$y_1 + y_2 + y_3 + y_4 + y_5 <= 3$

- if an oil (vegetable or non-vegetable) is used, at least 30 tonnes of that oil must be used

$x_i >= 30y_i \qquad i=1,...,5$

- if either of VEG1 or VEG2 are used then OIL2 must also be used

$y_4 >= y_1$
$y_4 >= y_2$

Objective

The objective is unchanged by the addition of these extra constraints and variables.

Integer programming example 1985 UG exam

A factory works a 24 hour day, 7 day week in producing four products. Since only one product can be produced at a time the factory operates a system where, throughout one day, the same product is produced (and then the next day either the same product is produced or the factory produces a different product). The rate of production is:

Product	1	2	3	4
No. of units produced per hour worked	100	250	190	150

The only complication is that in changing from producing product 1 one day to producing product 2 the next day five working hours are lost (from the 24 hours available to produce product 2 that day) due to the necessity of cleaning certain oil tanks.

To assist in planning the production for the next week the following data is available:

Product	Current stock (units)	Demand (units) for each day of the week 1	2	3	4	5	6	7
1	5000	1500	1700	1900	1000	2000	500	500
2	7000	4000	500	1000	3000	500	1000	2000
3	9000	2000	2000	3000	2000	2000	2000	500
4	8000	3000	2000	2000	1000	1000	500	500

Product 3 was produced on day 0. The factory is not allowed to be idle (i.e. one of the four products must be produced each day). Stockouts are not allowed. At the end of day 7 there must be (for each product) at least 1750 units in stock.

If the cost of holding stock is £1.50 per unit for products 1 and 2 but £2.50 per unit for products 3 and 4 (based on the stock held at the end of each day) formulate the problem of planning the production for the next week as an integer program in which all the constraints are linear.

Solution

Variables

The decisions that have to be made relate to the type of product to produce each day. Hence let:

- x_{it} = 1 if produce product i (i=1,2,3,4) on day t (t=1,2,3,4,5,6,7) = 0 otherwise

In fact, for this problem, we can ease the formulation by defining two additional variables - namely let:

- I_{it} be the closing inventory (amount of stock left) of product i (i=1,2,3,4) on day t (t=1,2,...,7)
- P_{it} be the number of units of product i (i=1,2,3,4) produced on day t (t=1,2,...,7)

Constraints

- only produce one product per day

$x_{1t} + x_{2t} + x_{3t} + x_{4t} = 1 \quad t=1,2,...,7$

- no stockouts

$I_{it} >= 0 \quad i=1,...,4 \ t=1,...,7$

- we have an inventory continuity equation of the form

closing stock = opening stock + production - demand

Letting D_{it} represent the demand for product i (i=1,2,3,4) on day t (t=1,2,...,7) we have

$I_{10} = 5000$
$I_{20} = 7000$
$I_{30} = 9000$
$I_{40} = 8000$

representing the initial stock situation and

$I_{it} = I_{it-1} + P_{it} - D_{it} \quad i=1,...,4 \ t=1,...,7$

for the inventory continuity equation.

Note here that we assume that we can meet demand in month t from goods produced in month t and also that the opening stock in month t = the closing stock in month $_{t-1}$.

- production constraint

Let R_i represent the work rate (units/hour) for product i (i=1,2,3,4) then the production constraint is

$P_{it} = x_{it}[24R_i] \quad i=1,3,4 \ t=1,...,7$

which covers the production for all except product 2 and

$P_{2t} = [24R_2]x_{2t} - [5R_2]x_{2t}x_{1t-1} \quad t=1,...,7$

i.e. for product 2 we lose 5 hours production if we are producing product 2 in period t and we produced product 1 the previous period. Note here that we initialise by

$x_{10} = 0$

since we know we were not producing product 1 on day 0. Plainly the constraint involving P_{2t} is non-linear as it involves a term which is the product of two variables. However we can linearise it by using the trick that given three zero-one variables (A,B,C say) the non-linear constraint

- A = BC

can be replaced by the two linear constraints

- A <= (B + C)/2 and
- A >= B + C - 1

Hence introduce a new variable Z_t defined by the verbal description

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Zt = 1 if produce product 2 on day t and product 1 on day t-1
   = 0 otherwise
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Then

$Z_t = x_{2t}x_{1t-1}$ t=1,...,7

and our non-linear equation becomes

$P_{2t} = [24R_2]x_{2t} - [5R_2]Z_t$ t=1,...,7

and applying our trick the non-linear equation for Z_t can be replaced by the two linear equations

$Z_t <= (x_{2t} + x_{1t-1})/2$ t=1,...,7

$Z_t >= x_{2t} + x_{1t-1} - 1$ t=1,...,7

- closing stock

$I_{i7} >= 1750$ i=1,...,4

- all variables >= 0 and integer, (x_{it}) and (Z_t) zero-one variables

Note that, in practise, we would probably regard (I_{it}) and (P_{it}) as taking fractional values and round to get integer values (since they are both quite large this should be acceptable).

Objective

We wish to minimise total cost and this is given by

$SUM\{t=1,...,7\}(1.50I_{1t} + 1.50I_{2t} + 2.50_{I3t} + 2.50I_{4t})$

Note here that this program may not have a feasible solution, i.e. it may simply not be possible to satisfy all the constraints. This is irrelevant to the process of constructing the model however. Indeed one advantage of the model may be that it will tell us (once a computational solution technique is applied) that the problem is infeasible.

Integer programming example 1987 UG exam

A company is attempting to decide the mix of products which it should produce next week. It has seven products, each with a profit (£) per unit and a production time (man-hours) per unit as shown below:

Product	Profit (£ per unit)	Production time (man-hours per unit)
1	10	1.0
2	22	2.0
3	35	3.7
4	19	2.4