject to):

$$[\max]\, z = 3x_1 + 5x_2$$

s.t.

{1}
$$
\begin{aligned}
x_1 &\ \ \qquad \le 4 \\
2x_2 &\ \le 12 \\
3x_1 + 2x_2 &\ \le 18
\end{aligned}
$$

and $x_1 \ge 0$, $x_2 \ge 0$.

Augmented form of the model:

$$[\max]\, z = 3x_1 + 5x_2 + 0x_3 + 0x_4 + 0x_5$$

s. to

{2}
$$
\begin{aligned}
x_1 \ \ \qquad + x_3 \qquad\qquad &= 4 \\
2x_2 \ \qquad + x_4 \ \qquad &= 12 \\
3x_1 + 2x_2 \ \qquad\qquad + x_5 &= 18
\end{aligned}
$$

and $x_i \ge 0$, $i = 1..5$. The new variables x_i, $i = 3..5$, are the "slack variables".

       A **basic solution** has the following properties:
1. Each variable is designated as either a nonbasic variable or a basic variable.
2. The number of basic variables equals the number of functional constraints (now equations). Therefore, the number of nonbasic variables equals the total number of variables *minus* the number of functional constraints.
3. The **nonbasic variables** are set equal to zero.
4. The values of the **basic variables** are obtained as the simultaneous solution of the system of equations (functional constraints in augmented form). The set of basic variables is often referred to as **the basis**.
5. If the basic variables satisfy the *nonnegativity constraints*, the basic solution is a **basic feasible** (BF) **solution**.

Adjacent solutions:
    Two BF solutions are **adjacent** if all but one of their nonbasic variables are the same (so *all but one* of their *basic variables* also are the same, although perhaps with different numerical values).

Consequently, moving from the current BF solution to an adjacent one involves switching one variable from nonbasic to basic and vice versa for one other variable (and then adjusting the values of the basic variables to continue satisfying the system of equations).

       It is convenient for the simplex method to rewrite the problem in the following equivalent way:

$$\text{Maximize } Z$$

subject to

{3}

$$
\begin{array}{llllll}
(0) & Z = & 3x_1 & +5x_2 & +0x_3 & +0x_4 & +0x_5 \\
(1) & & x_1 & & +x_3 & & & =4 \\
(2) & & & 2x_2 & & +x_4 & & =12 \\
(3) & & 3x_1 & +2x_2 & & & +x_5 & =18
\end{array}
$$

The model for the Wyndor Glass Co. problem fits *our standard form*, and all its functional constraints have nonnegative right-hand sides b_i. If this had not been the case, then additional adjustments would have been needed (see later).

4.3 The algebra of the Simplex Method
(H&L 111)

Initialization
The choice of x_1 and x_2 to be the nonbasic variables (the variables set equal to zero) for the initial BF solution is obvious (and it will always be so). This choice eliminates the work required to solve for the *basic variables* (x_3, x_4, x_5).

{4}

$$
\begin{array}{lllll}
(1) & x_1 & & +x_3 & & =4 \\
(2) & & 2x_2 & & +x_4 & =12 \\
(3) & 3x_1 & +2x_2 & & +x_5 & =18
\end{array}
$$

$$x_1 = 0 \quad x_2 = 0 \quad x_3 = 4 \quad x_4 = 12 \quad x_5 = 18$$

Optimality test
The objective function is

{5}
$$Z = 3x_1 + 5x_2 + 0x_3 + 0x_4 + 0x_5 = \begin{bmatrix} 3 & 5 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 4 \\ 12 \\ 18 \end{bmatrix} = 0$$

As none of the basic variables (x_3, x_4, x_5) have a *nonzero* coefficient in this objective function, the coefficients of each nonbasic variable (x_1, x_2) gives the rate of improvement —i.e., $\dfrac{\partial Z}{\partial x_1}$ and $\dfrac{\partial Z}{\partial x_2}$ — in Z if that variable were to be increased from zero (with adjustments in the basic variables). These rates of improvement are *positive*. Therefore, this solution is not optimal.

