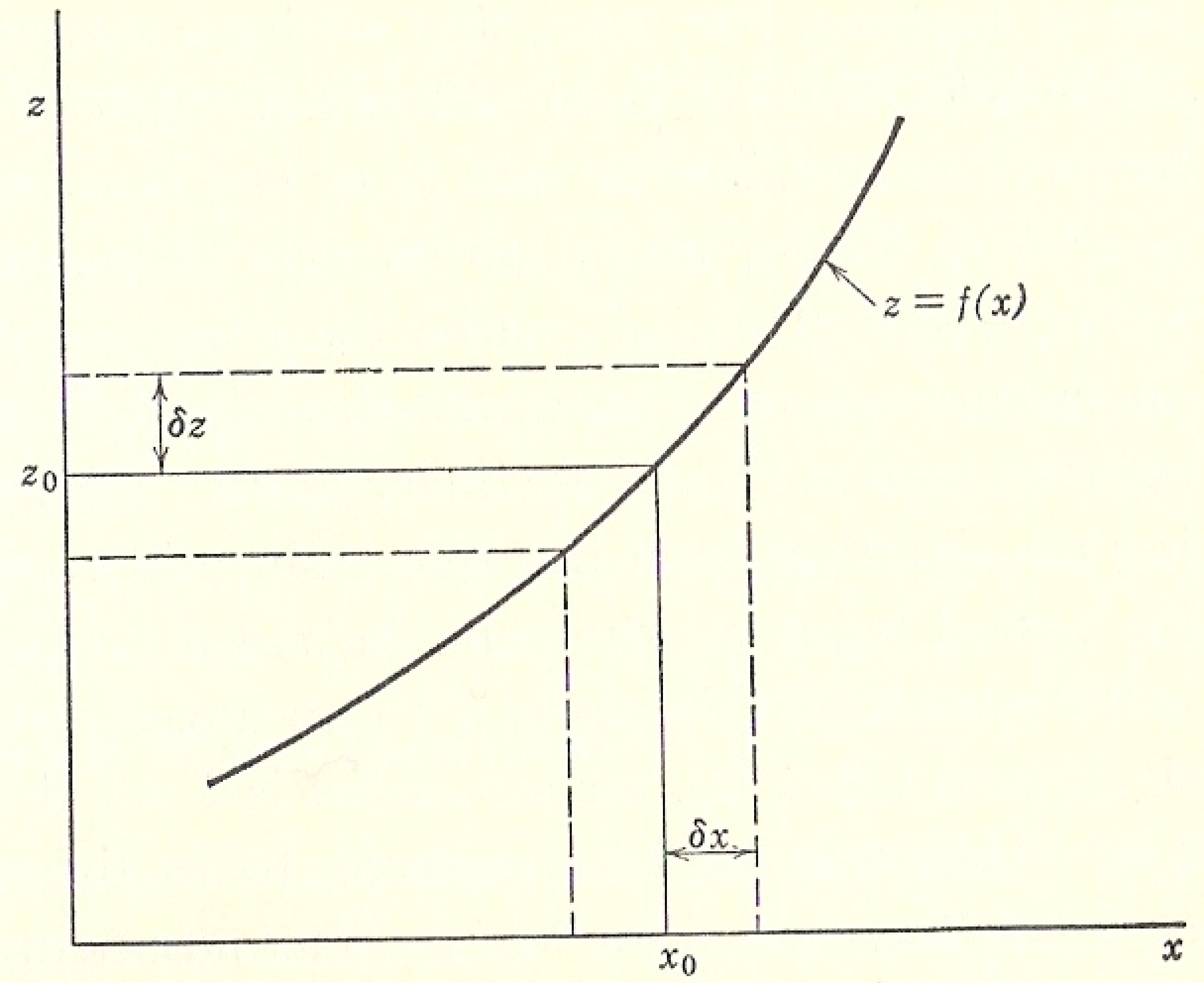

Fig. 3.1 Propagation of uncertainty from one variable to another.

rise to the computed result z_0, and how the range $\pm\delta x$ about x_0 gives a corresponding range $\pm\delta z$ about z_0.

