A “scientific application” (!) of Linear Programming

In Ecker & Kupferschmid [1988], Ch. 2, “LP models and applications”, 2.3, “Some scientific applications of LP”, pp 24–25; example problem from Guttman *et al.* [1982], Ch. 15, “Regression analysis”, 15.5, An example, pp 361–365

A study was instituted to determine the percent of waste solids removed in a filtration system as a function of flow rate, x, of the effluentbeing fed into the system. It was decided to use flow rate of 2 (2) 14 gal/min and to observe y^e (“experimental”), the percent of waste solid removed,when each of these flow rates was used. The study yielded the data displayed in Table 1.

The mathematical model $E(y|x) = ax + b$ was proposed.

Find the parameters, a and b, of the model [Guttman, *et al.,* 1982, p 361].

Table 1

i	x	y^e
1	2	24,3
2	4	19,7
3	6	17,8
4	8	14,0
5	10	12,3
6	12	7,2
7	14	5,5

→ Resolution

a) Classical solution

(We will use only points *1*, *4* and *7* of Table 1. With all the points, the source cited gives $\hat{y}=26{,}81-1{,}55x$, “in the sense of least squares”.)

As is well known, the parameters of the problem are obtained minimizing a sum of errors (squared, for convenience), of the form

$$z = \sum_{i=1}^{n} \left(y_i - y_i^e \right)^2 \qquad \{1\}$$

with

- z – measure (a sum) of the n errors. ([z] = ψ^2, see below)
- n – number of experiments
- y_i – theoretical (or “calculated”) value, $y = ax + b$, of the measured variable, corresponding to x_i (i integer, i = 1..n)
- a, b – process parameters. (With χ and ψ the dimensions of x and y, respectively, it is [a] = $\psi\chi^{-1}$ and [b] = ψ.)
- y_i^e – experimental value (a constant, thus) of the measured variable, corresponding to x_i

So, z [1] is a function of only a and b, whose minimum is easy to find by differentiation,

[1] The use of $\sqrt{z}$, as may be concluded, would be more logical, although indifferent from the viewpoint of optimization.

giving for these parameters, as is known,

$$\begin{bmatrix} \hat{a} \\ \\ \hat{b} \end{bmatrix} = \begin{bmatrix} \dfrac{\sum (x_i - \bar{x}) y_i^e}{\sum (x_i - \bar{x})^2} \\ \\ \bar{y}^e - \hat{a}\bar{x} \end{bmatrix} \qquad \{2\}$$

while the optimum of z is not relevant.

Table 2

i	x_i	y_i^e	$x_i - \bar{x}$	$(x_i - \bar{x}) y_i^e$	$(x_i - \bar{x})^2$
1	**2**	**24,3**	-6	-145,8	36,00
2	4	19,7			
3	6	17,8			
4	**8**	**14,0**	0	0,0	0,00
5	10	12,3			
6	12	7,2			
7	**14**	**5,5**	6	33	36,00
Sum	24	43,8	(0)	-112,8	72,00
Average	$\bar{x} = 8$	$\bar{y} = 14,6$			

From Table 2, for the points selected, it is

$$\hat{a} = -1{,}5(6) \text{ (\% removed) / (gal/min)}$$
$$\hat{b} = 27{,}1(3) \text{ (\% removed)}$$

(These values are near the values reported for the 7 points, $\hat{a} = -1{,}55$ and $\hat{b} = 26{,}81$.)

The calculated y's [2] give: $y_1 = 24$ ("low", *vs.* 24,3), $y_4 = 14{,}6$ ("high", *vs.* 14,0), $y_7 = 5{,}2$ ("low", *vs.* 5,5).

b) Solution by Linear Programming

(We use the same points, *1*, *4* e *7*, from Table 1.)

We propose, now, to obtain the parameters a and b, of the same model, by another criterion: make the "errors" or deviations be small individually (not as a sum). To this end, we will try to minimize the "worst" (largest) deviation, i.e.,

$$[\min] \quad z = \max_i (|d_i|) \qquad \{3\}$$

with

$$d_i = y_i^e - y_i \qquad \{4\}$$

or, since (in this case) it is $y = ax + b$,

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[2] The plural in the form "x's" seems appropriate (better than "xx").

$$d_i = y_i^e - ax_i - b \qquad \{5\}$$

Remember that any value of z depends only of a and b, as all other values are constants (experimental values). (This time, the physical dimensions of z are, of course, the same as those of the measured variable, y.)