Determining the direction of movement (Step 1 of an iteration)
The choice of which nonbasic variable is increased is as follows:

{6}
$$Z = 3x_1 + 5x_2$$

Increase x_1 ?	Rate of improvement in $Z = \dfrac{\partial Z}{\partial x_1} = 3$
Increase x_2 ?	Rate of improvement in $Z = \dfrac{\partial Z}{\partial x_2} = 5$

$5 > 3$, so choose x_2 to increase.

We call x_2 the *entering variable* for iteration 1 (it is entering the basis). (For it to enter, another must leave… Mnemonic: the *struggle* in Darwin's book[1].)

Determining where to stop (Step 2 of an iteration)

Increasing the entering variable x_2 increases Z, so we want to go as far as possible without leaving the feasibility region. The requirement to satisfy the functional constraints in augmented form means that increasing x_2 (while keeping nonbasic $x_1 = 0$) changes the values of some of the basic variables as shown on the right.

$$x_1 = 0, \text{ so}$$

$$\{7\}\qquad \begin{aligned} &(1)\quad x_1 && + x_3 && && = 4 && x_3 = 4 - 0x_2 \\ &(2)\quad && 2x_2 && + x_4 && = 12 && x_4 = 12 - 2x_2 \\ &(3)\quad 3x_1 && + 2x_2 && && + x_5 = 18 && x_5 = 18 - 2x_2 \end{aligned}$$

The other requirement for feasibility is that all the variables be *nonnegative*. The nonbasic variables (including the entering variable) are nonnegative, but we need to check **how far** x_2 can be increased without violating the nonnegativity constraints for the basic variables.

$$\{8\}\qquad \begin{aligned} x_3 &= 4 - 0x_2 \\ x_4 &= 12 - 2x_2 \\ x_5 &= 18 - 2x_2 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} x_2 &\le \infty \\ x_2 &\le 12/2 = 6 \leftarrow \min \\ x_2 &\le 18/2 = 9 \end{aligned}$$

Thus, x_2 can be increased just to 6, at which point x_4 has dropped to 0. Increasing x_2 beyond 6 would cause x_4 to become negative.

These calculations are referred to as the **minimum ratio test**.

At any iteration of the simplex method, Step 2 uses the *minimum test ratio* to determine which basic variable drops to zero first as the entering variable is increased. Decreasing this basic variable to zero will convert it to a *nonbasic variable* for the next BF solution. Therefore, this variable is called the **leaving variable** for the current iteration (because it is leaving the basis).

Thus, x_4 is the leaving variable for iteration 1 of the example.

Solving for the new BF solution (Step 3 of an iteration)

Increasing $x_2 = 0$ to $x_2 = 6$ moves us from the *initial* BF solution on the left to the *new* BF solution on the right:

[1] DARWIN, Charles, 1859, "On the origin of species by means of natural selection, or the preservation of favoured races in the struggle for life", ed. John Murray, London [or 2006, Dover, Mineola, NY (USA)].

	Initial BF solution	New BF solution
Nonbasic variables	$x_1 = 0,$ $x_2^{\text{E}} = 0$	$x_1 = 0,$ $x_4 = 0$
Basic variables	$x_3 = 4,$ $x_4^{\text{L}} = 12,$ $x_5 = 18$	$x_3 = ?,$ $x_2 = 6,$ $x_5 = ?$

The purpose of Step 3 is to convert the system of equations to a more convenient form (proper form from Gaussian elimination) for conducting the optimality test and (if needed) the next iteration with this new BF solution. In the process, this form also will identify the values of x_3 and x_5 for the new solution.

