Table 12.7

	Operating level	Cost per hour	Reservoir depth reduction per hour	Start-up cost
Hydro A	900 MW	£90	0·31 metres	£1500
Hydro B	1400 MW	£150	0·47 metres	£1200

generators: one of type A and one of type B. When a hydro generator is running, it operates at a fixed level and the depth of the reservoir decreases. The costs associated with each hydro generator are a fixed start-up cost and a running cost per hour. The characteristics of each type of generator are shown in Table 12.7.

For environmental reasons, the reservoir must be maintained at a depth of between 15 and 20 metres. Also, at midnight each night, the reservoir must be 16 metres deep. Thermal generators can be used to pump water into the reservoir. To increase the level of the reservoir by 1 metre requires 3000 MWh of electricity. You may assume that rainfall does not affect the reservoir level.

At any time it must be possible to meet an increase in demand for electricity of up to 15%. This can be achieved by any combination of the following: switching on a hydro generator (even if this would cause the reservoir depth to fall below 15 metres); using the output of a thermal generator which is used for pumping water into the reservoir; and increasing the operating level of a thermal generator to its maximum. Thermal generators cannot be switched on instantaneously to meet increased demand (although hydro generators can be).

Which generators should be working in which periods of the day, and how should the reservoir be maintained to minimize the total cost?

12.17 Three-dimensional Noughts and Crosses

Twenty-seven cells are arranged $3 \times 3 \times 3$ in a three-dimensional array as shown in Figure 12.5.

Three cells are regarded as lying in the same line if they are on the same horizontal or vertical line or the same diagonal. Diagonals exist on each

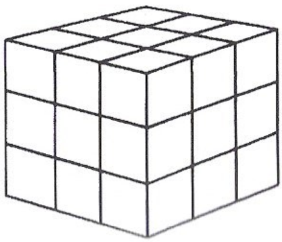

Figure 12.5

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horizontal and vertical section and connecting opposite vertices of the cube. (There are 49 lines altogether.)

Given 13 white balls (noughts) and 14 black balls (crosses), arrange them, one to a cell, so as to minimize the number of lines with balls all of one colour.

12.18 Optimizing a Constraint

In an integer programming problem the following constraint occurs:

$$9x_1 + 13x_2 - 14x_3 + 17x_4 + 13x_5 - 19x_6 + 23x_7 + 21x_8 \leqslant 37.$$

All the variables occurring in this constraint are 0–1 variables, i.e. they can only take the value of 0 or 1.

Find the 'simplest' version of this constraint. The objective is to find another constraint involving these variables which is logically equivalent to the original constraint but which has the smallest possible absolute value of the right-hand side (with all coefficients of similar signs to the original coefficients).

If the objective were to find an equivalent constraint where the sum of the absolute values of the coefficients (apart from the right-hand side coefficient) were a minimum what would be the result?

12.19 Distribution 1

A company has two factories, one at Liverpool and one at Brighton. In addition it has four depots with storage facilities at Newcastle, Birmingham, London and Exeter. The company sells its product to six customers C1, C2, . . . , C6. Customers can be supplied either from a depot or from the factory direct (see Figure 12.6).

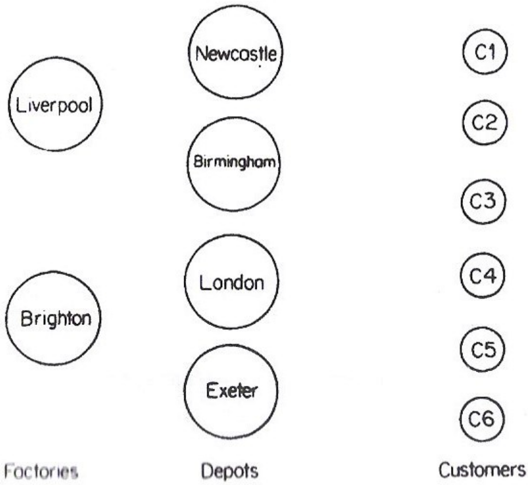

Figure 12.6

Table 12.7[a]

Supplied to	Liverpool factory	Brighton factory	Newcastle depot	Birmingham depot	London depot	Exeter depot
Depots						
Newcastle	0·5	—				
Birmingham	0·5	0·3				
London	1·0	0·5				
Exeter	0·2	0·2				
Customers						
C1	1·0	2·0	—	1·0	—	—
C2	—	—	1·5	0·5	1·5	—
C3	1·5	—	0·5	0·5	2·0	0·2
C4	2·0	—	1·5	1·0	—	1·5
C5	—	—	—	0·5	0·5	0·5
C6	1·0	—	1·0	—	1·5	1·5

[a] A dash indicates the impossibility of certain suppliers for certain depots or customers.