Before proceeding to any general methods of evaluating δz, it is instructive to see how finite perturbations are propagated in simple functions. For example, consider the function

$$z = x^2$$

If x can range between $x_0 + \delta x$ and $x_0 - \delta x$ then z can range between $z_0 + \delta z$ and $z_0 - \delta z$ where

$$\begin{aligned} z_0 \pm \delta z &= (x_0 \pm \delta x)^2 \\ &= x_0^2 \pm 2x_0\,\delta x + (\delta x)^2 \end{aligned}$$

we can ignore $(\delta x)^2$, since δx is assumed to be small compared with x_0, and equate z_0 to x_0^2, giving us the value of δz as

$$\delta z = 2x_0\,\delta x$$

This can more conveniently be expressed in terms of the relative uncertainty $\delta z/z_0$ as

$$\delta z/z_0 = 2x_0\,\delta x/x_0^2 = 2\,\delta x/x_0$$

Thus, the relative uncertainty of the computed result is twice that of the initial measurement.

Although it is essential to bear in mind the nature of the propagation of uncertainty, as illustrated by this example with finite differences, a considerable simplification of the formulation can result from the use of the techniques of the differential calculus.

3.3 General Method for Uncertainty in Functions of a Single Variable

It will be noticed that these finite differences δz and δx are merely an expression of the derivative dz/dx. We can therefore obtain our value of δz by using standard techniques to obtain

$$\frac{dz}{dx} = f'(x)$$

and then writing

$$\delta z = f'(x)\,\delta x \tag{3.1}$$

This is a relatively simple procedure and will work in cases where the elementary finite difference approach would lead to algebraic complexity.

Thus, if

$$\begin{aligned} z &= \frac{x}{x^2+1} \\ \frac{dz}{dx} &= \frac{x^2+1-x\cdot 2x}{(x^2+1)^2} \\ &= \frac{1-x^2}{(1+x^2)^2} \\ \delta z &= \frac{1-x^2}{(1+x^2)^2}\,\delta x \end{aligned}$$

This would have been very awkward by any other approach. It gives δz generally as a function of x and δx, and the particular value desired would be obtained by setting $x = x_0$. Let us now use this technique to evaluate the uncertainty for some common functions.

(a) *Powers*

Consider
$$z = x^n$$
$$\frac{dz}{dx} = nx^{n-1}$$
$$\delta z = nx^{n-1}\,\delta x$$

The significance of this result becomes a little more obvious when expressed in terms of the relative uncertainty. Thus,

$$\frac{\delta z}{z} = n\frac{\delta x}{x}$$

This will hold for either powers or roots, so that the precision diminishes as a quantity is raised to powers or improves on taking roots. This is a situation which must be carefully watched in an experiment in which powers are involved. The higher the power, the greater is the initial precision that is needed.

(b) *Trigonometric Functions*

We shall do only one example since all the others can be treated in a similar fashion.

Consider
$$z = \sin x$$
$$\frac{dz}{dx} = \cos x$$
$$\delta z = \cos x\,\delta x$$

This is one case where the elementary method of inserting

$x_0 \pm \delta x$ shows the nature of the result more clearly. This substitution can be easily verified to give

$$\delta z = \cos x \sin \delta x$$

showing that the δx in the previous result is really $\sin \delta x$ in the limit. Only in the case of a very large uncertainty would this difference be significant, but it is best to understand the nature of the result. Clearly δx should be expressed in radian measure. The result will normally have straightforward application when dealing with apparatus such as the spectrometer.

(c) *Logarithmic and Exponential Functions*

Consider
$$z = \log x$$
$$\frac{dz}{dx} = \frac{1}{x}$$
$$\delta z = \frac{1}{x}\,\delta x$$

and the relative uncertainty can be calculated as usual.

If
$$z = e^x$$
$$\frac{dz}{dx} = e^x$$
$$\delta z = e^x\,\delta x$$

This is a rather more important case since the exponential function is one of common occurrence in physics and engineering. These functions can become very sensitive to the exponent when it takes values much over unity, and the uncertainty δz can be seen to have potentially large values. This will be familiar to anyone who has watched the current fluctuations in a thermionic diode which can result from quite small filament temperature variations.

As stated above, the method can be easily applied to any function not listed above by evaluating the appropriate derivative and using Equation (3.1).

3.4 Uncertainty in Functions of Two or More Variables

If the result is to be computed from two or more measured quantities, x and y, the uncertainty in the result can be regarded in two different ways. We can, first, be as pessimistic as possible and suppose that the actual deviations of x and y happen to combine additively in such a way that the value of z is driven as far as possible from the central value. We shall, in this way, calculate a δz which gives the extreme width of the range of possible z values. It is possible to argue against this that the probability is small of a number of uncertainties combining in magnitude and direction to give the worst possible result for z. This is true, and we shall deal later with the matter of the *probable* uncertainty in z. For the moment, however, let us calculate the δz which represents the widest range of possibility of z. This is certainly a safe, though pessimistic, approach since if δx, δy etc. represent limits within which we are "almost certain" the actual value lies, then this δz will give limits within which we are equally certain that the actual value of z lies.