As z must be the ***maximum*** deviation, it has, equivalently, to satisfy each of the following inequalities

$$z \geq \left| y_i^e - ax_i - b \right| \qquad i = 1..n \qquad \{6\}$$

with n the number of points.

The problem becomes, thus, to find a, b and z such that all the inequalities should be satisfied and z be as small as possible. So, we need to find

$$[\min]\ z \qquad \{7\}$$

subject to

$$z \geq \left| y_i^e - ax_i - b \right| \qquad i = 1..n \qquad \{8\}$$

Now, this problem has the disadvantage of not being a linear programming, in this form, because, of course, the *'absolute value'* [3] *of a linear function is not a linear function* [Ecker *et al.,* 1988]. We can, however, convert it into a linear program through the following elementary fact:

For any z and w, $z \geq \lvert w \rvert$ iff $z \geq w$ and $z \geq -w$.

So, let us replace each (non-linear) inequality by two linear inequalities, to get a linear program:

$$[\min]\ z \qquad \{9\}$$

subject to

$$z \geq +\left(y_i^e - ax_i - b\right) \qquad i = 1..n \qquad \{10a\}$$

$$z \geq -\left(y_i^e - ax_i - b\right) \qquad i = 1..n \qquad \{10b\}$$

$$z \geq 0; \quad a, b\text{: of free sign}$$

As is known, a and b can be replaced by *differences of non-negative variables*, say, $a' - a''$ e $b' - b''$. Incidentally, as we have (possibly good) approximations of the optimum values of a and b, from the previous section, we can simply just replace a by $-a'$ (a' non-negative) —an artifice that must be verified in the end (and which would be under suspicion in case we obtained the boundary value $a' = 0$).

The problem then becomes:

$$[\min]\ z \qquad \{11\}$$

subject to

[3] Or "modulus".

$$x_i a' - b + z \geq -y_i^e \qquad i = 1..n \qquad \{12a\}$$

$$-x_i a' + b + z \geq y_i^e \qquad i = 1..n \qquad \{12b\}$$

or, finally, introducing the numerical values,

$$[\min]\ z \qquad \{13\}$$

subject to

$$\begin{aligned} 2a' - b + z &\geq -24{,}3 \\ 8a' - b + z &\geq -14{,}0 \\ 14a' - b + z &\geq -5{,}5 \\ -2a' + b + z &\geq 24{,}3 \\ -8a' + b + z &\geq 14{,}0 \\ -14a' + b + z &\geq 5{,}5 \end{aligned} \qquad \{14\}$$

In matrix form, it is

$$\begin{array}{lccl} [\min] & w & = & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to:} & & \mathbf{A}\mathbf{x} & \geq \mathbf{b} \\ & & \mathbf{x} & \geq \mathbf{0} \end{array} \qquad \{15\}$$

with

$$\mathbf{x} = \begin{bmatrix} a' & b & z \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{c}^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 8 & -1 & 1 \\ 14 & -1 & 1 \\ -2 & 1 & 1 \\ -8 & 1 & 1 \\ -14 & 1 & 1 \end{bmatrix} \mathbf{b} = \begin{bmatrix} -24{,}3 \\ -14{,}0 \\ -5{,}5 \\ 24{,}3 \\ 14{,}0 \\ 5{,}5 \end{bmatrix} \qquad \{16\}$$

i) **Direct resolution**

The problem, as just formulated, has 3 structural variables and 6 constraints. Its manual resolution, thus, faces the practically unfeasible handling of square matrices of order 6, among others. The computer resolution took 5 iterations and gave (as structural variables and objective function):

$$\begin{bmatrix} a' \\ b \\ z \end{bmatrix} = \begin{bmatrix} 1{,}56667 \\ 26{,}9833 \\ 0{,}45 \end{bmatrix} \qquad (z = 0{,}45) \qquad \{17\}$$

So, we have $a = -a' = -1{,}56667$ and $b = 26{,}9833$. Notice that it is $a' \neq 0$ (and, inevitably, $a' > 0$), as expected, which validates the hypothesis made to ease the calculations, so this result is not "suspect".