The distribution costs (which are borne by the company) are known; they are given in Table 12.7 (in £ per ton delivered).

Certain customers have expressed preferences for being supplied from factories or depots which they are used to. The preferred suppliers are

C1	Liverpool (factory)
C2	Newcastle (depot)
C3	No preferences
C4	No preferences
C5	Birmingham (depot)
C6	Exeter or London (depots)

Each factory has a monthly capacity given below which cannot be exceeded:

Liverpool	150 000 tons
Brighton	200 000 tons

Each depot has a maximum monthly throughput given below which cannot be exceeded:

Newcastle	70 000 tons
Birmingham	50 000 tons
London	100 000 tons
Exeter	40 000 tons

Each customer has a monthly requirement given below which must be met:

C1	50 000 tons
C2	10 000 tons
C3	40 000 tons
C4	35 000 tons
CS	60 000 tons
C6	20 000 tons

The company would like to determine:

(1) What distribution pattern would minimize overall cost?
(2) What the effect of increasing factory and depot capacities would be on distribution costs?
(3) What the effects of small changes in costs, capacities and requirements would be on the distribution pattern?
(4) Would it be possible to meet all customers preferences regarding suppliers and if so what would the extra cost of doing this be?

12.20 Depot Location (Distribution 2)

In the distribution problem there is a possibility of opening new depots at Bristol and Northampton as well as of enlarging the Birmingham depot.

It is not considered desirable to have more than four depots and if necessary Newcastle or Exeter (or both) can be closed down.

The monthly costs (in interest charges) of the possible new depots and expansion at Birmingham are given in Table 12.8 together with the potential monthly throughputs.

The monthly savings of closing down the Newcastle and Exeter depots are given in Table 12.9.

Table 12.8

	Cost (£1000)	Throughput (1000 tons)
Bristol	12	30
Northampton	4	25
Birmingham (expansion)	3	20

Table 12.9

	Saving (£1000)
Newcastle	10
Exeter	5

Table 12.10

Supplied to	Supplier: Liverpool factory	Brighton factory	Bristol depot	Northampton depot
New depots				
Bristol	0·6	0·4		
Northampton	0·4	0·3		
Customers				
C1	As given for distribution problem		1·2	—
C2			0·6	0·4
C3			0·5	—
C4			—	0·5
C5			0·3	0·6
C6			0·8	0·9

The distribution costs involving the new depots are given in Table 12.10 (in £ per ton delivered).

Which new depots should be built? Should Birmingham be expanded? Should Exeter or Newcastle be closed down? What would be the best resultant distribution pattern to minimize overall costs?

12.21 Agricultural Pricing

The government of a country wants to decide what prices should be charged for its dairy products, milk, butter and cheese. All these products arise directly or indirectly from the country's raw milk production. This raw milk is usefully divided into the two components of fat and dry matter. After subtracting the quantities of fat and dry matter which are used for making products for export or consumption on the farms there is a total yearly availability of 600 000 tons of fat and 750 000 tons of dry matter. This is all available for producing milk, butter and two kinds of cheese for domestic consumption.

The percentage compositions of the products are given in Table 12.11.

For the previous year the domestic consumption and prices for the products are given in Table 12.12.

Table 12.11

	Fat	Dry matter	Water
Milk	4	9	87
Butter	80	2	18
Cheese 1	35	30	35
Cheese 2	25	40	35

Table 12.12

	Milk	Butter	Cheese 1	Cheese 2
Domestic consumption (1000 tons)	4820	320	210	70
Price (£/ton)	297	720	1050	815

Price elasticities of demand, relating consumer demand to the prices of each product, have been calculated on the basis of past statistics. The price elasticity E of a product is defined by

$$E = \frac{\text{percentage decrease in demand}}{\text{percentage increase in price}}.$$

For the two makes of cheese there will be some degree of substitution in consumer demand depending on relative prices. This is measured by cross-elasticity of demand with respect to price. The cross-elasticity E_{AB} from a product A to a product B is defined by

$$E_{\mathrm{AB}} = \frac{\text{percentage increase in demand for A}}{\text{percentage increase in price of B}}.$$

The elasticities and cross-elasticities are given in Table 12.13.

