Problem TANK WAGONS

In a factory, there are 3 reactors in parallel, the product of each being different, but with the same volume, *V*, though subject to inevitable random fluctuations. It is verified also that each of the reactors does not always have its charge ready upon the passing of the tank wagon, supposed to receive all the charges, according to Figure 1, i.e, one or more charges may fail.

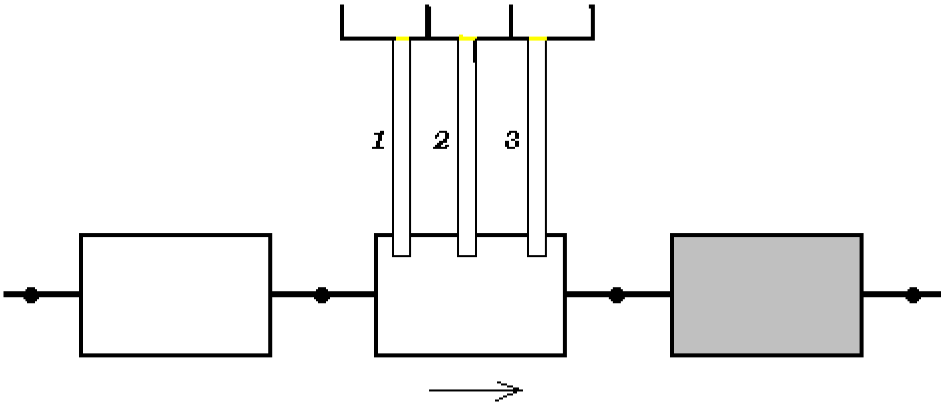

Figure 1

The volume of each charge has the following probability density:

$$g(v) = 0{,}004\,v + 0{,}02 \qquad [g] = \mathrm{L}^{-1}$$

(or $g(v) = \frac{1}{250}v + \frac{1}{50}$), with $15 < v < 25$ L*. Also, the 3 reactors fail, respectively, in 15, 20 and 5 % of the cases.

Simulate the volumes charged in each of a series of tank wagons.

* We shall not dwell on the distinction between "<" and "≤" in this context, because it is meaningless.

➲ Resolution

It is necessary to simulate *how many* (in general, also *which*) charges are produced for each tank wagon (without distinction as regards volumes, as the volumes have common characteristics) and, then, simulate the *volumes*.

It is suggested that the problem be studied on the given Excel file, but part **b)** may conveniently be read before.

a) Simulation of how many charges

In this problem, the determination of the number of charges is analogous to the following:

> *Three "unfair" coins are cast, with probabilities of, say, heads, respectively, 0,85, 0,80 and 0,95. What are the probabilities of getting 0, 1, 2 or 3 heads ?*

This phase of the problem can be solved *purely* by simulation, based on the elementary phenomena (production or not of the charge in the reactor), whose probability is simply, for each charge, a *point binomial* function, $f(x) = p^x q^{1-x}$, x = 0, 1, $0 \le p \le 1$ $(q = 1 - p)$ [Mood *et al.*, 1963, p 65]. The resolution in basic Excel benefits from this simple approach. In the case, *e.g.*, of a Poisson distribution, an adequate programming language[1] would, of course, be preferable.

The problem is solved here "manually". In Table 1 are shown (numbered 0 to 7) the 8 cases (2^3) that result from the production of a charge by the reactors (2 outcomes, *yes* or *no*, and 3 variables), meaning, *e.g.*, $x_3 = 1$ that the reactor 3 *produces* its charge. The column "Prod" contains the product that is the probability, *e.g.*: in case no. 3, it is $x_1 = 0$, $x_2 = 1$ and $x_3 = 1$ and the probability (based on given $\mathbf{p} = [0{,}85 \quad 0{,}8 \quad 0{,}95]$) is

$$(1 - 0{,}85) \times 0{,}8 \times 0{,}95 = 0{,}114 \qquad (1)$$

Column "Sum(**x**)" contains $\sum_{i=1}^{3} x_i$, which is, thus, the number of charges in that case. The probabilities of all the cases with 2 charges, $r = 2$ (mutually exclusive) give the corresponding $p(r)$,

$$p(2) = 0{,}114 + 0{,}1615 + 0{,}034 = 0{,}3095 \qquad (2)$$

Table 1

$p_i =$	0,85	0,8	0,95	Prod	Sum(**x**)	*r*	*p*(*r*)	*P*(*r*)
0	0	0	0	0,0015	0	**0**	0,0015	**0,0015**
1	0	0	1	0,0285	1	**1**	0,0430	**0,0445**
2	0	1	0	0,006	1	**2**	0,3095	**0,3540**
3	0	1	1	0,114	2	**3**	0,6460	**1**
4	1	0	0	0,0085	1		Soma: ***1***	
5	1	0	1	0,1615	2			
6	1	1	0	0,034	2			
7	1	1	1	0,646	3			
				Sum: ***1***				

From $p(r)$, the accumulated function is obtained, which permits to simulate the number of charges per tank wagon. (The mean of r, analytically, is 2,6.) For instance, the random $u_R = 0{,}2501$ is below $P(\mathbf{2}) = 0{,}3540$, so $r = 2$.