To solve the original system of equations for Z, x_2, x_3, and x_5, we need to perform some **elementary algebraic operations** (multiply or divide an equation by a nonzero constant; add or subtract a multiple of one equation to another equation) to reproduce the current pattern of coefficients of x_4 (0, 0, 1, 0) as the new coefficients of x_2. So, **divide Eq. (2) by 2** to obtain

{9}   (2') $$x_2 + \frac{1}{2}x_4 = 6$$

Next, add 5 times this new Eq. (2) to Eq. (0), and subtract 2 times this new Eq. (2) from Eq. (3). The resulting complete new system of equations is

{10}
$$
\begin{aligned}
(0)\quad & Z = 30 & +3x_1 & +0x_2 & +0x_3 & -\frac{5}{2}x_4 & +0x_5 & \quad \triangleright (0)-5(2') \\
(1)\quad & & x_1 & & +\underline{x_3} & & & = 4 \;\triangleright (1)-0(2') \\
(2)\quad & & & \underline{x_2} & & +\frac{1}{2}x_4 & & = 6 \;\triangleright (2') \\
(3)\quad & & 3x_1 & & & -x_4 & +\underline{x_5} & = 6 \;\triangleright (3)-2(2')
\end{aligned}
$$

Since $x_1 = 0$ and $x_4 = 0$, the equations in this form immediately yield the new BF solution, $X = (0, 6, 4, 0, 6)$, which yields $Z = 30$.

If the columns are written in "another" order everywhere, the identity matrix is —as promised— again recognized:

{11a}
$$
z - 30 = \begin{bmatrix} 3 & -\dfrac{5}{2} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \\ x_3 \\ x_2 \\ x_5 \end{bmatrix} = 0
$$

{11b}
$$
\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1/2 & 0 & 1 & 0 \\ 3 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \\ x_3 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}
$$

This procedure for obtaining the simultaneous solution of a system of linear equations is called the *Gauss-Jordan method of elimination*, or **Gaussian elimination** for short. The key concept for this method is the use of elementary algebraic operations to reduce the original system of equations to proper form from Gaussian elimination, where each basic variable has been eliminated from all but one equation (*its* equation) and has a coefficient of +1 in that equation.

Optimality test for the new BF solution

The current Eq. (0) gives the value of the objective function in terms of just the current nonbasic variables

{12}
$$Z = 30 + \begin{bmatrix} 3 & -\dfrac{5}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix}$$

Because x_1 has a positive coefficient, increasing x_1 would lead to an adjacent BF solution that is better than the current BF solution, so the current solution is not optimal.

Iteration 2

Since it is $Z = 30 + 3x_1 - \dfrac{5}{2}x_4$, Z can be increased by increasing x_1, but not x_4. Therefore, Step 1 chooses x_1 to be the entering variable.

For Step 2, the current system of equations yields the following conclusions about how far x_1 can be increased (with $x_4 = 0$):

{13}
$$
\begin{array}{lll}
x_3 = 4 - x_1 \ge 0 & & x_1 \le 4/1 = 4 \\
x_2 = 6 - 0x_1 \ge 0 & \Rightarrow & x_1 \le 6/0 = \infty \\
x_5 = 6 - 3x_1 \ge 0 & & x_1 \le 6/3 = 2 \leftarrow \min
\end{array}
$$

Therefore, the minimum ratio test indicates that x_5 is the leaving variable.

For Step 3, with x_1 replacing x_5 as a basic variable, we perform elementary algebraic operations on the current system of equations to reproduce the current pattern of coefficients of x_5 (0, 0, 0, 1) as the new coefficients of x_1. This yields the following new system of equations:

{14}
$$
\begin{array}{llllllll}
(0) & Z & +0x_1 & +0x_2 & +0x_3 & +\dfrac{3}{2}x_4 & +x_5 & = 36 \\[2mm]
(1) & & & & +x_3 & +\dfrac{1}{3}x_4 & -\dfrac{1}{3}x_5 & = 2 \\[2mm]
(2) & & & x_2 & & +\dfrac{1}{2}x_4 & & = 6 \\[2mm]
(3) & & x_1 & & & -\dfrac{1}{3}x_4 & +\dfrac{1}{3}x_5 & = 2
\end{array}
$$