The objective is to determine what prices and resultant demand will maximize total revenue.

It is, however, politically unacceptable to allow a certain price index to rise. As a result of the way this index is calculated this limitation simply demands that the new prices must be such that the total cost of last year's consumption would not be increased. A particularly important additional requirement is to quantify the economic cost of this political limitation.

Table 12.13

Milk	Butter	Cheese 1	Cheese 2	Cheese 1 to Cheese 2	Cheese 2 to Cheese 1
0·4	2·7	1·1	0·4	0·1	0·4

12.22 Efficiency Analysis

A car manufacturer wants to evaluate the efficiencies of different garages who have received a franchise to sell its cars. The method to be used is Data Envelopment Analysis (DEA). References to this technique are given in Section 3.2. Each garage has a certain number of measurable 'inputs'. These are taken to be: *Staff*, *Showroom Space*, *Catchment Population* in different coefficients does not exceed the right-hand side coefficient. Such a subset is maximal in the sense that no subset properly containing it, or to the left in the implied lexicographical ordering, can also be a ceiling. For example the subset {1, 2, 4, 8} is a ceiling, $23 + 21 + 17 + 9 \leqslant 70$, but any subset property containing it (e.g. {1, 2, 4, 7, 8}) or to the 'left' of it (e.g. {1, 2, 4, 7}) is not a ceiling. 'Roofs' are 'minimal' subsets of the indices for which the sum of the corresponding coefficients exceeds the right-hand side coefficient. Such a subset is 'minimal' in the same sense as a subset is 'maximal'. For example {2, 3, 4, 5} is a roof, $21 + 19 + 17 + 14 > 70$, but any subset properly contained in it (e.g. {3, 4, 5}) or to the 'right' of it (e.g. {2, 3, 4, 6}) is not a roof.

If $\{i_1, i_2, \ldots, i_r\}$ is a 'ceiling' the following condition among the new coefficients a_i is implied:

$$a_{i1} + a_{i2} + \cdots a_{ir} \leqslant a_0$$

If $\{i_1, i_2, \ldots, i_r\}$ is a 'roof' the following condition among the new coefficients a_i is implied:

$$a_{i1} + a_{i2} + \cdots a_{ir} \geqslant a_0 + 1$$

It is also necessary to guarantee the ordering of the coefficients. This can be done by the series of constraints:

$$a_1 \geqslant a_2 \geqslant a_3 \geqslant \cdots \geqslant a_8$$

If these constraints are given together with each constraint corresponding to a roof or ceiling then this is a sufficient set of conditions to guarantee that the new 0–1 constraint has exactly the same set of feasible 0–1 solutions as the original 0–1 constraint.

In order to pursue the first objective we minimize $a_0 - a_3 - a_5$ subject to these constraints.

For the second objective we minimize $\sum_{i=1}^{8} a_i$.

For this example the set of ceilings is

$$\{1,2,3\},\ \{1,2,4,8\},\ \{1,2,6,7\},\ \{1,3,5,6\},\ \{2,3,4,6\},\ \{2,5,6,7,8\}$$

The set of roofs is

$$\{1,2,3,8\},\ \{1,2,5,7\},\ \{1,3,4,7\},\ \{1,5,6,7,8\},\ \{2,3,4,5\},\ \{3,4,6,7,8\}$$

The resultant model has 19 constraints and nine variables.

If the constraint were to involve *general integer* rather than 0–1 variables, then we could still formulate the simplification problem in a similar manner after first converting the constraint to one involving 0–1 variables in the way described in Section 10.1. It is, however, necessary to ensure, by extra constraints in our LP model, the correct relationship between the coefficients in the simplified 0–1 form. How this may be done is described in Section 10.2.

13.19 Distribution 1

This problem can be regarded as one of finding the minimum cost flow through a network. Such network flow problems have been extensively treated in the mathematical programming literature. A standard reference is Ford and Fulkerson (1962). Specialized algorithms exist for solving such problems and are described in Ford and Fulkerson (1962), Jensen and Barnes (1980), Glover and Klingman (1977), and Bradley (1975).

It is, however, always possible to formulate such problems as ordinary linear programming models. Such models have the total unimodularity property described in Section 10.1. This property guarantees that the optimal solution to the LP problem will be integer as long as the right-hand side coefficients are integer.