It is not evident whether this set (a, b) is better or worse than the former (otherwise, it happens that one of the values coincides), a fact that depends on the finalities.

ii) Resolution by the *dual* (brief note)

The LP problems can be grouped in pairs, where one of the problems is the ***primal*** and the other the ***dual*** —an assignment that is arbitrary, although usually the primal corresponds to the original problem. Duality —present in various areas of Mathematics— is important both theoretically and practically in LP, as both problems yield an ***identical optimum*** (if it exists) for the **objective function**. Moreover: indeed, in the complete solution of one of the problems, the complete solution of the other can be read, with the advantage that, frequently, **one of them is computationally** (much) **less difficult**.

The relationship between primal and dual, explored in the LP literature, may be shortly presented as follows, conveniently for theory and application:

Primal	**Dual**
$[\min] \quad z = \mathbf{c}^{\mathrm{T}}\mathbf{x}$ subject to: $\quad \mathbf{Ax} \geq \mathbf{b}$ $\mathbf{x} \geq \mathbf{0}$	$[\max] \quad z = \mathbf{b}^{\mathrm{T}}\mathbf{y}$ subject to: $\quad \mathbf{A}^{\mathrm{T}}\mathbf{y} \leq \mathbf{c}$ $\mathbf{y} \geq \mathbf{0}$

The case under study corresponds to the classification above; in other cases, the descriptions under the titles *primal* and *dual* would be exchanged.

Among other properties, it can be proved that:

- If one of the problems has an optimum vector (solution), then the other also has one, and the optimum objective function is identical.
- If one of the problems is possible but has no finite optimum, then the other is impossible.
- Both problems can be impossible.
- The optimum vector for the *maximization* has its elements equal to the coefficients of the slack variables of the optimum basis of the *minimization*, and reciprocally.[4]

Therefore, starting from the original problem under study, which has 3 structural variables and 6 constraints (two per each experimental point), its dual can be

[4] ***Caution:*** different authors use "equal to the *symmetric* of the coefficients (...)".

constructed, having 6 structural variables and only 3 constraints. So, in this case, the dual ***(a)*** evolves by much easier iterations (matrices of order 3, not 6), and ***(b)*** will be, expectedly, less cumbersome, as it will yield about half the iterations (about proportional to 3, not 6). Using the dual would still allow to easily consider *all the experimental points*, even if more numerous, as the number of iterations till the optimum depends essentially on the number of constraints.

The dual would be:[5]

$$[\max]\ (w =) - 24{,}3 s_1 - 14{,}0 s_2 - 5{,}5 s_3 + 24{,}3 s_4 + 14{,}0 s_5 + 5{,}5 s_6 \qquad \{18\}$$

subject to

$$\begin{bmatrix} 2 & 8 & 14 & -2 & -8 & -14 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \{19\}$$

The result, in 4 iterations (instead of 5), is (of course)

$$z = 0{,}45 \qquad \{20\}$$

and contains —in its so-called *dual variables*— the values

$$\boldsymbol{\Delta} = \begin{bmatrix} -1{,}567 & -26{,}98 & -0{,}45 \end{bmatrix}^{\mathrm{T}} \qquad \{21\}$$

Consequently, this vector (always *negative* —i.e., non-positive— in the optimum of a *maximization*, of course) has as elements the *symmetrical* of the results (a', b, z) of the primal, already known.

←

References

– ECKER, Joseph G., Michael KUPFERSCHMID, 1988, “Introduction to Operations Research”, John Wiley & Sons, New York, NY (USA), ISBN 0-471-63362-3.

– GUTTMAN, Irwin, Samuel S. WILKS, J. Stuart HUNTER, 1982, “Introduction to Engineering Statistics”, 3.rd ed., John Wiley & Sons, New York, NY (USA), ISBN 0-471-86956-2.

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[5] *Warning:* this equation had wrong (opposite) signs in the previous edition.