The most instructive approach initially is to use the elementary substitution method, and we shall use this for the first two functions

(a) *Sum of Two or More Variables*

Consider

$$z = x + y$$

The uncertainty in z will be obtained from

$$z_0 \pm \delta z = x_0 \pm \delta x + y_0 \pm \delta y$$

and the maximum value of δz is given by choosing similar signs throughout. As might be expected, the uncertainty in the sum is just the sum of the individual uncertainties. This can be expressed in terms of relative uncertainties

$$\frac{\delta z}{z} = \frac{\delta x + \delta y}{x + y}$$

but no increased clarification is achieved.

(b) *Difference of Two Variables*

Consider

$$z = x - y$$

As in the case above, δz will be obtained from

$$z_0 \pm \delta z = (x_0 \pm \delta x) - (y_0 \pm \delta y)$$

Thus, we can obtain the maximum value of δz by choosing the negative sign for δy giving, once again,

$$\delta z = \delta x + \delta y$$

The significance of this is more clearly apparent if we consider the relative uncertainty given by

$$\frac{\delta z}{z} = \frac{\delta x + \delta y}{x - y}$$

This shows that, if x_0 and y_0 are close together, $x - y$ is small, and this relative uncertainty can rise to very large values. This is, at best, an unsatisfactory situation and it can become sufficiently bad to destroy the value of the measurement. It is a particularly dangerous condition since it can arise unnoticed. It is perfectly obvious that no one would attempt to measure the distance between two points a millimeter apart by measuring the distance of each from a third point a meter away, and then subtracting the two lengths. However, it can happen that a desired result is to be obtained by subtraction of two measurements made separately (two thermometers, clocks, etc.) and the character of the measurement as a difference may not be strikingly obvious. All measurements involving differences should be treated with the greatest caution. Clearly the way to avoid this difficulty is to measure the difference directly, rather than obtain it by subtraction between two measured quantities. For example if one has an apparatus within which two points are at potentials above ground of $V_1 = 1500$ v and $V_2 = 1510$ v respectively, and the required quantity is $V_2 - V_1$, only a very high quality voltmeter would permit the values of V_1 and V_2 to be measured to give $V_2 - V_1$ with even say 10 per cent. But an ordinary 10 v table voltmeter connected between the two points and measuring $V_2 - V_1$ directly will immediately give the answer with 2–3 per cent precision.

3.5 General Method for Uncertainty in Functions of Two or More Variables

These last two examples, treated by the elementary method, suggest that, once more, the differential calculus may offer a considerable simplification of the treatment. It is clear that if we have

$$z = f(x, y)$$

the appropriate quantity required in order to calculate δz is the total differential dz, given by

$$dz = \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy$$

We shall take this differential and treat it as a finite difference, δz, given, in terms of the uncertainties δx and δy, by

$$\delta z = \left(\frac{\partial f}{\partial x}\right) \delta x + \left(\frac{\partial f}{\partial y}\right) \delta y \qquad (3.2)$$

where the derivatives $\partial f/\partial x$ and $\partial f/\partial y$ will normally be evaluated for the values x_0, y_0 at which δz is required. We shall find that the sign of $\partial f/\partial x$ or $\partial f/\partial y$ may be negative, in which case, using our pessimistic requirement for a maximum value of δz, we shall choose negative values for the appropriate δx or δy giving a wholly positive contribution to the sum.

(a) *Product of Two or More Variables*

Suppose

$$z = xy$$

Using Equation (3.2) we need

$$\frac{\partial z}{\partial x} = y \quad \text{and} \quad \frac{\partial z}{\partial y} = x$$

Thus, the value of δz is given by

$$\delta z = y\,\delta x + x\,\delta y$$

The significance of this is more clearly seen in the relative uncertainty

$$\frac{\delta z}{z} = \frac{\delta x}{x} + \frac{\delta y}{y}$$

i.e., when the result is a product of two variables, the relative uncertainty of the result is the sum of the relative uncertainties of the components.