We choose to formulate this problem as an ordinary LP model in order that we may use the standard revised simplex algorithm. There would be virtue in using a specialized algorithm. The special features of this sort of problem which make the use of a specialized algorithm worthwhile also, fortunately, make the problem fairly easy to solve as an ordinary LP problem. Sometimes, however, when formulated in this way the resultant model is very large. The use of a specialized algorithm then also becomes desirable as it results in a compact representation of the problem. As the example presented is very small, such considerations do not arise here.

The factories, depots and customers will be numbered as below:

Factories	1	Liverpool
	2	Brighton
Depots	1	Newcastle
	2	Birmingham
	3	London
	4	Exeter
Customers	C1	to C6

Variables

$$x_{ij} = \text{quantity sent from factory } i \text{ to depot } j,$$
$$i = 1, 2, \quad j = 1, 2, 3, 4;$$
$$y_{ik} = \text{quantity sent from factory } i \text{ to customer } k,$$
$$i = 1, 2, \quad k = 1, 2, \ldots, 6;$$
$$z_{jk} = \text{quantity sent from depot } j \text{ to customer } k,$$
$$j = 1, 2, 3, 4, \quad k = 1, 2, \ldots, 6.$$

There are 44 such variables.

Constraints

Factory Capacities

$$\sum_{j=1}^{2} x_{ij} + \sum_{k=1}^{6} y_{ik} \leqslant \text{capacity}, \qquad i = 1, 2.$$

Quantity into Depots

$$\sum_{i=1}^{2} x_{ij} \leqslant \text{capacity}, \qquad j = 1, 2, 3, 4.$$

Quantity out of Depots

$$\sum_{k=1}^{6} z_{jk} = \sum_{i=1}^{2} x_{ij} \qquad j = 1, 2, 3, 4$$

Customer Requirements

$$\sum_{i=1}^{2} y_{ik} + \sum_{j=1}^{4} z_{jk} = \text{requirement}, \qquad k = 1, 2, \ldots, 6.$$

The capacity, quantity and requirement figures are given with the statement of the problem in Part 2.

There are 16 such constraints.

Objectives

The first objective is to minimize cost. This is given by

$$\sum_{\substack{i=1\\j=1}}^{\substack{i=2\\j=4}} c_{ij}x_{ij} + \sum_{\substack{i=1\\k=1}}^{\substack{i=2\\k=6}} d_{ik}y_{ik} + \sum_{\substack{j=1\\k=1}}^{\substack{j=2\\k=6}} e_{jk}z_{jk},$$

where the coefficients c_{ij}, d_{ik}, and e_{jk} are given with the problem in Part 2.

The second objective will take the same form as that above, but this time the c_{ij}, d_{ik}, and e_{jk} will be defined as below:

$$d_{ik} = \begin{cases} 0 & \text{if customer } k \text{ prefers factory } i, \\ 1 & \text{otherwise} \end{cases}$$

$$e_{ik} = \begin{cases} 0 & \text{if customer } k \text{ prefers depot } j, \\ 1 & \text{otherwise} \end{cases}$$

$$c_{ij} = 0 \qquad \text{for all } i, j.$$

This objective is to be minimized.

13.20 Depot Location (Distribution 2)

The linear programming formulation of the distribution problem can be extended to a *mixed integer* model to deal with the extra decisions of whether to build or close down depots. Extra 0–1 integer variables are introduced with the following interpretations:

$$\delta_1 = \begin{cases} 1 & \text{if the Newcastle depot is retained,} \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta_2 = \begin{cases} 1 & \text{if the Birmingham depot is expanded,} \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta_4 = \begin{cases} 1 & \text{if the Exeter depot is retained,} \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta_5 = \begin{cases} 1 & \text{if a depot is built at Bristol,} \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta_6 = \begin{cases} 1 & \text{if a depot is built at Northampton,} \\ 0 & \text{otherwise.} \end{cases}$$

In addition extra continuous variables x_{i5}, x_{i6}, z_{5k}, and z_{6k} are introduced to represent quantities sent to and from the new depots.

The following constraints are added to the model.

If a depot is closed down or not built then nothing can be supplied to it or from it:

$$\sum_{i=1}^{2} x_{ij} \leqslant T_j \delta_j,$$

where T_j is the capacity of depot j.