Therefore, the next BF solution is $X = (2, 6, 2, 0, 0)$, yielding $Z = 36$. To apply the optimality test to this new BF solution, we use the current Eq. (0) to express Z in terms of just the current nonbasic variables,

{15}
$$Z = 36 + \begin{bmatrix} -\dfrac{3}{2} & -1 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \end{bmatrix}$$

Increasing either x_4 or x_5 would *decrease* Z, so neither adjacent BF solution is as good as the current one. Therefore, the current BF solution must be **optimal**.

In terms of the original form of the problem (no slack variables), the optimal solution is $x_1 = 2$, $x_2 = 6$, which yields $Z = 36$.

The next section shows a more convenient tabular form.

4.4 The Simplex Method in tabular form

(H&L 117)

The tabular form of the simplex method records only the essential information, namely, (1) the coefficients of the variables, (2) the constants on the right-hand sides of the equations, and (3) the basic variable appearing in each equation.

Table 3 compares the initial, algebraic form with a new, tabular form.

Summary of the simplex method (and iteration 1 for the Example)

INITIALIZATION: Introduce slack variables. Select the *decision variables* to be the *initial nonbasic variables* (set equal to zero). And the *slack variables* to be the *initial basic variables*. (Adjust if the model is not in our standard form: maximization, only ≤ functional constraints, all nonnegativity constraints, and if any b_i values are negative.)

For the Example: the initial BF solution is (0, 0, 4, 12, 18).

Table 3a Initial system of equations for the Wyndor Glass Co. problem
in **algebraic form**

$$
\begin{array}{llllllll}
(0) & Z & -3x_1 & -5x_2 & +0x_3 & +0x_4 & +0x_5 & = 0 \\
(1) & & x_1 & & +x_3 & & & = 4 \\
(2) & & & 2x_2 & & +x_4 & & = 12 \\
(3) & & 3x_1 & +2x_2 & & & +x_5 & = 18
\end{array}
$$

Table 3b Initial system of equations for the Wyndor Glass Co. problem
in **tabular form**

Basic variable	Eq.	Coefficient of						Right side
		Z	x_1	x_2	x_3	x_4	x_5	
Z	(0)	1	–3	–5	0	0	0	0
x_3	(1)	0	1	0	1	0	0	4
x_4	(2)	0	0	2	0	1	0	12
x_5	(3)	0	3	2	0	0	1	18

OPTIMALITY TEST: The current BF solution is optimal if and only if *every* coefficient in row 0 is nonnegative (≥ 0). If it is, stop; otherwise, go to an iteration to obtain the next BF solution, which involves changing one nonbasic variable to a basic variable (Step 1) and vice versa (Step 2) and then solving for the new solution (Step 3).

For the Example: just as $Z = 3x_1 + 5x_2$ indicates that increasing either x_1 or x_2 will increase Z, so the current BF solution is not optimal, the same conclusion being drawn from the equation $Z - 3x_1 - 5x_2 = 0$. These coefficients of –3 and –5 are shown in row 0 of Table 3b.

ITERATION

Step 1:

Determine the *entering variable* by selecting the variable (automatically nonbasic variable) with the *negative coefficient* having the largest absolute value (i.e.,

the "most negative" coefficient) in Eq. (0). Mark the column below this coefficient, and call this the **pivot column**.

 For the Example: the most negative coefficient is –5 for x_2 (5 > 3), so x_2 is to be changed to a basic variable. (This change is indicated in Table 4 by marking the x_2 column below –5.)

