```
= 0 otherwise
```

and add the constraint

- $z_1+z_2+z_3+z_4 >= 3$ (at least 3 constraints satisfied)

and alter the nutrient constraints to be

- $100x_1 + 200x_2 >= 90z_1$
- $80x_1 + 150x_2 >= 50z_2$
- $40x_1 + 20x_2 >= 20z_3$
- $10x_1 >= 2z_4$

The logic behind this change is that if a $z_i=1$ then the constraint becomes the original nutrient constraint which needs to be satisfied. However if a $z_i=0$ then the original nutrient constraint becomes

- same left-hand side >= zero

which (for the four left-hand sides dealt with above) is always true and so can be neglected - meaning the original nutrient constraint need not be satisfied. Hence the complete (MIP) formulation of the problem is given by

```
minimise 40x1 + 60x2 + 15y
subject to
x1 + x2 + x3 = 1
100x1 + 200x2 >= 90z1
80x1 + 150x2 >= 50z2
40x1 + 20x2 >= 20z3
10x1 >= 2z4
z1 + z2 + z3 + z4 >= 3
x2 <= y
zi = 0 or 1    i=1,2,3,4
y = 0 or 1
xi >= 0    i=1,2,3
```

Integer programming example

In the planning of the monthly production for the next six months a company must, in each month, operate either a normal shift or an extended shift (if it produces at all). A normal shift costs £100,000 per month and can produce up to 5,000 units per month. An extended shift costs £180,000 per month and can produce up to 7,500 units per month. Note here that, for either type of shift, the cost incurred is fixed by a union guarantee agreement and so is independent of the amount produced.

It is estimated that changing from a normal shift in one month to an extended shift in the next month costs an extra £15,000. No extra cost is incurred in changing from an extended shift in one month to a normal shift in the next month.

The cost of holding stock is estimated to be £2 per unit per month (based on the stock held at the end of each month) and the initial stock is 3,000 units (produced by a normal shift). The amount in stock at the end of month 6 should be at least 2,000 units. The demand for the company's product in each of the next six months is estimated to be as shown below:

Month	1	2	3	4	5	6
Demand	6,000	6,500	7,500	7,000	6,000	6,000

Production constraints are such that if the company produces anything in a particular month it must produce at least 2,000 units. If the company wants a production plan for the next six months that avoids stockouts, formulate their problem as an integer program.

Hint: first formulate the problem allowing non-linear constraints and then attempt to make all the constraints linear.

Solution

Variables

The decisions that have to be made relate to:

- whether to operate a normal shift or an extended shift in each month; and
- how much to produce each month.

Hence let:

x_t = 1 if we operate a normal shift in month t (t=1,2,...,6)
   = 0 otherwise

y_t = 1 if we operate an extended shift in month t (t=1,2,...,6)
   = 0 otherwise

P_t (>= 0) be the amount produced in month t (t=1,2,...,6)

In fact, for this problem, we can ease the formulation by defining three additional variables - namely let:

z_t = 1 if we switch from a normal shift in month t-1 to an extended
     shift in month t (t=1,2,...,6)
   = 0 otherwise

I_t be the closing inventory (amount of stock left) at the end
   of month t (t=1,2,...,6)

w_t = 1 if we produce in month t, and hence from the production
     constraints P_t >= 2000 (t=1,2,...,6)
   = 0 otherwise (i.e. P_t = 0)

The motivation behind introducing the first two of these variables (z_t, I_t) is that in the objective function we will need terms relating to shift change cost and inventory holding cost. The motivation behind introducing the third of these variables (w_t) is the production constraint "either $P_t = 0$ or $P_t >= 2000$", which needs a zero-one variable so that it can be dealt with using the standard trick for "either/or" constraints.

In any event formulating an IP tends to be an iterative process and if we have made a mistake in defining variables we will encounter difficulties in formulating the constraints/objective. At that point we can redefine our variables and reformulate.

Constraints

We first express each constraint in words and then in terms of the variables defined above.

- only operate (at most) one shift each month

$x_t + y_t <= 1$ t=1,2,...,6

Note here that we could not have made do with just one variable (x_t say) and defined that variable to be one for a normal shift and zero for an extended shift (since in that case what if we decide not to produce in a particular month?).

Although we could have introduced a variable indicating no shift (normal or extended) operated in a particular month this is not necessary as such a variable is equivalent to $1-x_t-y_t$.

- production limits not exceeded

$P_t <= 5000x_t + 7500y_t \quad t=1,2,...,6$

Note here the use of addition in the right-hand side of the above equation where we are making use of the fact that at most one of x_t and y_t can be one and the other must be zero.

- no stockouts

$I_t >= 0 \quad t=1,2,...,6$

- we have an inventory continuity equation of the form

closing stock = opening stock + production - demand

where I_0 = 3000. Hence letting D_t = demand in month t (t=1,2,...,6) (a known constant) and assuming

- that opening stock in period t = closing stock in period t-1 and
- that production in period t is available to meet demand in period t

we have that

$I_t = I_{t-1} + P_t - D_t \quad t=1,2,...,6$

As noted above this equation assumes that we can meet demand in the current month from goods produced that month. Any time lag between goods being produced and becoming available to meet demand is easily incorporated into the above equation. For example for a 2 month time lag we replace P_t in the above equation by P_{t-2} and interpret I_t as the number of goods in stock at the end of month t which are available to meet demand i.e. goods are not regarded as being in stock until they are available to meet demand. Inventory continuity equations of the type shown are common in production planning problems.