From Birmingham the quantity supplied to and from the depot must lie within the extension:

$$\sum_{i=1}^{2} x_{12} \leqslant 50 + 20\delta_2.$$

There can be no more than four depots (including Birmingham and London):

$$\delta_1 + \delta_4 + \delta_5 + \delta_6 \leqslant 2.$$

In the objective function the new x_{ij} and z_{jk} variables are given their appropriate costs. The additional expression involving the δ_j variables is added to the objective function:

$$10\delta_1 + 3\delta_2 + 5\delta_4 + 12\delta_5 + 4\delta_6 - 15.$$

This model has 21 constraints and 65 variables (five are integer and 0–1).

Pumping should take place in the following periods at the given levels:

Period 1	815 MW
Period 3	950 MW
Period 5	350 MW

Although it may seem paradoxical to both pump and run Hydro B in period 5, this is necessary to meet the requirement of the reservoir being at 16 metres at the beginning of period 1, given that the Hydro can work only at a fixed level. It would be possible to use the model to cost this environmental requirement.

The height of the reservoir at the beginning of each period should be:

Period 1	12 metres
Period 2	17.63 metres
Period 3	17.63 metres
Period 4	19.53 metres
Period 5	18.12 metres

The cost of these operations is £986 630.

In solving this model it is valuable to exploit the fact that the optimal objective value must be less than or equal to that reported in Section 14.15. This is for two reasons. Firstly the optimal solution in Section 14.15 using only thermal generators is a feasible solution to this model. Secondly the 15% extra output guarantee can be met at no start-up cost using the hydro generators. When solving this model by the branch and bound method the optimal objective value in Section 14.15 could be used as an 'objective cut-off' to prune the tree search.

Planning the use of hydro power by means of Stochastic Programming in order to model uncertainty is described by Archibald, Buchanan, McKinnon and Thomas (1999).

14.17 Three-dimensional Noughts and Crosses

The minimum number of lines of the same colour is four. There are many alternative solutions, one of which is given in Figure 14.5, where the top, middle and bottom sections of the cube are given. Cells with black balls are shaded.

This solution was obtained in 15 nodes. A total of 1367 nodes were needed to prove optimality.

14.18 Optimizing a Constraint

The 'simplest' version of this constraint (with minimum right-hand side coefficient) is

$$6x_1 + 9x_2 - 10x_3 + 12x_4 + 9x_5 - 13x_6 + 16x_7 + 14x_8 \leqslant 25.$$

This is also the equivalent constraint with the minimum sum of absolute values of the coefficients.

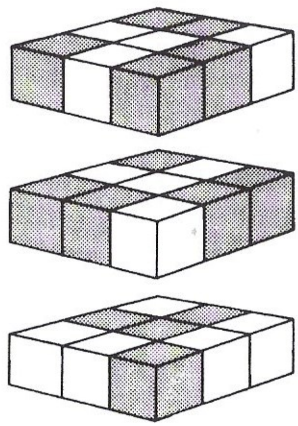

Figure 14.5

14.19 Distribution 1

The minimum cost distribution pattern is shown in Figure 14.6 (with quantities in thousands of tons).

There is an alternative optimal solution in which the 40 000 tons from Brighton to Exeter come from Liverpool instead.

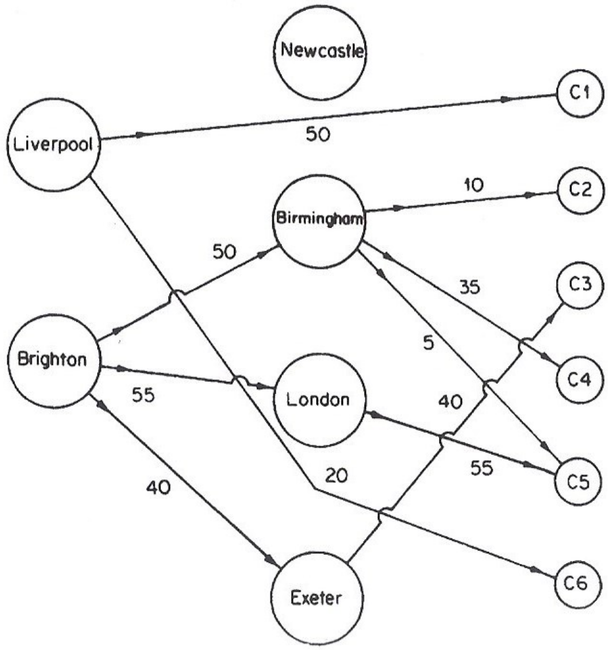

Figure 14.6

This distribution pattern costs £198 500 per month.