The most general case of a compound function, and one very commonly found in physics, is the one in which an algebraic product has components raised to powers in the form

$$z = x^a y^b$$

where a and b may be positive or negative, integral or fractional powers. In this case the formulation is greatly simplified by taking logs of both sides before doing the differentiating.
Thus,

$$\log z = a \log x + b \log y$$

Therefore, differentiating implicity,

$$\frac{dz}{z} = a\frac{dx}{x} + b\frac{dy}{y}$$

As usual, we take the differentials to be finite differences, giving

$$\frac{\delta z}{z} = a\frac{\delta x}{x} + b\frac{\delta y}{y}$$

Note that this process gives the relative uncertainty directly. This is frequently convenient but, if the absolute uncertainty δz is required, it is simply evaluated by multiplying by the computed value z_0, which is normally available. This form of implicit differentiation is still the simplest even when z is itself raised to some power. For if the equation reads

$$z^2 = xy$$

it is unnecessary to rewrite it

$$z = x^{1/2}y^{1/2}$$

and work from there because, by taking logs,

$$2 \log z = \log x + \log y$$

i.e.,

$$2\frac{\delta z}{z} = \frac{\delta x}{x} + \frac{\delta y}{y}$$

giving $\delta z/z$ as required.

(b) *Quotients*

These come under the heading of the previous section, which permits negative values, and we repeat that the maximum value of δz will be obtained by neglecting the negative sign in the differential.

If a function other than those already listed is encountered, some kind of a differentiation will usually be found to work. It is frequently a convenience to differentiate an equation implicitly, thus simplifying the working by avoiding the necessity for calculating the unknown explicitly as a function of the other variables. For example, consider the lens equation

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}$$

where f is a function of the measured quantities s and s'. We can differentiate the equation implicitly to obtain

$$-\frac{df}{f^2} = -\frac{ds}{s^2} - \frac{ds'}{s'^2}$$

It is now possible to calculate df or df/f directly and more easily than would have been the case by writing f explicitly as a function of s and s'. Thus, a formula may be prepared for the uncertainty into which all the unknowns can be inserted directly. Make sure that the appropriate signs are used so that the contributions to the resultant uncertainty all add positively to give the outer limits of possibility for the answer.

If the function is too big and complicated to work out a value of δz in general, one can always take the measured values x_0, y_0 and work out z_0. Then if one evaluates the result by substituting the actual numerical values of $x_0 + \delta x$, $y_0 + \delta y$ (or $y_0 - \delta y$ if appropriate) to give one of the outer values of z and then repeating the other way, the limits on z have been determined and δz obtained.

3.6 Compensating Errors

A special situation can arise when compound variables are involved. Consider, for example, the well-known relation for the angle of minimum deviation D in a prism of refractive index μ and vertical angle A

$$\mu = \frac{\sin \frac{1}{2}(A + D)}{\sin \frac{1}{2} A}$$

If A and D are measured variables with uncertainties δA and δD, the quantity μ will be the required answer, with an uncertainty $\delta\mu$. It would be fallacious, however, to calculate the uncertainty in $A + D$, then in $\sin \frac{1}{2}(A + D)$, and combine it with the uncertainty in $\sin \frac{1}{2} A$, treating the function as a quotient of two variables. This can be seen by thinking of the effect on μ of an increase in A. Both $\sin \frac{1}{2}(A + D)$ *and* $\sin \frac{1}{2} A$ increase, and the change in μ is not correspondingly large. The fallacy is in the application of the particular methods of the previous sections to variables which are not independent (e.g., $A + D$ and A). The cure is either to reduce the equation to a form in which the

variables are all independent, or else to go back to first principles and use the equation of Sec. 3.5 directly.

Cases which involve compensation of errors should be watched carefully since they can, if treated incorrectly, give rise to large errors in uncertainty calculations.

3.7 Standard Deviation of Computed Values: General Methods

As has been frequently stressed, this last section has been concerned with outer limits of possibility for the computed value z. We have already suggested that this represents an unrealistically pessimistic approach and that the more useful quantity would be a *probable* value for δz, provided we can attach a numerical meaning to "probable." The limits given by this quantity will be smaller than $\pm\delta z$, but we have the hope of an actual numerical significance for them. Such statistical validity will be possible only if the uncertainties in x and y have such validity, and we shall, therefore, assume that the measurements have been sufficiently numerous to justify a calculation of the standard deviation of the x values s_x, and correspondingly, of s_y. We then hope to be able to calculate an s_z.

