- A = BC

can be replaced by the two linear constraints

- A <= (B + C)/2 and
- A >= B + C - 1

Hence introduce a new variable Z_t defined by the verbal description

Z_t = 1 if produce product 2 on day t and product 1 on day t-1
   = 0 otherwise

Then

$Z_t = x_{2t}x_{1t-1} \quad t=1,...,7$

and our non-linear equation becomes

$P_{2t} = [24R_2]x_{2t} - [5R_2]Z_t \quad t=1,...,7$

and applying our trick the non-linear equation for Z_t can be replaced by the two linear equations

$Z_t <= (x_{2t} + x_{1t-1})/2 \quad t=1,...,7$

$Z_t >= x_{2t} + x_{1t-1} - 1 \quad t=1,...,7$

- closing stock

$I_{i7} >= 1750 \quad i=1,...,4$

- all variables >= 0 and integer, (x_{it}) and (Z_t) zero-one variables

Note that, in practise, we would probably regard (I_{it}) and (P_{it}) as taking fractional values and round to get integer values (since they are both quite large this should be acceptable).

Objective

We wish to minimise total cost and this is given by

$SUM\{t=1,...,7\}(1.50I_{1t} + 1.50I_{2t} + 2.50_{I3t} + 2.50I_{4t})$

Note here that this program may not have a feasible solution, i.e. it may simply not be possible to satisfy all the constraints. This is irrelevant to the process of constructing the model however. Indeed one advantage of the model may be that it will tell us (once a computational solution technique is applied) that the problem is infeasible.

Integer programming example 1987 UG exam

A company is attempting to decide the mix of products which it should produce next week. It has seven products, each with a profit (£) per unit and a production time (man-hours) per unit as shown below:

Product	Profit (£ per unit)	Production time (man-hours per unit)
1	10	1.0
2	22	2.0
3	35	3.7
4	19	2.4

5	55	4.5
6	10	0.7
7	115	9.5

The company has 720 man-hours available next week.

- Formulate the problem of how many units (if any) of each product to produce next week as an integer program in which all the constraints are linear.

Incorporate the following additional restrictions into your integer program (retaining linear constraints and a linear objective):

- If any of product 7 are produced an additional fixed cost of £2000 is incurred.
- Each unit of product 2 that is produced over 100 units requires a production time of 3.0 man-hours instead of 2.0 man-hours (e.g. producing 101 units of product 2 requires 100(2.0) + 1(3.0) man-hours).
- If both product 3 and product 4 are produced 75 man-hours are needed for production line set-up and hence the (effective) number of man-hours available falls to 720 - 75 = 645.

Solution

Let x_i (integer >=0) be the number of units of product i produced then the integer program is

maximise

$10x_1 + 22x_2 + 35x_3 + 19x_4 + 55x_5 + 10x_6 + 115x_7$

subject to

$1.0x_1 + 2.0x_2 + 3.7x_3 + 2.4x_4 + 4.5x_5 + 0.7x_6 + 9.5x_7 <= 720$

$x_i >= 0$ integer $i=1,2,...,7$

Let

$z_7 = 1$ if produce product 7 ($x_7 >= 1$)
$= 0$ otherwise

then

- subtract $2000z_7$ from the objective function and
- add the constraint $x_7 <=$ [most we can make of product 7]z_7

Hence

$x_7 <= (720/9.5)z_7$
i.e. $x_7 <= 75.8z_7$
so $x_7 <= 75z_7$ (since x_7 is integer)

Let y_2 = number of units of product 2 produced in excess of 100 units then add the constraints

- $x_2 <= 100$
- $y_2 >= 0$ integer
- and amend the work-time constraint to be $1.0x_1 + [2.0x_2 + 3.0y_2] + 3.7x_3 + 2.4x_4 + 4.5x_5 + 0.7x_6 + 9.5x_7 <= 720$

- and add $+22y_2$ to the objective function.

This will work because x_2 and y_2 have the same objective function coefficient but y_2 requires longer to produce so will always get more flexibility by producing x_2 first (up to the 100 limit) before producing y_2.

Introduce

```
Z = 1 if produce both product 3 and product 4 (x3 >= 1 and x4 >= 1)
  = 0 otherwise

z3 = 1 if produce product 3 (x3 >= 1)
   = 0 otherwise

z4 = 1 if produce product 4 (x4 >= 1)
   = 0 otherwise
```

and

- subtract from the rhs of the work-time constraint 75Z

and add the two constraints

- x_3 <= [most we can make of product 3]z_3
- x_4 <= [most we can make of product 4]z_4

i.e.

```
x3 <= 194z3 and x4 <= 300z4
```

and relate Z to z_3 and z_4 with the non-linear constraint

- $Z = z_3 z_4$

which we linearise by replacing the non-linear constraint by the two linear constraints

- $Z >= z_3 + z_4 - 1$
- $Z <= (z_3 + z_4)/2$