Depot capacity is exhausted at Birmingham and Exeter. The value (in reducing distribution costs) of an extra ton per month capacity in these depots is £0·20 and £0·30 respectively.

This distribution pattern will remain the same as long as the unit distribution costs remain within certain ranges. These are given below (for routes which are to be used):

Route	Cost range
Liverpool to C1	$-\infty$ to 1·5
Liverpool to C6	$-\infty$ to 1·2
Brighton to Birmingham	$-\infty$ to 0·5
Brighton to London	0·3 to 0·8
Brighton to Exeter	$-\infty$ to 0·2
Birmingham to C2	$-\infty$ to 1·2
Birmingham to C4	$-\infty$ to 1·2
Birmingham to C5	0·3 to 0·7
London to C5	0·3 to 0·8
Exeter to C3	0 to 0·5

Depot capacities can be altered within certain limits. For the not fully utilized depots of Newcastle and London changing capacity within these limits has no effect on the optimal distribution pattern. For Birmingham and Exeter the effect on total cost will be £0·2 and £0·3 per ton per month within the limits. Outside certain limits the prediction of the effect requires resolving the problem. The limits are:

Depot	Capacity range
Birmingham	45 000 to 105 000 tons
Exeter	40 000 to 95 000 tons

N.B. All the above effects of changes are only valid if *one* thing is changed at a time within the permitted ranges. Clearly the above solution does not satisfy the customer preferences for suppliers.

By minimizing the second objective it is possible to reduce the number of goods sent by non-preferred suppliers to a customer to a minimum. This was done and revealed that it is impossible to satisfy all preferences. The best that could be done resulted in the distribution pattern shown in Figure 14.7, where customer C5 receives 10 000 tons from his non-preferred depot of London. This is the minimum cost such distribution pattern. (There are alternative patterns which also minimize the number of non-preferences but which cost more.) The minimum cost here is £246 000, showing that the extra cost of satisfying more customers preferences is £47 500.

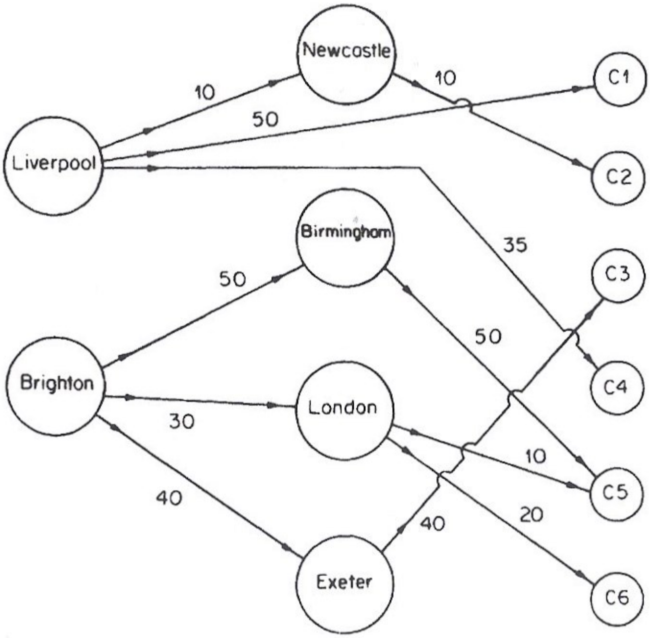

Figure 14.7

14.20 Depot Location (Distribution 2)

The minimum cost solution is to close down the Newcastle depot and open a depot in Northampton. The Birmingham depot should be expanded. The total monthly cost (taking account of the saving from closing down Newcastle) resulting from these changes and the new distribution pattern is £174 000. Figure 14.8 shows the new distribution pattern (with quantities in thousands of tons).

This solution was obtained in 40 iterations. The continuous optimal solution was integer. Therefore no tree search was necessary.