However, we must first inquire what we mean by s_z. We assume that the measurement has taken the form of pairs of observations x, y (for example, the current through and the potential across a resistor, which have been measured with the aim of obtaining the resistance) obtained by repetition under the same conditions. Each pair will define a value of z and, if the repetition had yielded n pairs, we shall have a set of n values of z showing statistical fluctuations. The quantity we require, s_z, is the standard deviation of this set of z values. Now these individual z values may never be calculated, because one would calculate the means $\bar{x}$ and $\bar{y}$ and obtain $\bar{z}$ directly using the assumption (valid if s_x, s_y and s_z are small compared, respectively, with x, y, and z) that

$$\bar{z} = f(\bar{x}, \bar{y})$$

Nevertheless, that is the significance of the s_z we are about to calculate.

If we assume that the universes of the x, y, and z values have a Gaussian distribution, the quantity σ_z (of which we are about to calculate the best estimate in terms of s_z) will have the usual significance that any z value will stand a 68 per cent chance of falling within $\pm\sigma_z$ of the true value.

As before, let

$$z = f(x, y)$$

and consider perturbations δx, δy which lead to a perturbation δz given by

$$\delta z = \left(\frac{\partial z}{\partial x}\right) \delta x + \left(\frac{\partial z}{\partial y}\right) \delta y$$

This perturbation can be used to calculate a standard deviation for the n different z values since

$$s_z = \sqrt{\Sigma\, (\delta z)^2/n}$$

Thus

$$s_z^2 = \frac{1}{n} \Sigma \left[\left(\frac{\partial z}{\partial x}\right) \delta x + \left(\frac{\partial z}{\partial y}\right) \delta y\right]^2$$

$$= \frac{1}{n} \Sigma \left[\left(\frac{\partial z}{\partial x}\right)^2 (\delta x)^2 + \left(\frac{\partial z}{\partial y}\right)^2 (\delta y)^2 + 2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) \delta x\, \delta y\right]$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 \frac{\Sigma\, (\delta x)^2}{n} + \left(\frac{\partial z}{\partial y}\right)^2 \frac{\Sigma\, (\delta y)^2}{n} + \frac{2}{n}\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) \delta x\, \delta y$$

But $$\frac{\Sigma\, (\delta x)^2}{n} = s_x^2 \quad \text{and} \quad \frac{\Sigma\, (\delta y)^2}{n} = s_y^2$$

and, since δx, δy may be considered for the present purpose to be independent perturbations,

$$\Sigma\, \delta x\, \delta y = 0$$

Thus, finally

$$s_z = \sqrt{(\partial z/\partial x)^2 s_x^2 + (\partial z/\partial y)^2 s_y^2} \qquad (3.3)$$

If z is a function of more than two variables the equation is extended by adding similar terms.

Thus, if the components of a calculation have standard deviations of some degree of reliability, a value can be found for the probable uncertainty of the answer where "probable" has a real numerical significance.

The calculation has been carried out in terms of the variance or standard deviation of the x and y distributions. However, in actual practice the quantities we want are the best estimates of σ_x, σ_y, etc., and so we would use the modified value with denominator $n - 1$ in accordance with Equation (2.9). The result would then be a best estimate for σ_z. The standard deviation of the mean for z can then be calculated by direct use of Equation (2.7) and this will give the limits within which the *mean* value of z, $\bar{z}$, stands a 68 per cent chance of falling.

Note that most actual experiments do not accord with the assumptions of the development just given. If we are measuring the flow rate of water through a pipe, we shall measure the flow rate, pipe radius and pipe length independently and each one with a number of readings dictated by the intrinsic precision of the measurement. We cannot, therefore, use Equation (3.3) directly, since the various s's are different types of quantity. The solution is to calculate the standard deviation of the mean for each of the elementary quantities first. If these are used in Equation (3.3), the result of the calculation will be immediately a standard deviation of the mean for z.

3.8 Standard Deviation of Computed Values: Special Cases

Let us now apply Equation (3.3) to a few common examples. In all the following cases the various s's are all assumed to be best estimates of the appropriate universe value σ.

