SCHAUM'S OUTLINE SERIES

THEORY AND PROBLEMS OF

OPERATIONS RESEARCH

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INCLUDING 310 SOLVED PROBLEMS

SCHAUM'S OUTLINE SERIES IN ENGINEERING

McGRAW-HILL BOOK COMPANY

INTEGER PROGRAMS

An *integer program* is a linear program with the additional restriction that the input variables be integers. It is not necessary that the coefficients in (*1.2*) and (*1.3*), and the constants in (*1.1*), also be integers, but this will very often be the case.

QUADRATIC PROGRAMS

A *quadratic program* is a mathematical program in which each constraint is linear—that is, each constraint function has the form (*1.3*)—but the objective is of the form

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j + \sum_{i=1}^{n} d_i x_i \tag{1.4}$$

where c_{ij} and d_i are known constants.

The program given in Example 1.1 is quadratic. Both constraints are linear, and the objective has the form (*1.4*), with $n = 2$ (two variables), $c_{11} = 1$, $c_{12} = c_{21} = 0$, $c_{22} = 1$, and $d_1 = d_2 = 0$.

PROBLEM FORMULATION

Optimization problems most often are stated verbally. The solution procedure is to model the problem with a mathematical program and then solve the program by the techniques described in Chapters 2 through 15. The following approach is recommended for transforming a word problem into a mathematical program:

STEP 1 Determine the quantity to be optimized and express it as a mathematical function. Doing so serves to define the input variables.

STEP 2 Identify all stipulated requirements, restrictions, and limitations, and express them mathematically. These requirements constitute the constraints.

STEP 3 Express any hidden conditions. Such conditions are not stipulated explicitly in the problem but are apparent from the physical situation being modeled. Generally they involve nonnegativity or integer requirements on the input variables.

SOLUTION CONVENTION

In any mathematical program, we seek *a* solution. If a number of equally optimal solutions exist, then any one will do. *There is no preference between equally optimal solutions if there is no preference stipulated in the constraints.*

Solved Problems

1.1 The Village Butcher Shop traditionally makes its meat loaf from a combination of lean ground beef and ground pork. The ground beef contains 80 percent meat and 20 percent fat, and costs the shop 80¢ per pound; the ground pork contains 68 percent meat and 32 percent fat, and costs 60¢ per pound. How much of each kind of meat should the shop use in each pound of meat loaf if it wants to minimize its cost and to keep the fat content of the meat loaf to no more than 25 percent?

The objective is to minimize the cost (in cents), z, of a pound of meat loaf, where

$z = 80$ times the poundage of ground beef used plus 60 times the poundage of ground pork used

Defining

x_1 = poundage of ground beef used in each pound of meat loaf
x_2 = poundage of ground pork used in each pound of meat loaf

we express the objective as

$$\text{minimize:} \quad z = 80x_1 + 60x_2 \tag{1}$$

Each pound of meat loaf will contain $0.20x_1$ pound of fat contributed from the beef and $0.32x_2$ pound of fat contributed from the pork. The total fat content of a pound of meat loaf must be no greater than 0.25 lb. Therefore,

$$0.20x_1 + 0.32x_2 \le 0.25 \tag{2}$$

The poundages of beef and pork used in each pound of meat loaf must sum to 1; hence,

$$x_1 + x_2 = 1 \tag{3}$$

Finally, the butcher shop may not use negative quantities of either meat, so that two hidden constraints are $x_1 \ge 0$ and $x_2 \ge 0$. Combining these conditions with (1), (2), and (3), we obtain

$$\begin{aligned} \text{minimize:} \quad & z = 80x_1 + 60x_2 \\ \text{subject to:} \quad & 0.20x_1 + 0.32x_2 \le 0.25 \\ & x_1 + \quad x_2 = 1 \\ \text{with:} \quad & \text{all variables nonnegative} \end{aligned} \tag{4}$$

System (4) is a linear program. As there are only two variables, a graphical solution may be given.

1.2 Solve the linear program (4) of Problem 1.1 graphically.

See Fig. 1-1. The *feasible region*—the set of points (x_1, x_2) satisfying all the constraints, including the nonnegativity conditions—is the heavy line segment in the figure. To determine z^*, the minimal value of z, we arbitrarily choose values of z and plot the graphs of the associated objectives. By choosing $z = 70$ and then $z = 75$, we obtain the objectives

$$70 = 80x_1 + 60x_2 \qquad \text{and} \qquad 75 = 80x_1 + 60x_2$$

respectively. Their graphs are the dashed lines in Fig. 1-1. It is seen that z^* will be assumed at the upper endpoint of the feasible segment, which is the intersection of the two lines

$$0.20x_1 + 0.32x_2 = 0.25 \qquad \text{and} \qquad x_1 + x_2 = 1$$

Simultaneous solution of these equations gives $x_1^* = 7/12$, $x_2^* = 5/12$; hence,

$$z^* = 80(7/12) + 60(5/12) = 71.67¢$$

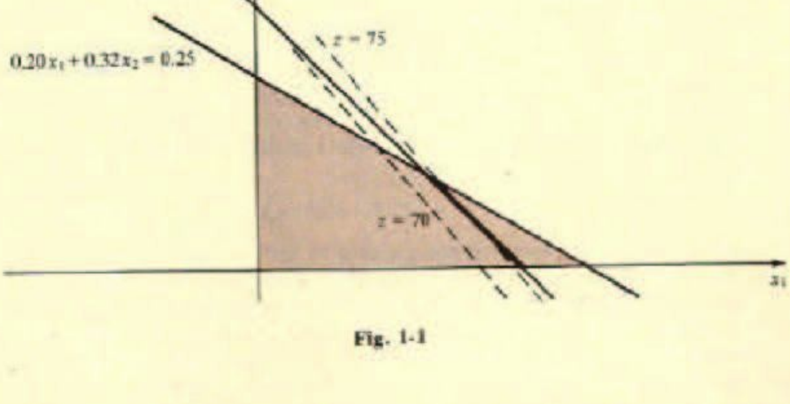

Fig. 1-1

1.3 A furniture maker has 6 units of wood and 28 h of free time, in which he will make decorative screens. Two models have sold well in the past, so he will restrict himself to those two. He estimates that model I requires 2 units of wood and 7 h of time, while model II requires 1 unit of wood and 8 h of time. The prices of the models are \$120 and \$80, respectively. How many screens of each model should the furniture maker assemble if he wishes to maximize his sales revenue?

The objective is to maximize revenue (in dollars), which we denote as z:

z = 120 times the number of model I screens produced plus 80 times the number of model II screens produced

Letting

x_1 = number of model I screens to be produced
x_2 = number of model II screens to be produced

we express the objective as

$$\text{maximize:} \quad z = 120x_1 + 80x_2 \tag{1}$$

The furniture maker is subject to a wood constraint. As each model I requires 2 units of wood, $2x_1$ units must be allocated to them; likewise, $1x_2$ units of wood must be allocated to the model II screens. Hence the wood constraint is

$$2x_1 + x_2 \le 6 \tag{2}$$

The furniture maker also has a time constraint. The model I screens will consume $7x_1$ hours and the model II screens $8x_2$ hours; and so

$$7x_1 + 8x_2 \le 28 \tag{3}$$

It is obvious that negative quantities of either screen cannot be produced, so two hidden constraints are $x_1 \ge 0$ and $x_2 \ge 0$. Furthermore, since there is no revenue derived from partially completed screens, another hidden condition is that x_1 and x_2 be integers. Combining these hidden conditions with (*1*), (*2*), and (*3*), we obtain the mathematical program

$$\begin{aligned} \text{maximize:} \quad & z = 120x_1 + 80x_2 \\ \text{subject to:} \quad & 2x_1 + x_2 \le 6 \\ & 7x_1 + 8x_2 \le 28 \\ \text{with:} \quad & \text{all variables nonnegative and integral} \end{aligned} \tag{4}$$

System (*4*) is an integer program. As there are only two variables, a graphical solution may be given.