Table 2

Wag.	u_A	r_A
1	0,7502	3
2	0,2501	2
3	0,4023	3
4	0,7327	3
5	0,3983	3
6	0,6728	3
7	0,5278	3
8	0,0202	1
9	0,7161	3
10	0,7855	3
Aver.:	*0,5256*	*2,7*

[1] Such as Fortran, C, Matlab, Mathematica, etc..

b) Simulation of the volumes

We begin with the 1.st tank wagon, into which (Table 2) are put 3 charges.

To simulate the volumes, it is necessary to have their cumulative distribution function (cdf). Starting from the given pdf,

$$g(v) = \frac{1}{250} v + \frac{1}{50} \qquad [g] = \mathrm{L}^{-1} \tag{3}$$

with 15 < v < 25 L, and integrating, we have:

$$\begin{aligned} G(v) &= \int_{v_{\min}}^{v} g(w)\,\mathrm{d}w = \int_{15}^{v} \left(\frac{1}{250} w + \frac{1}{50} \right) \mathrm{d}w = \\ &= \left[\frac{w^2}{500} + \frac{w}{50} \right]_{15}^{v} = \frac{v^2 - 15^2}{500} + \frac{v - 15}{50} = \\ &= \frac{v^2}{500} + \frac{v}{50} - \frac{225 + 10 \times 15}{500} = \frac{v^2}{500} + \frac{v}{50} - \frac{375}{500} = \\ &= \frac{v^2}{500} + \frac{v}{50} - \frac{3}{4} \end{aligned} \tag{4}$$

This function, $G(v)$, is easily invertible, so, from $G = \frac{1}{500} v^2 + \frac{1}{50} v - \frac{3}{4}$, it is

$$\begin{aligned} v &= \frac{-\frac{1}{50} \pm \sqrt{\left(\frac{1}{50}\right)^2 - 4 \times \frac{1}{500} \left(-\frac{3}{4} - G\right)}}{2/500} = \\ &= -5 \pm 250 \sqrt{\frac{1 + 5(3 + 4G)}{2500}} = -5 \pm 5\sqrt{16 + 20G} = \\ &= -5 \pm 20\sqrt{1 + 1{,}25G} \end{aligned} \tag{5}$$

For G = 0, we must get the lower limit, v = 15, i.e., $15 = -5 \pm 20$, so the correct sign is +. (Also, for G = 1, it must be $v = -5 + 20\sqrt{1 + 1{,}25}$ = 25, as expected.) Using, thus, $v_\mathrm{R} = -5 + 20\sqrt{1 + 1{,}25 u_\mathrm{R}}$, we finally have the simulation of the volumes of the charges and their totals.

Table 3

Wagon	Char.^{s}	Random numbers, u_R			Random volumes, v_R (L)			Total vol. (L)
1	3	0,3658	0,2095	0,5774	**19,14332**	**17,46664**	**21,24309**	***57,85306***
2	2	0,9099	0,8493		**24,23953**	**23,71672**		***47,95625***
3	3	0,069	0,0831	0,293	**15,84466**	**16,01309**	**18,37734**	***50,23509***
4	3	0,432	0,8302	0,4006	**19,81935**	**23,54996**	**19,50102**	***62,87032***
5	3	0,194	0,9175	0,2087	**17,2935**	**24,30444**	**17,45774**	***59,05567***
6	3	0,4905	0,4489	0,8878	**20,40177**	**19,989**	**24,04996**	***64,44073***
7	3	0,3817	0,3245	0,4573	**19,30741**	**18,71181**	**20,07289**	***58,09211***
8	1	0,9049			**24,19675**			***24,19675***
9	3	0,8751	0,7701	0,6301	**23,94046**	**23,01874**	**21,74042**	***68,69962***
10	3	0,642	0,9861	0,0263	**21,85144**	**24,88394**	**15,32609**	***62,06148***
							Average:	*55,54611*

The behaviour of the final (total) volumes in the tank wagons might now be examined, finding their average, maximum, minimum, standard deviation, etc.. A frequency histogram of the values simulated —hopefully, many more than those used above—, namely, would give an idea of the pdf of the total volume, supposedly difficult to obtain analytically.

– MOOD, Alexander M., Franklin A. GRAYBILL, “Introduction to the Theory of Statistics”, 2nd. ed., McGraw-Hill Book Company, Inc., New York, Kōgakusha Company, Ltd., Tokyo, 1963.