Table 4 Applying the minimum ratio test to determine the first leaving basic
   variable for the Wyndor Glass Co. problem

Basic variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right side	Ratio
Z	(0)	1	–3	–5	0	0	0	0	
x_3	(1)	0	1	0	1	0	0	4	∞
x_4	(2)	0	0	2	0	1	0	12	**6**
x_5	(3)	0	3	2	0	0	1	18	9

Step 2:

 Determine the *leaving basic variable* by applying the minimum ratio test. Mark its row and call it the **pivot row**. Also call the number that is in both cases the **pivot number**.

Table 5 Simplex tableaux for the Wyndor Glass Co. problem after the
   first pivot row is divided by the first pivot number

Basic variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right side
Z	(0)	1	–3	–5	0	0	0	0
x_3	(1)	0	1	0	1	0	0	4
x_4	(2)	0	0	2	0	1	0	12
x_5	(3)	0	3	2	0	0	1	18
Z	(0)	1						
x_3	(1)	0						
x_4	(2)	0	0	1	0	½	0	6
x_5	(3)	0						

 For the Example: the calculations for the minimum ratio test are shown at the right of Table 4. Thus, row 2 is the pivot row (shown in the first half of Table 5), and x_4 is the leaving basic variable. In the next simplex tableau (in Table 5), x_2 replaces x_4 as the basic variable for row 2.

Step 3:

 Solve for the *new BF solution* by using **elementary row operations** to construct a new simplex tableau in proper form from Gaussian elimination below the current one, and then return to the optimality test.

 For the Example: since x_2 is replacing x_4 as a basic variable, we need to reproduce the first tableau's pattern of coefficients in the column of x_4 (0, 0, 1, 0) in the second tableau's column of x_2. To start, divide the pivot row (row 2) by the pivot number (2), which gives the new row 2 shown in Table 5. Next, we add to row 0 the product, 5 times the new row 2. Then we substract from row 3 the product, 2 times the new row 2 (or equivalently, subtract from row 3 the *old* row 2). These calculations yield the new tableau shown in Table 6 for iteration 1. Thus, the new BF solution is (0, 6, 4, 0, 6), with $Z = 30$. We next return to the optimality test to check if

the new BF solution is optimal. Since the new row 0 still has a negative coefficient (-3 for x_1), the solution is not optimal, and so at least one more iteration is needed.

Table 6 Second simplex tableau for the Wyndor Glass Co. problem

Basic variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right side
Z	(0)	1	–3	0	0	5/2	0	30
x_3	(1)	0	1	0	1	0	0	4
x_2	(2)	0	0	1	0	1/2	0	6
x_5	(3)	0	3	0	0	–1	1	6

Iteration 2 for the Example

The second iteration starts anew from the second tableau of Table 6 to find the next BF solution. Following the instructions for Steps 1 and 2, we find x_1 as the entering basic variable and x_5 as the leaving basic variable, as shown in Table 7.

Table 7 Steps 1 and 2 of Iteration 2 for the Wyndor Glass Co. problem

Basic variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right side	Ratio
Z	(0)	1	–3	0	0	5/2	0	30	
x_3	(1)	0	1	0	1	0	0	4	4
x_2	(2)	0	0	1	0	1/2	0	6	∞
x_5	(3)	0	3	0	0	–1	1	6	2

(See Table 8 for the whole process.)

Table 8 Simplex tableaux for the Wyndor Glass Co. problem

Basic variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right side	Ratio
Z	(0)	1	–3	**–5**	0	0	0	0	
x_3	(1)	0	1	0	1	0	0	4	∞
x_4	(2)	0	0	**2**	0	1	0	12	6
x_5	(3)	0	3	2	0	0	1	18	9
Z	(0)	1	**–3**	0	0	5/2	0	30	
x_3	(1)	0	1	0	1	0	0	4	4
x_2	(2)	0	0	1	0	1/2	0	6	∞
x_5	(3)	0	**3**	0	0	–1	1	4	4/3
Z	(0)	1	0	0	0	3/2	**1**	36	
x_3	(1)	0	0	0	1	1/3	–1/3	2	
x_2	(2)	0	0	1	0	1/2	0	6	
x_1	(3)	0	1	0	0	–1/3	1/3	2	

4.5 Tie breaking in the Simplex Method
(H&L 121)

Tie for the entering basic variable

The selection between two or more nonbasic variables having the largest coefficients is *arbitrary*. The optimum will be reached eventually, although the path

(and the number of iterations) will be different. There is no convenient method for predicting which choice will lead there sooner.