1.4 Give a graphical solution of the integer program (*4*) of Problem 1.3.

See Fig. 1-2. The feasible region is the set of integer points (marked by crosses) within the shaded area. The dashed lines are the graphs of the objective function when z is arbitrarily given the values 240, 330, and 380. It is seen that the z-line through the point (3, 0) will furnish the desired maximum; thus, the furniture maker should assemble three model I screens and no model II screens, for a maximum revenue of

$$z^* = 120(3) + 80(0) = \$360$$

Observe that this optimal answer is *not* achieved by first solving the associated linear program (the same problem without the integer constraints) and then moving to the closest feasible integer point. In fact, the feasible region for the associated linear program is the shaded area of Fig. 1-2; so the optimal solution occurs at the circled corner point. But at the closest feasible integer point, (2, 1), the objective function has the value $z = 120(2) + 80(1) = \$320$ or \$40 less than the true optimum.

An alternate solution procedure for Problem 1.3 is given in Problem 7.8.

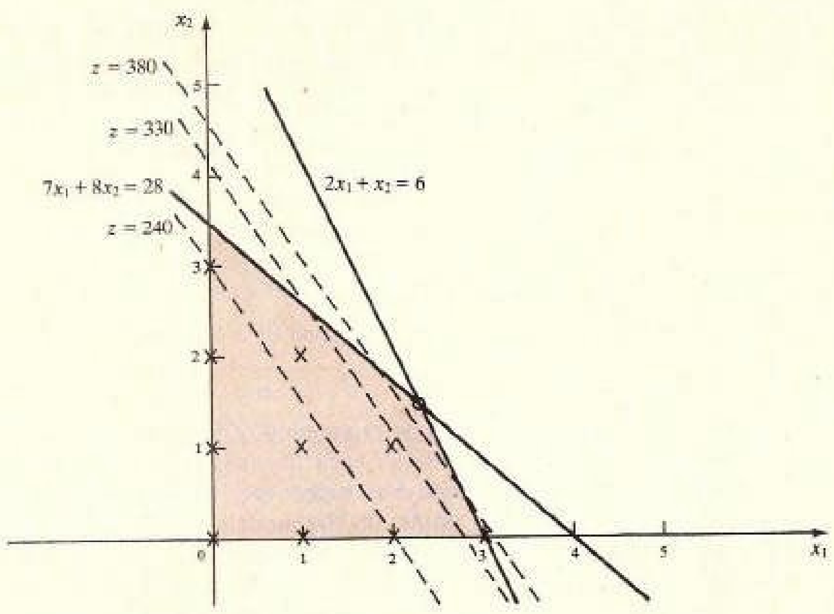

Fig. 1-2

1.5 Universal Mines Inc. operates three mines in West Virginia. The ore from each mine is separated into two grades before it is shipped; the daily production capacities of the mines, as well as their daily operating costs, are as follows:

	High-Grade Ore, tons/day	Low-Grade Ore, tons/day	Operating Cost, \$1000/day
Mine I	4	4	20
Mine II	6	4	22
Mine III	1	6	18

Universal has committed itself to deliver 54 tons of high-grade ore and 65 tons of low-grade ore by the end of the week. It also has labor contracts that guarantee employees in each mine a full day's pay for each day or fraction of a day the mine is open. Determine the number of days each mine should be operated during the upcoming week if Universal Mines is to fulfill its commitment at minimum total cost.

Let x_1, x_2, and x_3, respectively, denote the numbers of days that mines I, II, and III will be operated during the upcoming week. Then the objective (measured in units of \$1000) is

$$\text{minimize:} \quad z = 20x_1 + 22x_2 + 18x_3 \tag{1}$$

The high-grade ore requirement is

$$4x_1 + 6x_2 + x_3 \geq 54 \tag{2}$$

and the low-grade ore requirement is

$$4x_1 + 4x_2 + 6x_3 \geq 65 \tag{3}$$

As no mine may operate a negative number of days, three hidden constraints are $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$. Moreover, as no mine may operate more than 7 days in a week, three other hidden constraints are $x_1 \leq 7$, $x_2 \leq 7$, and $x_3 \leq 7$. Finally, in view of the labor contracts, Universal Mines has nothing to gain in operating a mine for part of a day; consequently, x_1, x_2, and x_3 are required to be integral. Combining the hidden conditions with (*1*), (*2*), and (*3*), we obtain the mathematical program

$$\begin{aligned} \text{minimize:}\quad & z = 20x_1 + 22x_2 + 18x_3 \\ \text{subject to:}\quad & 4x_1 + 6x_2 + x_3 \geq 54 \\ & 4x_1 + 4x_2 + 6x_3 \geq 65 \\ & x_1 \leq 7 \\ & x_2 \leq 7 \\ & x_3 \leq 7 \\ \text{with:}\quad & \text{all variables nonnegative and integral} \end{aligned} \tag{4}$$

System (*4*) is an integer program; its solution is determined in Problem 7.4.

1.6 A manufacturer is beginning the last week of production of four different models of wooden television consoles, labeled I, II, III, and IV, each of which must be assembled and then decorated. The models require 4, 5, 3, and 5 h, respectively, for assembling and 2, 1.5, 3, and 3 h, respectively, for decorating. The profits on the models are \$7, \$7, \$6, and \$9, respectively. The manufacturer has 30 000 h available for assembling these products (750 assemblers working 40 h/wk) and 20 000 h available for decorating (500 decorators working 40 h/wk). How many of each model should the manufacturer produce during this last week to maximize profit? Assume that all units made can be sold.

The objective is to maximize profit (in dollars), which we denote as z. Setting

$x_1 \equiv$ number of model I consoles to be produced in the week
$x_2 \equiv$ number of model II consoles to be produced in the week
$x_3 \equiv$ number of model III consoles to be produced in the week
$x_4 \equiv$ number of model IV consoles to be produced in the week

we can formulate the objective as

$$\text{maximize:}\quad z = 7x_1 + 7x_2 + 6x_3 + 9x_4 \tag{1}$$

There are constraints on the total time available for assembling and the total time available for decorating. These are, respectively, modeled by

$$4x_1 + 5x_2 + 3x_3 + 5x_4 \leq 30\,000 \tag{2}$$

$$2x_1 + 1.5x_2 + 3x_3 + 3x_4 \leq 20\,000 \tag{3}$$

As negative quantities may not be produced, four hidden constraints are $x_i \geq 0$ $(i = 1, 2, 3, 4)$. Additionally, since this is the last week of production, partially completed models at the week's end would remain unfinished and so would generate no profit. To avoid such possibilities, we require an integral value for each variable. Combining the hidden conditions with (*1*), (*2*), and (*3*), we obtain the mathematical program

$$\begin{aligned} \text{maximize:}\quad & z = 7x_1 + 7x_2 + 6x_3 + 9x_4 \\ \text{subject to:}\quad & 4x_1 + 5x_2 + 3x_3 + 5x_4 \leq 30\,000 \\ & 2x_1 + 1.5x_2 + 3x_3 + 3x_4 \leq 20\,000 \\ \text{with:}\quad & \text{all variables nonnegative and integral} \end{aligned} \tag{4}$$

System (*4*) is an integer program; its solution is determined in Problem 6.4.