(a) *Sum of Two Variables*

$$z = x + y$$

hence
$$\frac{\partial z}{\partial x} = 1, \quad \frac{\partial z}{\partial y} = 1$$

and
$$s_z = \sqrt{s_x^2 + s_y^2}$$

Note that this result affords a justification for Equation (2.7) on page 33. The mean value for the sample, $\Sigma\, x_i/n$, is just a function such as $z = x + y$, where x and y happen to be independent measurements of the *same* quantity. Thus if

$$z = \frac{1}{n}(x_1 + x_2 + x_3 + \cdots)$$

$$\frac{\partial z}{\partial x_1} = \frac{1}{n}, \quad \frac{\partial z}{\partial x_2} = \frac{1}{n}, \quad \text{etc.}$$

and
$$s_z = \sqrt{\left(\frac{1}{n}\right)^2 s_x^2 + \left(\frac{1}{n}\right)^2 s_x^2 + \cdots}$$

$$= \sqrt{n s_x^2/n^2} = \frac{s_x}{\sqrt{n}}$$

(b) *Difference of Two Variables*

$$z = x - y$$

Here
$$\frac{\partial z}{\partial x} = 1, \quad \frac{\partial z}{\partial y} = -1$$

but again
$$s_z = \sqrt{s_x^2 + s_y^2}$$

As dealt with in Sec. 3.4 on page 56, the previous considerations regarding measurements of differences are still valid.

(c) *Product of Two Variables*

$$z = xy$$

hence
$$\frac{\partial z}{\partial x} = y, \quad \frac{\partial z}{\partial y} = x$$

thus
$$s_z = \sqrt{y^2 s_x^2 + x^2 s_y^2}$$

and the specific value for s_z at the particular values x_0, y_0 of x and y would be obtained by substituting x_0 and y_0 in the equations.

Just as in the previously treated case of products, the equation is more clearly expressed in terms of relative values of s i.e. s_z/z. We obtain

$$\frac{s_z}{z} = \sqrt{s_x^2/x^2 + s_y^2/y^2}$$

(d) *Variables Raised to Powers*

$$z = x^a$$

$$\frac{\partial z}{\partial x} = a x^{a-1}$$

$$s_z = \sqrt{a^2 x^{2(a-1)} s_x^2}$$

Again this is more instructive when expressed in terms of the relative value

$$\frac{s_z}{z} = \sqrt{a^2 \frac{s_x^2}{x^2}}$$

$$= a \frac{s_x}{x}$$

(e) *The General Case of Powers and Products*

$$z = x^a y^b$$

Obviously the results of (c) and (d) can be extended to give the result

$$\frac{s_z}{z} = \sqrt{\left(a \frac{s_x}{x}\right)^2 + \left(b \frac{s_y}{y}\right)^2}$$

In this result note that the presence of negative indices in the original function is unimportant, since they occur only squared in the expression for s_z.

If a function other than those listed above is encountered, the use of Equation (3.3) will yield the required result. It can be seen that, for the case of a function of a single variable, $z = f(x)$, Equation (3.3) reduces to the same form as that for uncertainties, Equation (3.1). The result is, therefore, the same for standard deviations as it was for uncertainties in the case of the trigonometric, exponential and logarithmic functions treated in Sec. 3.3.

Note that, although we listed in Secs. 3.2 to 3.5 a number of different approaches to the problem of outside limits to uncertainty, the standard deviation of z is a uniquely defined quantity and there is no alternative to the use of Equation (3.3).

3.9 Combination of Different Types of Uncertainty

Unfortunately for the mathematical elegance of the development, it very frequently occurs that the uncertainty in a computed result is required when the component quantities have different types of uncertainty. Thus we may require the uncertainty in

$$z = f(x, y)$$

where x is a quantity to which have been assigned outer limits $\pm\delta x$ within which we are "almost certain" that the actual value lies and y is a quantity whose uncertainty is statistical in nature, and for which a sample standard deviation s_y and a standard deviation of the mean $s_y/\sqrt{n}$ have been calculated. We require the uncertainty in z. The problem is that the uncertainty in z is a difficult thing even to define. We are trying to combine two quantities which have, in effect, completely different distribution curves. One is the standard Gaussian function but the other is a rectangle, bounded by the outer limits of uncertainty, and flat on top because the actual value of the unknown x is equally likely to be anywhere between the outer limits $x_0 \pm \delta x$. Any general method of solving this problem is likely to be far too complex for general use, but particular solutions can be found following a method suggested by Dr. T. M. Brown.

In the calculation for z one uses the sample mean $\bar{y}$ for the y value. This has the significance that it stands approximately a $\frac{2}{3}$ chance of coming within $\pm s_y/\sqrt{n}$ of the true value. Let us therefore calculate limits for x which, similarly, give a $\frac{2}{3}$ probability of enclosing the true value. Since the probability distribution for x is rectangular, $\frac{2}{3}$ of the