Tie for the leaving basic variable —degeneracy
        If two or more basic variables tie for being the leaving basic variable in an iteration, does it matter which one is chosen? Theoretically it does. First, all the tied variables reach zero simultaneously as the entering basic variable is increased. Therefore, the one or ones *not* chosen to be the leaving basic variable also will have a value of zero in the new solution. Note that basic variables with a value of *zero* are called **degenerate**, and the same term is applied to the correspondent solution. Second, if one of these degenerate basic variables retains its value of zero until it is chosen as leaving at a subsequent iteration, the corresponding entering variable also must remain zero, so the value of Z will remain unchanged. Third, if Z may remain the same rather than increase at each iteration, the simplex method may then go around in a loop. In fact, examples have been artificially constructed so that they do become entrapped in just such a perpetual loop.
        Fortunately, although a perpetual loop is theoretically possible, it has rarely been known to occur in practice. If a loop were to occur, one could always get out by changing the choice of the leaving basic variable. Furthermore, special rules have been constructed for breaking ties so that such loops are always avoided. However, these rules frequently are ignored in actual application.

No leaving basic variable —unbounded Z
        If the smallest nonnegative ratio does not exist, the solution for the objective function is unbounded (infinite). Because even linear programming has not discovered a way of making infinite profit, the real message for practical problems is that a mistake has been made! The model probably has been misformulated either by omitting relevant constraints or by stating them incorrectly. Alternatively, a computational mistake may have occurred.
        In the matrix method of resolution (to be seen later), let it be said that, even making some mistakes during the resolution of a linear programming problem, the optimum will be reached.

Multiple optimal solutions
        If the last solution shows a zero in the *final* row 0, at least one of the nonbasic variables has a coefficient of zero, and so increasing any such variable will not change the value of Z. Therefore, the other optimal solutions can be identified (if desired) by performing additional iterations, each time choosing a nonbasic variable with a zero coefficient as the entering variable. (An unbounded solution with the same Z can occur.)
        If there are more than one solution, any weighted average of two or more solutions is called a **linear convex combination** and is also an optimal solution. (There are, thus, infinite such solutions.)

4.6 Adapting to other model forms
(H&L 124)

Equality constraints
        Instead of replacing an equality ($f(\mathbf{x}) = b$) by two complementary inequalities ($f(\mathbf{x}) \le b, \quad f(\mathbf{x}) \ge b$), which would increase the number of constraints, it is more

convenient to use the artificial-variable technique, attributing to this type of variable an "infinite" coefficient, usually called *M* —really, a very large number— whose sign will be opposite to the direction of the optimization: –*M* for maximization and +*M* for minimization.

When the artificial-variable technique is used, the presence of one or more artificial variables in the final solution means that the problem indeed has no solution. This situation implies no complication, so any problem "without" solution may be treated just like any other.

Negative right-hand sides

The usual solution is to multiply both sides by –1 and reverse the direction of the inequality. An artificial variable will normally be needed.

Variables allowed to be negative (free variables)

Any variable allowed to be negative, or "free" variable, say x_j, can be replaced by the difference of two nonnegative ones. A usual notation (not adopted later) is

{16}
$$x_j = x_j^+ - x_j^-$$

Every occurrence of x_j will, then, be replaced by this expression, including, of course, the objective function.

III. Epilogue

The *simplex method* for solving Linear Programming problems was presented, attempting to show its basic idea and features, from an essentially practical standpoint. Special cases, easily reduced to the standard form, were also addressed.

IV. Acknowledgements

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