1.7 The Aztec Refining Company produces two types of unleaded gasoline, regular and premium, which it sells to its chain of service stations for \$12 and \$14 per barrel, respectively. Both types are blended from Aztec's inventory of refined domestic oil and refined foreign oil, and must meet the following specifications:

	Maximum Vapor Pressure	Minimum Octane Rating	Maximum Demand, bbl/wk	Minimum Deliveries, bbl/wk
Regular	23	88	100 000	50 000
Premium	23	93	20 000	5 000

The characteristics of the refined oils in inventory are as follows:

	Vapor Pressure	Octane Rating	Inventory, bbl	Cost, $/bbl
Domestic	25	87	40 000	8
Foreign	15	98	60 000	15

What quantities of the two oils should Aztec blend into the two gasolines in order to maximize weekly profit?

Set

x_1 = barrels of domestic blended into regular
x_2 = barrels of foreign blended into regular
x_3 = barrels of domestic blended into premium
x_4 = barrels of foreign blended into premium

An amount $x_1 + x_2$ of regular will be produced and generate a revenue of $12(x_1 + x_2)$; an amount $x_3 + x_4$ of premium will be produced and generate a revenue of $14(x_3 + x_4)$. An amount $x_1 + x_3$ of domestic will be used, at a cost of $8(x_1 + x_3)$; an amount $x_2 + x_4$ of foreign will be used, at a cost of $15(x_2 + x_4)$. The total profit, z, is revenue minus cost:

$$\text{maximize:} \quad z = 12(x_1 + x_2) + 14(x_3 + x_4) - 8(x_1 + x_3) - 15(x_2 + x_4)$$
$$= 4x_1 - 3x_2 + 6x_3 - x_4 \qquad (1)$$

There are limitations imposed on the production by demand, availability of supplies, and specifications on the blends. From the demands,

$$x_1 + x_2 \leq 100\,000 \quad \text{(maximum demand for regular)} \qquad (2)$$
$$x_3 + x_4 \leq 20\,000 \quad \text{(maximum demand for premium)} \qquad (3)$$
$$x_1 + x_2 \geq 50\,000 \quad \text{(minimum regular required)} \qquad (4)$$
$$x_3 + x_4 \geq 5\,000 \quad \text{(minimum premium required)} \qquad (5)$$

From the availability,

$$x_1 + x_3 \leq 40\,000 \quad \text{(domestic)} \qquad (6)$$
$$x_2 + x_4 \leq 60\,000 \quad \text{(foreign)} \qquad (7)$$

The constituents of a blend contribute to the overall octane rating according to their percentages by weight; likewise for the vapor pressure. Thus, the octane rating of regular is

$$87\frac{x_1}{x_1 + x_2} + 98\frac{x_2}{x_1 + x_2}$$

and the requirement that this be at least 88 leads to

$$x_1 - 10x_2 \leq 0 \qquad (8)$$

Similarly, we obtain:

$$6x_3 - 5x_4 \leq 0 \quad \text{(premium octane constraint)} \qquad (9)$$

$$2x_1 - 8x_2 \le 0 \quad \text{(regular vapor-pressure constraint)} \tag{10}$$

$$2x_3 - 8x_4 \le 0 \quad \text{(premium vapor-pressure constraint)} \tag{11}$$

Combining *(1)* through *(11)* with the four (hidden) nonnegativity constraints on the four variables, we obtain the mathematical program

$$\begin{aligned}
\text{maximize:} \quad & z = 4x_1 - 3x_2 + 6x_3 - x_4 \\
\text{subject to:} \quad & x_1 + x_2 \le 100\,000 \\
& x_3 + x_4 \le 20\,000 \\
& x_1 + x_3 \le 40\,000 \\
& x_2 + x_4 \le 60\,000 \\
& x_1 - 10x_2 \le 0 \\
& 6x_3 - 5x_4 \le 0 \\
& 2x_1 - 8x_2 \le 0 \\
& 2x_3 - 8x_4 \le 0 \\
& x_1 + x_2 \ge 50\,000 \\
& x_3 + x_4 \ge 5\,000 \\
\text{with:} \quad & \text{all variables nonnegative}
\end{aligned} \tag{12}$$

System *(12)* is a linear program; its solution is determined in Problem 4.7.

1.8 A hiker plans to go on a camping trip. There are five items the hiker wishes to take with her, but together they exceed the 60-lb weight limit she feels she can carry. To assist herself in the selection process, she has assigned a value to each item in ascending order of importance:

Item	1	2	3	4	5
Weight, lb	52	23	35	15	7
Value	100	60	70	15	15

Which items should she take to maximize the total value without exceeding the weight restriction?

Letting x_i $(i = 1, 2, 3, 4, 5)$ designate the amount of item i to be taken, we can formulate the objective as

$$\text{maximize:} \quad z = 100x_1 + 60x_2 + 70x_3 + 15x_4 + 15x_5 \tag{1}$$

The weight limitation is

$$52x_1 + 23x_2 + 35x_3 + 15x_4 + 7x_5 \le 60 \tag{2}$$

Since an item either will or will not be taken, each variable must be either 1 or 0. Such conditions are enforced if we require each variable to be nonnegative, no greater than 1, and integral. Combining these constraints with *(1)* and *(2)*, we obtain the mathematical program

$$\begin{aligned}
\text{maximize:} \quad & z = 100x_1 + 60x_2 + 70x_3 + 15x_4 + 15x_5 \\
\text{subject to:} \quad & 52x_1 + 23x_2 + 35x_3 + 15x_4 + 7x_5 \le 60 \\
& x_1 \le 1 \\
& x_2 \le 1 \\
& x_3 \le 1 \\
& x_4 \le 1 \\
& x_5 \le 1 \\
\text{with:} \quad & \text{all variables nonnegative and integral}
\end{aligned} \tag{3}$$

System (3) is an integer program; its solution is determined in Problem 6.7 and again in Problem 14.16.

1.9 A 24-hour supermarket has the following minimal requirements for cashiers:

Period	1	2	3	4	5	6
Time of day (24-h clock)	3–7	7–11	11–15	15–19	19–23	23–3
Minimum No.	7	20	14	20	10	5

Period 1 follows immediately after period 6. A cashier works eight consecutive hours, starting at the beginning of one of the six periods. Determine a daily employee worksheet which satisfies the requirements with the least number of personnel.

Setting x_i $(i = 1, 2, \ldots, 6)$ equal to the number of cashiers *beginning* work at the start of period i, we can model this problem by the mathematical program

$$\begin{aligned} \text{minimize:} \quad & z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\ \text{subject to:} \quad & x_1 + x_6 \geq 7 \\ & x_1 + x_2 \geq 20 \\ & x_2 + x_3 \geq 14 \\ & x_3 + x_4 \geq 20 \\ & x_4 + x_5 \geq 10 \\ & x_5 + x_6 \geq 5 \end{aligned} \tag{1}$$

with: all variables nonnegative and integral

System (1) is an integer program; its solution is determined in Problem 6.3.

1.10 A cheese shop has 20 lb of a seasonal fruit mix and 60 lb of an expensive cheese with which it will make two cheese spreads, delux and regular, that are popular during Christmas week. Each pound of the delux spread consists of 0.2 lb of the fruit mix and 0.8 lb of the expensive cheese, while each pound of the regular spread consists of 0.2 lb of the fruit mix, 0.3 lb of the expensive cheese, and 0.5 lb of a filler cheese which is cheap and in plentiful supply. From past pricing policies, the shop has found that the demand for each spread depends on its price as follows:

$$D_1 = 190 - 25P_1 \qquad \text{and} \qquad D_2 = 250 - 50P_2$$

where D denotes demand (in pounds), P denotes price (in dollars per pound), and the subscripts 1 and 2 refer to the delux and regular spreads, respectively. How many pounds of each spread should the cheese shop prepare, and what prices should it establish, if it wishes to maximize income and be left with no inventory of either spread at the end of Christmas week?