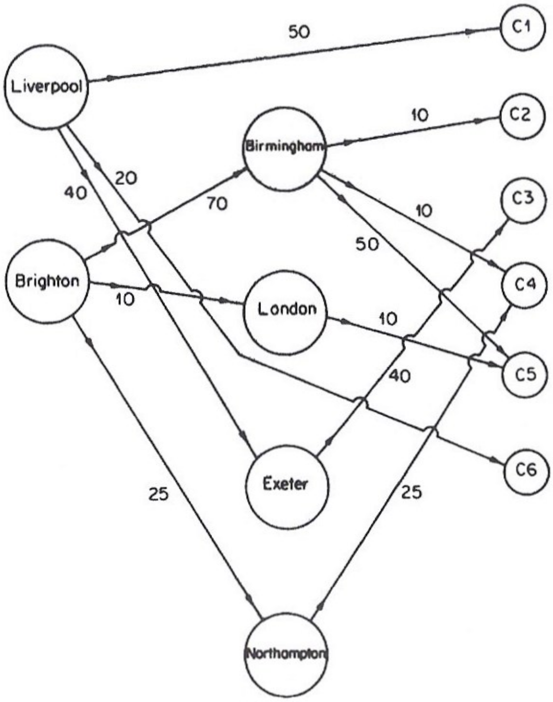

Figure 14.8

14.21 Agricultural Pricing

The optimal prices are

Milk	£303 per ton
Butter	£667 per ton
Cheese 1	£900 per ton
Cheese 2	£1085 per ton

The resultant yearly revenue will be £1992m. It is straightforward to calculate the yearly demands which will result from these prices. They are

Milk	4 781 000 tons
Butter	384 000 tons
Cheese 1	250 000 tons
Cheese 2	57 000 tons

The economic cost of imposing a constraint on the price index can be obtained from the shadow price on the constraint. For this example this shadow price in the optimal solution indicates that each £1 by which the new prices are allowed to increase the cost of last year's consumption would result in an increased revenue of £0·61.

14.22 Efficiency Analysis

The efficient garages turn out to be: 3 (Basingstoke), 6 (Newbury), 7 (Portsmouth), 8 (Alresford), 9 (Salisbury), 11 (Alton), 15 (Weymouth), 16 (Portland), 18 (Petersfield), 22 (Southampton), 23 (Bournemouth), 24 (Henley), 25 (Maidenhead), 26 (Fareham) and 27 (Romsey).

It should be observed that these garages may be efficient for different reasons. For example, Newbury has 12 times the staff of Basingstoke but only five times as much showroom space. It sells 10 times as many Alphas, 10.4 times as many Betas and makes nine times as much profit. This suggests it makes more efficient use of showroom space, but less of staff.

The other garages are deemed inefficient. They are listed in Table 14.8 in decreasing order of efficiency together with the multiples of the efficient garages which demonstrate them to be inefficient.

For example, the comparators to Petworth taken in the multiples given below use inputs of:

Staff	5.02
Showroom space	550 square metres
Category 1 population	2 (1000s)
Category 2 population	2 (1000s)
Alpha enquiries	7·35 (100s)
Beta enquiries	3·98 (100s)

to produce outputs of

1·518 (1000s)	Alpha sales
0·568 (1000s)	Beta sales
1·568 (£million)	Profit

Table 14.8

Garage	Efficiency number	Multiples of efficient garages
19 Petworth	0·988	0·066(6) + 0·015(18) + 0·034(25) + 0·675(26)
21 Reading	0·982	1·269(3) + 0·544(15) + 1·199(16) + 2·86(24) + 1·37(25)
14 Bridport	0·971	0·033(3) + 0.470(16) + 0.783(24) + 0·195(25)
2 Andover	0·917	0·857(15) + 0·214(25)
28 Ringwood	0·876	0·008(3) + 0·320(16) + 0·146(24)
5 Woking	0·867	0·952(8) + 0·021(11) + 0·009(22) + 0·148(25)
4 Poole	0·862	0·329(3) + 0·757(16) + 0·434(24) + 0·345(25)
12 Weybridge	0·854	0·797(15) + 0·145(25) + 0·018(26)
1 Winchester	0·840	0·005(7) + 0·416(8) + 0·403(9) + 0·333(15) + 0·096(16)
13 Dorchester	0·839	0·134(3) + 0·104(8) + 0·119(15) + 0·752(16) + 0·035(24) + 0·479(26)
20 Midhurst	0·829	0·059(9) + 0·066(15) + 0·472(16) + 0·043(18) + 0·009(25)
17 Chichester	0·824	0·058(3) + 0·097(8) + (0·335(15) + 0·166(16) + 0·236(24) + 0·154(26)
10 Guildford	0·814	0·425(3) + 0·150(7) + 0·623(8) + 0·192(15) + 0·168(16)

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