- the amount in stock at the end of month 6 should be at least 2000 units

$I_6 >= 2000$

- production constraints of the form "either P_t = 0 or P_t >= 2,000".

Here we make use of the standard trick we presented for "either/or" constraints. We have already defined appropriate zero-one variables w_t (t=1,2,...,6) and so we merely need the constraints

$P_t <= Mw_t \quad t=1,2,...,6$

$P_t >= 2000w_t \quad t=1,2,...,6$

Here M is a positive constant and represents the most we can produce in any period t (t=1,2,...,6). A convenient value for M for this example is M = 7500 (the most we can produce irrespective of the shift operated).

- we also need to relate the shift change variable z_t to the shifts being operated

The obvious constraint is

$z_t = x_{t-1}y_t \quad t=1,2,...,6$

since as both x_{t-1} and y_t are zero-one variables z_t can only take the value one if both x_{t-1} and y_t are one (i.e. we operate a normal shift in period t-1 and an extended shift in period t). Looking back to the verbal description of z_t it is clear that the mathematical description given above is equivalent to that verbal description. (Note here that we define $x_0 = 1$ ($y_0 = 0$)).

This constraint is non-linear. However we are told that we can first formulate the problem with non-linear constraints and so we proceed. We shall see later how to linearise (generate equivalent linear constraints for) this equation.

Objective

We wish to minimise total cost and this is given by

SUM{t=1,...,6}($100000x_t + 180000y_t + 15000z_t + 2I_t$)

Hence our formulation is complete.

Note that, in practise, we would probably regard I_t and P_t as taking fractional values and round to get integer values (since they are both large this should be acceptable). Note too here that this is a non-linear integer program.

Comments

In practice a model of this kind would be used on a "rolling horizon" basis whereby every month or so it would be updated and resolved to give a new production plan.

The inventory continuity equation presented is quite flexible, being able to accommodate both time lags (as discussed previously) and wastage. For example if 2% of the stock is wasted each month due to deterioration/pilfering etc then the inventory continuity equation becomes $I_t = 0.98I_{t-1} + P_t - D_t$. Note that, if necessary, the objective function can include a term related to $0.02I_{t-1}$ to account for the loss in financial terms.

In order to linearise (generate equivalent linear constraints) for our non-linear constraint we again use a standard trick. Note that that equation is of the form

- A = BC

where A, B and C are zero-one variables. The standard trick is that a non-linear constraint of this type can be replaced by the two linear constraints

- A <= (B+C)/2 and
- A >= B+C-1

To see this we use the fact that as B and C take only zero-one values there are only four possible cases to consider:

B	C	A = BC becomes	A <= (B+C)/2 becomes	A >= B+C-1 becomes
0	0	A = 0	A <= 0	A >= -1
0	1	A = 0	A <= 0.5	A >= 0

```
1   0   A = 0      A <= 0.5        A >= 0
1   1   A = 1      A <= 1          A >= 1
```

Then, recalling that A can also only take zero-one values, it is clear that in each of the four possible cases the two linear constraints (A <= (B+C)/2 and A >= B+C-1) are equivalent to the single non-linear constraint (A=BC).

Returning now to our original non-linear constraint

- $z_t = x_{t-1}y_t$

this involves the three zero-one variables z_t, x_{t-1} and y_t and so we can use our general rule and replace this non-linear constraint by the two linear constraints

$$z_t <= (x_{t-1} + y_t)/2 \quad t=1,2,...,6$$
$$\text{and } z_t >= x_{t-1} + y_t - 1 \quad t=1,2,...,6$$

Making this change transforms the non-linear integer program given before into an equivalent linear integer program.

Integer programming example 1996 MBA exam

A toy manufacturer is planning to produce new toys. The setup cost of the production facilities and the unit profit for each toy are given below. :

Toy	Setup cost (£)	Profit (£)
1	45000	12
2	76000	16

The company has two factories that are capable of producing these toys. In order to avoid doubling the setup cost only *one* factory could be used.

The production rates of each toy are given below (in units/hour):

	Toy 1	Toy 2
Factory 1	52	38
Factory 2	42	23

Factories 1 and 2, respectively, have 480 and 720 hours of production time available for the production of these toys. The manufacturer wants to know *which* of the new toys to produce, *where* and *how many* of each (if any) should be produced so as to maximise the total profit.

- Introducing 0-1 decision variables *formulate* the above problem as an integer program. (Do *not* try to solve it).
- Explain briefly how the above mathematical model can be used in production planning.

Solution

Variables

We need to decide whether to setup a factory to produce a toy or not so let f_{ij} = 1 if factory i (i=1,2) is setup to produce toys of type j (j=1,2), 0 otherwise

We need to decide how many of each toy should be produced in each factory so let x_{ij} be the number of