Let x_1 pounds of delux spread and x_2 pounds of regular spread be made. If all product can be sold, the objective is to

$$\text{maximize:} \quad z = P_1x_1 + P_2x_2 \tag{1}$$

Now, all product will indeed be sold (and none will be left over in inventory) if production does not exceed demand, i.e., if $x_1 \leq D_1$ and $x_2 \leq D_2$. This gives the constraints

$$x_1 + 25P_1 \leq 190 \qquad \text{and} \qquad x_2 + 50P_2 \leq 250 \tag{2}$$

From the availability of fruit mix,

$$0.2x_1 + 0.2x_2 \leq 20 \tag{3}$$

and from the availability of expensive cheese,

$$0.8x_1 + 0.3x_2 \leq 60 \tag{4}$$

There is no constraint on the filler cheese, since the shop has as much as it needs. Finally, neither production nor price can be negative; so four hidden constraints are $x_1 \geq 0$, $x_2 \geq 0$, $P_1 \geq 0$, and $P_2 \geq 0$. Combining these conditions with (1) through (4), we obtain the mathematical program

$$\begin{aligned} \text{maximize:}\quad & z = P_1x_1 + P_2x_2 \\ \text{subject to:}\quad & 0.2x_1 + 0.2x_2 \leq 20 \\ & 0.8x_1 + 0.3x_2 \leq 60 \\ & x_1 + 25P_1 \leq 190 \\ & x_2 + 50P_2 \leq 250 \\ \text{with:}\quad & \text{all variables nonnegative} \end{aligned} \tag{5}$$

System (5) is a quadratic program in the variables x_1, x_2, P_1, and P_2. It can be simplified if we note that for any fixed positive x_1 and x_2 the objective function increases as either P_1 or P_2 increases. Thus, for a maximum, P_1 and P_2 must be such that the constraints (2) become equations, whereby P_1 and P_2 may be eliminated from the objective function. We then have a quadratic program in x_1 and x_2,

$$\begin{aligned} \text{maximize:}\quad & z = (7.6 - 0.04x_1)x_1 + (5 - 0.02x_2)x_2 \\ \text{subject to:}\quad & 0.2x_1 + 0.2x_2 \leq 20 \\ & 0.8x_1 + 0.3x_2 \leq 60 \\ \text{with:}\quad & x_1 \text{ and } x_2 \text{ nonnegative} \end{aligned} \tag{6}$$

which is easily solved graphically.

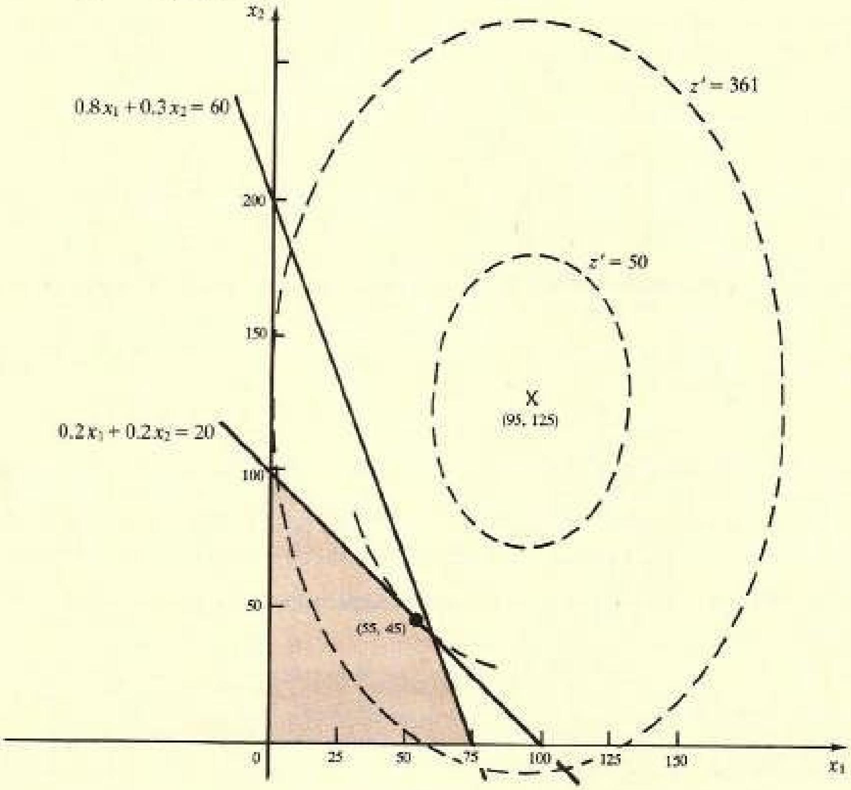

Fig. 1-3

1.11 Give a graphical solution of the quadratic program (*6*) of Problem 1.10.

For graphing purposes, it is convenient to complete the square in the objective function, yielding

$$\text{maximize:} \quad z = 673.5 - 0.04(x_1 - 95)^2 - 0.02(x_2 - 125)^2$$

which is equivalent to

$$\text{minimize:} \quad z' = 0.04(x_1 - 95)^2 + 0.02(x_2 - 125)^2 \tag{1}$$

Since the constraints are linear, the feasible region is bounded by straight lines; it appears shaded in Fig. 1-3. For any particular value of z', (*1*) defines an ellipse centered at (95, 125), and two such ellipses are shown in Fig. 1-3 as dashed curves. The minimum value of z' will correspond to that ellipse defined by (*1*) which is tangent to the line

$$0.2x_1 + 0.2x_2 = 20 \tag{2}$$

To find the point of tangency, we equate the slopes of the line and the ellipse,

$$\frac{dx_2}{dx_1} = -1 \qquad \text{and} \qquad \frac{dx_2}{dx_1} = -\frac{2(x_1 - 95)}{x_2 - 125}$$

obtained by implicit differentiation of (*2*) and (*1*), respectively; this gives

$$x_2 = 2x_1 - 65 \tag{3}$$

Solving (*2*) and (*3*) simultaneously gives the optimal solution to Problem 1.10:

$$x_1^* = 55 \text{ lb of delux spread} \qquad x_2^* = 45 \text{ lb of regular spread}$$

1.12 A plastics manufacturer has 1200 boxes of transparent wrap in stock at one factory and another 1000 boxes at its second factory. The manufacturer has orders for this product from three different retailers, in quantities of 1000, 700, and 500 boxes, respectively. The unit shipping costs (in cents per box) from the factories to the retailers are as follows:

	Retailer 1	Retailer 2	Retailer 3
Factory 1	14	13	11
Factory 2	13	13	12

Determine a minimum-cost shipping schedule for satisfying all demands from current inventory.

