toys of type j (j=1,2) produced in factory i (i=1,2) where x_{ij} >=0 and integer.

Constraints

- at each factory cannot exceed the production time available

$x_{11}/52 + x_{12}/38 <= 480$
$x_{21}/42 + x_{22}/23 <= 720$

- cannot produce a toy unless we are setup to do so

$x_{11} <= 52(480)f_{11}$
$x_{12} <= 38(480)f_{12}$
$x_{21} <= 42(720)f_{21}$
$x_{22} <= 23(720)f_{22}$

Objective

The objective is to maximise total profit, i.e.

maximise $12(x_{11}+ x_{21}) + 16(x_{12}+ x_{22}) - 45000(f_{11} + f_{21}) - 76000(f_{12} + f_{22})$

Note here that the question says that in order to avoid doubling the setup costs only one factory could be used. This is not a constraint. We can argue that if it is only cost considerations that prevent us using more than one factory these cost considerations have already been incorporated into the model given above and the model can decide for us how many factories to use, rather than we artificially imposing a limit via an explicit constraint on the number of factories that can be used.

The above mathematical model could be used in production planning in the following way:

- enables us to maximise profit, rather than relying on an ad-hoc judgemental approach
- can be used for sensitivity analysis - for example to see how sensitive our production planning decision is to changes in the production rates
- enables us to see the effect upon production of (say) increasing the profit per unit on toy 1
- enables us to easily replan production in the event of a change in the system (e.g. a reduction in the available production hours at factory one due to increased work from other products made at that factory)
- can use on a rolling horizon basis to plan production as time passes (in which case we perhaps need to introduce a time subscript into the above model)

Integer programming example 1995 MBA exam

A project manager in a company is considering a portfolio of 10 large project investments. These investments differ in the estimated long-run profit (net present value) they will generate as well as in the amount of capital required.

Let P_j and C_j denote the estimated profit and capital required (both given in units of millions of £) for investment opportunity j (j=1,...,10) respectively. The total amount of capital available for these investments is Q (in units of millions of £)

Investment opportunities 3 and 4 are mutually exclusive and so are 5 and 6. Furthermore, neither 5 nor 6 can be undertaken unless either 3 or 4 is undertaken. At least two and at most four investment opportunities have to be undertaken from the set {1,2,7,8,9,10}.

The project manager wishes to select the combination of capital investments that will maximise the total estimated long-run profit subject to the restrictions described above.

Formulate this problem using an *integer programming* model and comment on the difficulties of solving this model. (Do not actually solve it).

What are the advantages and disadvantages of using this model for portfolio selection?

Solution

Variables

We need to decide whether to use an investment opportunity or not so let $x_j = 1$ if we use investment opportunity j (j=1,...,10), 0 otherwise

Constraints

- total amount of capital available for these investments is Q

$SUM\{j=1,...,10\}C_jx_j <= Q$

- investment opportunities 3 and 4 are mutually exclusive and so are 5 and 6

$x_3 + x_4 <= 1$
$x_5 + x_6 <= 1$

- neither 5 nor 6 can be undertaken unless either 3 or 4 is undertaken

$x_5 <= x_3+ x_4$
$x_6 <= x_3+ x_4$

- at least two and at most four investment opportunities have to be undertaken from the set {1,2,7,8,9,10}

$x_1 + x_2 + x_7 + x_8 + x_9 + x_{10} >= 2$
$x_1 + x_2 + x_7 + x_8 + x_9 + x_{10} <= 4$

Objective

The objective is to maximise the total estimated long-run profit i.e.

maximise $SUM\{j=1,...,10\}P_jx_j$

The model given above is a very small zero-one integer programming problem with just 10 variables and 7 constraints and should be very easy to solve. For example even by complete (total) enumeration there are just $2^{10} = 1024$ possible solutions to be examined.

The advantages and disadvantages of using this model for portfolio selection are:

- enables us to maximise profit, rather than relying on an ad-hoc judgemental approach
- can be easily extended to deal with a larger number of potential investment opportunities

- can be used for sensitivity analysis, for example to see how sensitive our portfolio selection decision is to changes in the data
- the model fails to take into account any statistical uncertainly (risk) in the data, it is a completely deterministic model, for example project j might have a (known or estimated) statistical distribution for its profit P_j and so we might need a model that takes this distribution into account

---

Integer programming example 1994 MBA exam

A food is manufactured by refining raw oils and blending them together. The raw oils come in two categories:

- Vegetable oil:
  - VEG1
  - VEG2
- Non-vegetable oil:
  - OIL1
  - OIL2
  - OIL3

The prices for buying each oil are given below (in £/tonne)

```
VEG1  VEG2  OIL1  OIL2  OIL3
115   128   132   109   114
```

The final product sells at £180 per tonne. Vegetable oils and non-vegetable oils require different production lines for refining. It is not possible to refine more than 210 tonnes of vegetable oils and more than 260 tonnes of non-vegetable oils. There is no loss of weight in the refining process and the cost of refining may be ignored.

There is a technical restriction relating to the hardness of the final product. In the units in which hardness is measured this must lie between 3.5 and 6.2. It is assumed that hardness blends linearly and that the hardness of the raw oils is:

```
VEG1  VEG2  OIL1  OIL2  OIL3
8.8   6.2   1.9   4.3   5.1
```

It is required to determine what to buy and how to blend the raw oils so that the company maximises its profit.

- *Formulate* the above problem as a *linear program*. (Do not actually solve it).
- What assumptions do you make in solving this problem by linear programming?

The following extra conditions are imposed on the food manufacture problem stated above as a result of the production process involved:

- the food may never be made up of more than 3 raw oils
- if an oil (vegetable or non-vegetable) is used, at least 30 tonnes of that oil must be used
- if either of VEG1 or VEG2 are used then OIL2 must also be used

Introducing 0-1 integer variables extend the linear programming model you have developed to encompass these new extra conditions.

Solution