Writing x_{ij} ($i = 1, 2;\ j = 1, 2, 3$) for the number of boxes to be shipped from factory i to retailer j, we have as the objective (in cents):

$$\text{minimize:} \quad z = 14x_{11} + 13x_{12} + 11x_{13} + 13x_{21} + 13x_{22} + 12x_{23}$$

Since the amounts shipped from the factories cannot exceed supplies,

$$x_{11} + x_{12} + x_{13} \le 1200 \quad \text{(shipments from factory 1)}$$

$$x_{21} + x_{22} + x_{23} \le 1000 \quad \text{(shipments from factory 2)}$$

Additionally, the total amounts sent to the retailers must meet their demands; hence

$$x_{11} + x_{21} \ge 1000 \quad \text{(shipments to retailer 1)}$$

$$x_{12} + x_{22} \ge 700 \quad \text{(shipments to retailer 2)}$$

$$x_{13} + x_{23} \ge 500 \quad \text{(shipments to retailer 3)}$$

Since the total supply, 1200 + 1000, equals the total demand, 1000 + 700 + 500, each inequality constraint can be tightened to an equality. Doing so, and including the hidden conditions that no shipment be negative and no box be split for shipment, we obtain the mathematical program

$$\begin{array}{ll} \text{minimize:} & z = 14x_{11} + 13x_{12} + 11x_{13} + 13x_{21} + 13x_{22} + 12x_{23} \\ \text{subject to:} & x_{11} + x_{12} + x_{13} = 1200 \\ & x_{21} + x_{22} + x_{23} = 1000 \\ & x_{11} + x_{21} = 1000 \\ & x_{12} + x_{22} = 700 \\ & x_{13} + x_{23} = 500 \\ \text{with:} & \text{all variables nonnegative and integral} \end{array} \tag{1}$$

System (*1*) is an integer program; its solution is determined in Problem 7.3 and again in Problem 8.6.

1.13 A 400-meter medley relay involves four different swimmers, who successively swim 100 meters of the backstroke, breaststroke, butterfly, and freestyle. A coach has six very fast swimmers whose expected times (in seconds) in the individual events are given in Table 1-1.

Table 1-1

	Event 1 (backstroke)	Event 2 (breaststroke)	Event 3 (butterfly)	Event 4 (freestyle)
Swimmer 1	65	73	63	57
Swimmer 2	67	70	65	58
Swimmer 3	68	72	69	55
Swimmer 4	67	75	70	59
Swimmer 5	71	69	75	57
Swimmer 6	69	71	66	59

How should the coach assign swimmers to the relay so as to minimize the sum of their times?

The objective is to minimize total time, which we denote as z. Using double-subscripted variables x_{ij} $(i = 1, 2, \ldots, 6;\ j = 1, 2, 3, 4)$ to designate the number of times swimmer i will be assigned to event j, we can formulate the objective as

$$\text{minimize:} \quad z = 65x_{11} + 73x_{12} + 63x_{13} + 57x_{14} + 67x_{21} + \cdots + 66x_{63} + 59x_{64}$$

Since no swimmer can be assigned to more than one event,

$$\begin{array}{c} x_{11} + x_{12} + x_{13} + x_{14} \le 1 \\ x_{21} + x_{22} + x_{23} + x_{24} \le 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_{61} + x_{62} + x_{63} + x_{64} \le 1 \end{array}$$

Since each event must have one swimmer assigned to it, we also have

$$\begin{array}{c} x_{11} + x_{21} + x_{31} + x_{41} + x_{51} + x_{61} = 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{14} + x_{24} + x_{34} + x_{44} + x_{54} + x_{64} = 1 \end{array}$$

These 10 constraints, combined with the objective and the hidden conditions that each variable be nonnegative and integral, comprise an integer program. Its solution is determined in Problem 9.4.

1.14 A major oil company wants to build a refinery that will be supplied from three port cities. Port B is located 300 km east and 400 km north of Port A, while Port C is 400 km east and 100 km south of Port B. Determine the location of the refinery so that the total amount of pipe required to connect the refinery to the ports is minimized.

The objective is tantamount to minimizing the sum of the distances between the refinery and the three ports. As an aid to calculating this sum, we establish a coordinate system, Fig. 1-4, with Port A as the origin. In this system, Port B has coordinates (300, 400) and Port C has coordinates (700, 300).

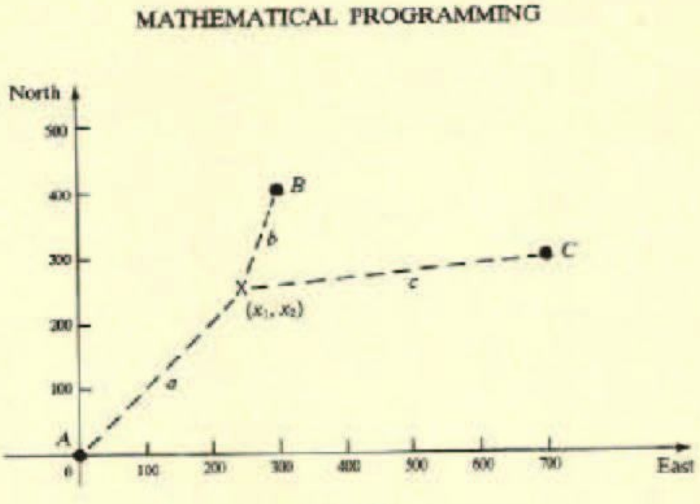

Fig. 1-4

With (x_1, x_2) designating the unknown coordinates of the refinery, the objective is

$$\text{minimize:} \quad z = \sqrt{x_1^2 + x_2^2} + \sqrt{(x_1 - 300)^2 + (x_2 - 400)^2} + \sqrt{(x_1 - 700)^2 + (x_2 - 300)^2} \tag{1}$$

There are no constraints on the coordinates of the refinery nor any hidden conditions: for example, a negative value of x_1 signifies only that the refinery should be placed west of Port A. Equation (*1*) is a nonlinear, unconstrained, mathematical program; its solution is determined in Problem 11.11. See also Problem 1.26.

1.15 An individual has $4000 to invest and three opportunities available to him. Each opportunity requires deposits in $1000 amounts; the investor may allocate all the money to just one opportunity or split the money between them. The expected returns are tabulated as follows.

	Dollars Invested				
	0	1000	2000	3000	4000
Return from Opportunity 1	0	2000	5000	6000	7000
Return from Opportunity 2	0	1000	3000	6000	7000
Return from Opportunity 3	0	1000	4000	5000	8000

How much money should be invested in each opportunity to obtain the greatest total return?

The objective is to maximize total return, denoted by z, which is the sum of the returns from each opportunity. All investments are restricted to be integral multiples of the unit $1000. Letting $f_i(x)$ $(i = 1, 2, 3)$ denote the return (in thousand-dollar units) from opportunity i when x units of money are invested in it, we can rewrite the returns table as Table 1-2.

Table 1-2

f \ x	0	1	2	3	4
$f_1(x)$	0	2	5	6	7
$f_2(x)$	0	1	3	6	7
$f_3(x)$	0	1	4	5	8

Defining x_i $(i = 1, 2, 3)$ as the number of units of money invested in opportunity i, we can formulate the objective as

$$\text{maximize:} \quad z = f_1(x_1) + f_2(x_2) + f_3(x_3) \tag{1}$$

Since the individual has only 4 units of money to invest,

$$x_1 + x_2 + x_3 \leq 4 \tag{2}$$

Augmenting (*1*) and (*2*) with the hidden conditions that x_1, x_2, and x_3 be nonnegative and integral, we obtain the mathematical program

$$\begin{aligned} \text{maximize:} \quad & z = f_1(x_1) + f_2(x_2) + f_3(x_3) \\ \text{subject to:} \quad & x_1 + x_2 + x_3 \leq 4 \\ \text{with:} \quad & \text{all variables nonnegative and integral} \end{aligned} \tag{3}$$

Plotting $f_i(x)$ against x for each function gives a graph that is not a straight line. Therefore, system (*3*) is a nonlinear program; its solution is determined in Problem 14.1.

Supplementary Problems

Formulate but do not solve mathematical programs that model Problems 1.16 through 1.25.

1.16 Fay Klein had developed two types of handcrafted, adult games that she sells to department stores throughout the country. Although the demand for these games exceeds her capacity to produce them, Ms. Klein continues to work alone and to limit her workweek to 50 h. Game I takes 3.5 h to produce and brings a profit of $28, while game II requires 4 h to complete and brings a profit of $31. How many games of each type should Ms. Klein produce weekly if her objective is to maximize total profit?

1.17 A pet store has determined that each hamster should receive at least 70 units of protein, 100 units of carbohydrates, and 20 units of fat daily. If the store carries the six types of feed shown in Table 1-3, what blend of feeds satisfies the requirements at minimum cost to the store?

Table 1-3

Feed	Protein, units/oz	Carbohydrates, units/oz	Fat, units/oz	Cost, ¢/oz
A	20	50	4	2
B	30	30	9	3
C	40	20	11	5
D	40	25	10	6
E	45	50	9	8
F	30	20	10	8

1.18 A local manufacturing firm produces four different metal products, each of which must be machined, polished, and assembled. The specific time requirements (in hours) for each product are as follows.

	Machining, h	Polishing, h	Assembling, h
Product I	3	1	2
Product II	2	1	1
Product III	2	2	2
Product IV	4	3	1

The firm has available to it on a weekly basis 480 h of machine time, 400 h of polishing time, and 400 h of assembly time. The unit profits on the products are \$6, \$4, \$6, and \$8, respectively. The firm has a contract with a distributor to provide 50 units of product I and 100 units of any combination of products II and III each week. Through other customers, the firm can sell each week as many units of products I, II, and III as it can produce, but only a maximum of 25 units of product IV. How many units of each product should the firm manufacture each week to meet all contractual obligations and maximize its total profit? Assume that any unfinished pieces can be completed the following week.

1.19 A caterer must prepare from five fruit drinks in stock 500 gal of a punch containing at least 20 percent orange juice, 10 percent grapefruit juice, and 5 percent cranberry juice. If inventory data are as shown below, how much of each fruit drink should the caterer use to obtain the required composition at minimum total cost?

	Orange Juice, %	Grapefruit Juice, %	Cranberry Juice, %	Supply, gal	Cost, \$/gal
Drink A	40	40	0	200	1.50
Drink B	5	10	20	400	0.75
Drink C	100	0	0	100	2.00
Drink D	0	100	0	50	1.75
Drink E	0	0	0	800	0.25

1.20 A town has budgeted \$250 000 for the development of new rubbish disposal areas. Seven sites are available, whose projected capacities and development costs are given below. Which sites should the town develop?

Site	A	B	C	D	E	F	G
Capacity, tons/wk	20	17	15	15	10	8	5
Cost, \$1000	145	92	70	70	84	14	47

1.21 A semiconductor corporation produces a particular solid-state module that it supplies to four different television manufacturers. The module can be produced at each of the corporation's three plants, although the costs vary because of differing production efficiencies at the plants. Specifically, it costs \$1.10 to produce a module at plant A, \$0.95 at plant B, and \$1.03 at plant C. Monthly production capacities of the plants are 7500, 10 000, and 8100 modules, respectively. Sales forecasts project monthly demand at 4200, 8300, 6300, and 2700 modules for television manufacturers I, II, III, and IV, respectively. If the cost (in dollars) for shipping a module from a factory to a manufacturer is as shown below, find a production schedule that will meet all needs at minimum total cost.

	I	II	III	IV
A	0.11	0.13	0.09	0.19
B	0.12	0.16	0.10	0.14
C	0.14	0.13	0.12	0.15

1.22 The manager of a supermarket meat department finds she has 200 lb of round steak, 800 lb of chuck steak, and 150 lb of pork in stock on Saturday morning, which she will use to make hamburger meat, picnic patties, and meat loaf. The demand for each of these items always exceeds the supermarket's supply. Hamburger meat must be at least 20 percent ground round and 50 percent ground chuck (by weight); picnic patties must be at least 20 percent ground pork and 50 percent ground chuck; and meat loaf must be at least 10 percent ground round, 30 percent ground pork, and 40 percent ground chuck. The remainder of each product is an inexpensive nonmeat filler which the store has in unlimited supply. How many pounds of each product should be made if the manager desires to minimize the amount of meat that must be stored in the supermarket over Sunday?

1.23 A legal firm has accepted five new cases, each of which can be handled adequately by any one of its five junior partners. Due to differences in experience and expertise, however, the junior partners would spend varying amounts of time on the cases. A senior partner has estimated the time requirements (in hours) as shown below:

	Case 1	Case 2	Case 3	Case 4	Case 5
Lawyer 1	145	122	130	95	115
Lawyer 2	80	63	85	48	78
Lawyer 3	121	107	93	69	95
Lawyer 4	118	83	116	80	105
Lawyer 5	97	75	120	80	111

Determine an optimal assignment of cases to lawyers such that each junior partner receives a different case and the total hours expended by the firm is minimized.

1.24 Recreational Motors manufactures golf carts and snowmobiles at its three plants. Plant A produces 40 golf carts and 35 snowmobiles daily; plant B produces 65 golf carts daily, but no snowmobiles; plant C produces 53 snowmobiles daily, but no golf carts. The costs of operating plants A, B, and C are respectively \$210 000, \$190 000, and \$182 000 per day. How many days (including Sundays and holidays) should each plant operate during September to fulfill a production schedule of 1500 golf carts and 1100 snowmobiles at minimum cost? Assume that labor contracts require that once a plant is opened, workers must be paid for the entire day.

1.25 The Futura Company produces two types of farm fertilizers, Futura Regular and Futura's Best. Futura Regular is composed of 25% active ingredients and 75% inert ingredients, while Futura's Best contains 40% active ingredients and 60% inert ingredients. Warehouse facilities limit inventories to 500 tons of active ingredients and 1200 tons of inert ingredients, and they are completely replenished once a week.

Futura Regular is similar to other fertilizers on the market and is competitively priced at \$250 per ton. At this price, the company has had no difficulty in selling all the Futura Regular it produces. Futura's Best, however, has no competition, and so there are no constraints on its price. Of course, demand does depend on price, and through past experience the company has determined that price P (in dollars) and demand D (in tons) are related by $P = 600 - D$. How many tons of each type of fertilizer should Futura produce weekly in order to maximize revenue?

1.26 *Explain why the following constitutes an analog solution to Problem 1.14.* Imagine that Fig. 1-4 represents the top of a tall table. Small holes are bored through the tabletop at points A, B, and C. The three ends of three lengths of string are joined in a knot, which lies on the tabletop; the three free ends are run through the holes, and, underneath the tabletop, three equal weights are hung from them. Then, assuming negligible friction, the equilibrium position of the knot gives the optimal location of the refinery.

$$\begin{aligned}
\text{minimize:}\quad & z = 25x_3 - 25x_4 + 30x_5 - 30x_6 + 0x_7 + 0x_8 + 0x_9 + Mx_{10} + Mx_{11} \\
\text{subject to:}\quad & 4x_3 - 4x_4 + 7x_5 - 7x_6 - x_7 + x_{10} = 1 \\
& 8x_3 - 8x_4 + 5x_5 - 5x_6 - x_8 + x_{11} = 3 \\
& -6x_3 + 6x_4 - 9x_5 + 9x_6 + x_9 = 2 \\
\text{with:}\quad & \text{all variables nonnegative}
\end{aligned}$$

An initial solution to this program in standard form is

$$x_{10} = 1 \qquad x_{11} = 3 \qquad x_9 = 2 \qquad x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = 0$$

Supplementary Problems

Put each of the following programs in matrix standard form.

2.7

$$\begin{aligned}
\text{minimize:}\quad & z = 2x_1 - x_2 + 4x_3 \\
\text{subject to:}\quad & 5x_1 + 2x_2 - 3x_3 \geq -7 \\
& 2x_1 - 2x_2 + x_3 \leq 8 \\
\text{with:}\quad & x_1 \text{ nonnegative}
\end{aligned}$$

2.8

$$\begin{aligned}
\text{maximize:}\quad & z = 10x_1 + 11x_2 \\
\text{subject to:}\quad & x_1 + 2x_2 \leq 150 \\
& 3x_1 + 4x_2 \leq 200 \\
& 6x_1 + x_2 \leq 175 \\
\text{with:}\quad & x_1 \text{ and } x_2 \text{ nonnegative}
\end{aligned}$$

2.9 Problem 2.8 with the three constraint inequalities reversed.

2.10

$$\begin{aligned}
\text{minimize:}\quad & z = 3x_1 + 2x_2 + 4x_3 + 6x_4 \\
\text{subject to:}\quad & x_1 + 2x_2 + x_3 + x_4 \geq 1000 \\
& 2x_1 + x_2 + 3x_3 + 7x_4 \geq 1500 \\
\text{with:}\quad & \text{all variables nonnegative}
\end{aligned}$$

2.11

$$\begin{aligned}
\text{minimize:}\quad & z = 6x_1 + 3x_2 + 4x_3 \\
\text{subject to:}\quad & x_1 + 6x_2 + x_3 = 10 \\
& 2x_1 + 3x_2 + x_3 = 15 \\
\text{with:}\quad & \text{all variables nonnegative}
\end{aligned}$$

2.12

$$\begin{aligned}
\text{maximize:}\quad & z = 7x_1 + 2x_2 + 3x_3 + x_4 \\
\text{subject to:}\quad & 2x_1 + 7x_2 = 7 \\
& 5x_1 + 8x_2 + 2x_4 = 10 \\
& x_1 + x_3 = 11 \\
\text{with:}\quad & x_1,\ x_2, \text{ and } x_3 \text{ nonnegative}
\end{aligned}$$

2.13

$$
\begin{array}{rl}
\text{minimize:} & z = 10x_1 + 2x_2 - x_3 \\
\text{subject to:} & x_1 + x_2 \qquad\; \leq 50 \\
& x_1 + x_2 \qquad\; \geq 10 \\
& \qquad x_2 + x_3 \leq 30 \\
& \qquad x_2 + x_3 \geq \;\;7 \\
& x_1 + x_2 + x_3 = 60 \\
\text{with:} & \text{all variables nonnegative}
\end{array}
$$

Supplementary Problems

Use the simplex or two-phase method to solve the following problems.

4.9 maximize: $z = x_1 + x_2$

subject to: $x_1 + 5x_2 \le 5$

$2x_1 + x_2 \le 4$

with: x_1, x_2 nonnegative

4.10 maximize: $z = 3x_1 + 4x_2$

subject to: $2x_1 + x_2 \le 6$

$2x_1 + 3x_2 \le 9$

with: x_1, x_2 nonnegative

4.11 minimize: $z = x_1 + 2x_2$

subject to: $x_1 + 3x_2 \ge 11$

$2x_1 + x_2 \ge 9$

with: x_1, x_2 nonnegative

4.12 maximize: $z = -x_1 - x_2$

subject to: $x_1 + 2x_2 \ge 5\,000$

$5x_1 + 3x_2 \ge 12\,000$

with: x_1, x_2 nonnegative

4.13 maximize: $z = 2x_1 + 3x_2 + 4x_3$

subject to: $x_1 + x_2 + x_3 \le 1$

$x_1 + x_2 + 2x_3 = 2$

$3x_1 + 2x_2 + x_3 \ge 4$

with: all variables nonnegative

4.14 minimize: $z = 14x_1 + 13x_2 + 11x_3 + 13x_4 + 13x_5 + 12x_6$

subject to:

$$\begin{aligned}
x_1 + x_2 + x_3 \qquad\qquad\qquad &= 1200 \\
x_4 + x_5 + x_6 &= 1000 \\
x_1 \qquad + x_4 \qquad\qquad &= 1000 \\
x_2 \qquad + x_5 \qquad &= 700 \\
x_3 \qquad + x_6 &= 500
\end{aligned}$$

with: all variables nonnegative

4.15 Problem 2.8.

4.16 Problem 2.10.

4.17 Problem 2.9.

4.18 Problem 2.11.

4.19 Problem 2.13.

4.20 Problem 1.7, but with inventories of 80 000 bbl of domestic oil and 20 000 bbl of foreign oil.

4.21 Problem 1.17.

4.22 Problem 1.18.

4.23 Problem 1.19.

4.24 Problem 1.22.

Answers to Supplementary Problems

CHAPTER 1

1.16

$$\begin{aligned} \text{maximize:}\quad & z = 28x_1 + 31x_2 \\ \text{subject to:}\quad & 3.5x_1 + 4x_2 \le 50 \\ \text{with:}\quad & \text{both variables nonnegative} \end{aligned}$$

Note: Integer constraints on the variables are not required, since partially completed games can be finished in following weeks.

1.17

$$\begin{aligned} \text{minimize:}\quad & z = 2x_1 + 3x_2 + 5x_3 + 6x_4 + 8x_5 + 8x_6 \\ \text{subject to:}\quad & 20x_1 + 30x_2 + 40x_3 + 40x_4 + 45x_5 + 30x_6 \ge 70 \\ & 50x_1 + 30x_2 + 20x_3 + 25x_4 + 50x_5 + 20x_6 \ge 100 \\ & 4x_1 + 9x_2 + 11x_3 + 10x_4 + 9x_5 + 10x_6 \ge 20 \\ \text{with:}\quad & \text{all variables nonnegative} \end{aligned}$$

Note: Since feed F is no better than feed C, which is cheaper, no feed F will be used in the optimal mix. Thus, the program can be simplified by substituting $x_6 = 0$.

1.18

$$\begin{aligned} \text{maximize:}\quad & z = 6x_1 + 4x_2 + 6x_3 + 8x_4 \\ \text{subject to:}\quad & 3x_1 + 2x_2 + 2x_3 + 4x_4 \le 480 \\ & x_1 + x_2 + 2x_3 + 3x_4 \le 400 \\ & 2x_1 + x_2 + 2x_3 + x_4 \le 400 \\ & x_1 \ge 50 \\ & x_2 + x_3 \ge 100 \\ & x_4 \le 25 \\ \text{with:}\quad & \text{all variables nonnegative} \end{aligned}$$

1.19

$$\begin{aligned} \text{minimize:}\quad & z = 1.50x_1 + 0.75x_2 + 2.00x_3 + 1.75x_4 + 0.25x_5 \\ \text{subject to:}\quad & 0.2x_1 - 0.15x_2 + 0.8x_3 - 0.2x_4 - 0.2x_5 \ge 0 \\ & 0.3x_1 - 0.1x_3 + 0.9x_4 - 0.1x_5 \ge 0 \\ & -0.05x_1 + 0.15x_2 - 0.05x_3 - 0.05x_4 - 0.05x_5 \ge 0 \\ & x_1 + x_2 + x_3 + x_4 + x_5 \ge 500 \\ & x_1 \le 200 \\ & x_2 \le 400 \\ & x_3 \le 100 \\ & x_4 \le 50 \\ & x_5 \le 800 \\ \text{with:}\quad & \text{all variables nonnegative} \end{aligned}$$

1.20

$$\begin{aligned} \text{maximize:}\quad & z = 20x_1 + 17x_2 + 15x_3 + 15x_4 + 10x_5 + 8x_6 + 5x_7 \\ \text{subject to:}\quad & 145x_1 + 92x_2 + 70x_3 + 70x_4 + 84x_5 + 14x_6 + 47x_7 \le 250 \\ & x_i \le 1 \quad (i = 1, 2, \ldots, 7) \\ \text{with:}\quad & \text{all variables nonnegative and integral} \end{aligned}$$

1.21 The cost of delivering a module from a factory to a manufacturer is the production cost plus the shipping cost.

$$\begin{aligned}
\text{minimize:}\quad & z = (1.10+0.11)x_{11} + (1.10+0.13)x_{12} + \cdots + (1.03+0.15)x_{34} \\
\text{subject to:}\quad & x_{11}+x_{12}+x_{13}+x_{14} \le 7\,500 \\
& x_{21}+x_{22}+x_{23}+x_{24} \le 10\,000 \\
& x_{31}+x_{32}+x_{33}+x_{34} \le 8\,100 \\
& x_{11}+x_{21}+x_{31} = 4\,200 \\
& x_{12}+x_{22}+x_{32} = 8\,300 \\
& x_{13}+x_{23}+x_{33} = 6\,300 \\
& x_{14}+x_{24}+x_{34} = 2\,700 \\
\text{with:}\quad & \text{all variables nonnegative and integral}
\end{aligned}$$

1.22 Since the filler is inexpensive, no more meat will be be used in each product than is required. Let x_1, x_2, and x_3, respectively, designate the poundages of hamburger, picnic patties, and meat loaf to be made.

$$\begin{aligned}
\text{minimize:}\quad & (200-0.2x_1-0.1x_3)+(800-0.5x_1-0.5x_2-0.4x_3)+(150-0.2x_2-0.3x_3) \\
\text{subject to:}\quad & 0.2x_1 + 0.1x_3 \le 200 \\
& 0.5x_1+0.5x_2+0.4x_3 \le 800 \\
& 0.2x_2+0.3x_3 \le 150 \\
\text{with:}\quad & \text{all variables nonnegative}
\end{aligned}$$

The objective is equivalent to

$$\text{maximize:}\quad z = 0.7x_1+0.7x_2+0.8x_3$$

1.23

$$\begin{aligned}
\text{minimize:}\quad & z = 145x_{11}+122x_{12}+130x_{13}+\cdots+80x_{54}+111x_{55} \\
\text{subject to:}\quad & \sum_{i=1}^{5} x_{ij} = 1 \quad (j=1,2,3,4,5) \\
& \sum_{j=1}^{5} x_{ij} = 1 \quad (i=1,2,3,4,5) \\
\text{with:}\quad & \text{all variables nonnegative and integral}
\end{aligned}$$

1.24

$$\begin{aligned}
\text{minimize:}\quad & z = 210\,000x_1+190\,000x_2+182\,000x_3 \\
\text{subject to:}\quad & 40x_1+65x_2 \ge 1500 \\
& 35x_1 + 53x_3 \ge 1100 \\
& x_1 \le 30 \\
& x_2 \le 30 \\
& x_3 \le 30 \\
\text{with:}\quad & \text{all variables nonnegative and integral}
\end{aligned}$$

1.25

$$\begin{aligned}
\text{maximize:}\quad & z = 250x_1+(600-x_2)x_2 \\
\text{subject to:}\quad & 0.25x_1+0.40x_2 \le 500 \\
& 0.75x_1+0.60x_2 \le 1200 \\
\text{with:}\quad & \text{both variables nonnegative}
\end{aligned}$$

1.26 The gravitational potential energy of the system is (for a suitably chosen reference level) proportional to $a+b+c$, and this energy is a minimum at equilibrium.

3.18 (*b*) and (*c*) are basic feasible solutions; (*b*) is degenerate.

3.19
$$x_1\begin{bmatrix}1\\2\\-1\end{bmatrix}+x_2\begin{bmatrix}2\\1\\1\end{bmatrix}+x_3\begin{bmatrix}1\\0\\1\end{bmatrix}+x_4\begin{bmatrix}3\\3\\0\end{bmatrix}+x_5\begin{bmatrix}1\\0\\0\end{bmatrix}+x_6\begin{bmatrix}0\\1\\0\end{bmatrix}+x_7\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}9\\9\\0\end{bmatrix}$$

3.20 (*a*), (*c*), and (*d*) are basic feasible, degenerate solutions.

3.21 Let $f(\mathbf{X}) = \mathbf{C}^T\mathbf{X}$ assume its minimum, m, at $\mathbf{P}_1$ and $\mathbf{P}_2$. Then, for $\beta_1 \geq 0$, $\beta_2 \geq 0$, $\beta_1 + \beta_2 = 1$,
$$f(\beta_1\mathbf{P}_1 + \beta_2\mathbf{P}_2) = \beta_1 f(\mathbf{P}_1) + \beta_2 f(\mathbf{P}_2) = \beta_1 m + \beta_2 m = m$$

3.22 If the subset were linearly dependent, then the nonzero constants which satisfied (*3.1*) for this subset would also satisfy (*3.1*) for the entire set, with all extra constants taken as zero. This would imply that the set is linearly dependent, which it is not.

3.23 In (*3.1*), take the constant in front of the zero vector to be nonzero and all other constants as zero.

CHAPTER 4

4.9 $x_1^* = \frac{5}{3}$, $x_2^* = \frac{2}{3}$; $z^* = \frac{7}{3}$

4.10 $x_1^* = \frac{9}{4}$, $x_2^* = \frac{3}{2}$; $z^* = \frac{51}{4}$

4.11 $x_1^* = \frac{16}{5}$, $x_2^* = \frac{13}{5}$; $z^* = \frac{42}{5}$

4.12 $x_1^* = 1285.7$, $x_2^* = 1857.1$; $z^* = -3142.8$

4.13 No feasible solution exists.

4.14 $x_1^* = 0$, $x_2^* = 700$, $x_3^* = 500$, $x_4^* = 1000$, $x_5^* = 0$, $x_6^* = 0$; $z^* = 27\,600$. (Not only is this solution degenerate, but the solution includes a zero artificial variable among the basic variables. This may occur when one or more of the constraints is redundant. Here, the last constraint is the sum of the first two constraints minus the sum of the next two.)

4.15 $x_1^* = 23.8095$, $x_2^* = 32.1429$; $z^* = 591.667$.

4.16 $x_1^* = 0$, $x_2^* = 423.077$, $x_3^* = 0$, $x_4^* = 153.846$; $z^* = 1769.23$.

4.17 No maximum exists.

4.18 $x_1^* = 6.66667$, $x_2^* = 0.555556$, $x_3^* = 0$; $z^* = 41.6667$.

4.19 $x_1^* = 30$, $x_2^* = 0$, $x_3^* = 30$; $z^* = 270$.

4.20 $x_1^* = 69\,090.9$ bbl, $x_2^* = 17\,272.7$ bbl, $x_3^* = 2272.73$ bbl, $x_4^* = 2727.27$ bbl; $z^* = \$235\,454$.

4.21 $x_1^* = 0.90909$ oz, $x_2^* = 1.81818$ oz, $x_3^* = x_4^* = x_5^* = x_6^* = 0$; $z^* = 7.27273$¢.

4.22 $x_1^* = 50$, $x_2^* = 0$, $x_3^* = 145$, $x_4^* = 10$; $z^* = \$1250$.

4.23 $x_1^* = 93.75$ gal, $x_2^* = 125$ gal, $x_3^* = 56.25$ gal, $x_4^* = 0$, $x_5^* = 225$ gal; $z^* = \$403.125$.

4.24 $x_1^* = 937.5$ lb, $x_2^* = 562.5$ lb, $x_3^* = 125$ lb; $z^* = 0$ lb.