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Special relativity as a simple geometry problem

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Abstract

The null result of the Michelson–Morley experiment and the constancy of the one-way speed of light in the 'rest system' are used to formulate a simple problem, to be solved by elementary geometry techniques using a pair of compasses and non-graduated rulers. The solution consists of a drawing allowing a direct visualization of all the fundamental effects of standard relativistic kinematics, namely time dilation, length contraction and relativity of simultaneity. Moreover, it also provides an immediate image of other important and more subtle aspects, often passed by in relativity courses, such as the conventionality of simultaneity thesis, possible non-invariance of the one-way speed of light and compatibility between the Lorentz–Poincaré and Einstein–Minkowski philosophies. The geometric scheme so constructed constitutes a powerful tool to clearly illustrate both traditional and not-so-traditional aspects of special relativity teaching.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

In the last few years, we revisited the foundations of special relativity [1–3], ultimately suggesting a somewhat unconventional approach to the teaching of the theory [3]. In those works, John Bell's claim that special relativity should be taught using Lorentzian pedagogy first [4] was openly advocated. It was argued that this can be easily and effectively achieved by introducing the IST (inertial [5, 6]–synchronized [7]–Tangherlini [8]) transformation before establishing the symmetric Lorentz transformation [3]. In addition, it was noted that the usual presentation of special relativity relies on too strong a formulation of its postulates. In truth, both postulates can be expressed in more general terms, while keeping them fully consistent with experiment [3]. For this reason, and in order to widen the view of special relativity often

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presented in textbooks, it was proposed to give the introduction to the subject simply from the definition of a 'rest system', a system in which the *one-way* speed of light in empty space is c in all directions, independent of the speed of the source emitting the light. No assumptions about the possible uniqueness or not of this frame are required to start studying relativity [2, 3]. The next step is to impose the constancy of the *two-way* speed of light on all inertial frames. This can be motivated by the conceptualization of time [1] or by simply assuming a null result for the Michelson–Morley experiment in vacuum. For more details on the Michelson–Morley experiment, see, for instance, [9] and references therein, and note as well that a new variant of this experiment was recently proposed in [10]. Finally, it was recommended to discuss the principle of relativity only at a later stage, after all the standard relativistic effects have already been analysed [3]. Additional insight was subsequently given by Iyer [11].

Herein, we propose a relatively simple geometric exercise providing to a large extent an intuitive visual image of several relativistic effects and their interpretation. The only prior requirements are the definition of the 'rest system' as enunciated above and the assumption of a null result for the Michelson–Morley experiment in vacuum, which can be taken as an experimental fact. The problem is precisely formulated in brief in section 2. Section 3 details the construction of the solution. Finally, a window to less classic observations is opened, as discussed in section 4.

2. Formulation of the problem

The question to be solved can be introduced early in the first or second lectures on special relativity, which makes it relevant to undergraduates studying relativity for the first time. For instance, a traditional presentation of the idea of the aether as a supporting medium for propagation of electromagnetic waves, followed by a brief account of the quest for the aether wind and the puzzling failure of its detection, would constitute a motivation enough to have a close look at the suggested exercise.

The problem may then be put in words in the following way. Suppose the existence of a 'rest system' S, i.e. a frame in which the one-way speed of light in vacuum is c in all directions, independent of the speed of the source emitting the light. Furthermore, consider a moving frame S' going with speed v in the x-direction.

In the Michelson–Morley experiment, light is emitted from the origin of the moving frame in two perpendicular directions, say, in the direction of motion (x, x') and in the upright direction (y'), being reflected by two mirrors placed at equal distances L' (measured in S') from the origin of the moving frame. The outcome of the experiment when performed in vacuum is that both light rays arrive at the origin of S' at the same time.

From this so-called null result, find, in the rest system,

(i) the position of the front mirror when light hits it,
(ii) the position of the origin of the moving frame at the same instant.

Further assume that distances in the y-direction are not affected by motion.

3. Geometric solution

The beginning of the construction is quite well known. Let us identify the rest system S with a sheet of paper (or with the blackboard in class). The origins of both frames coincide at a certain initial instant. Note that it would make no difference if S' had emitted light in all directions (which would be seen as a spherical wave in S) instead of only two light rays in perpendicular directions in S'.

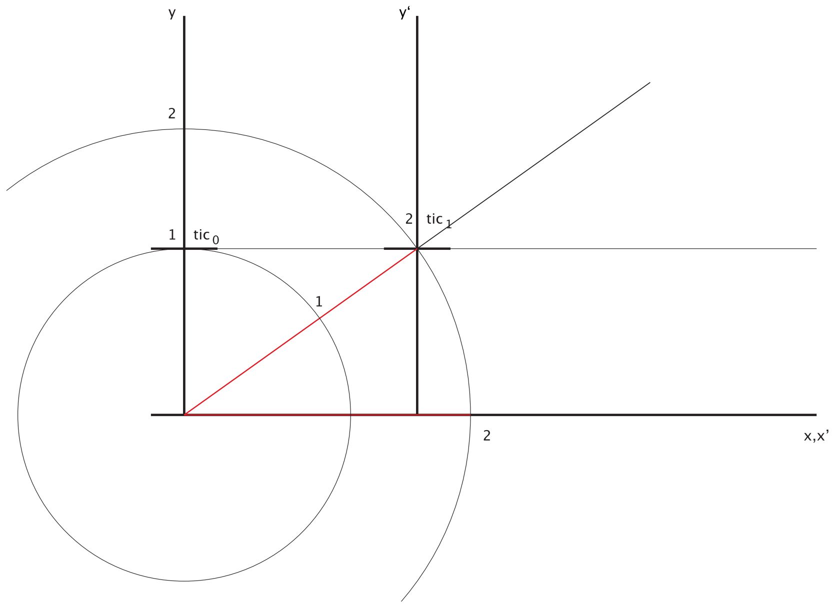

Figure 1. Light hits the top mirror.

A first instant of interest, denoted as instant 1, corresponds to the moment when light would reach the top mirror if S' was not moving, making tic_0. At a later instant 2, the light ray sent in the y'-direction hits the top mirror, making tic_1. In S this ray travels diagonally, as shown in figure 1, because the moving frame has advanced along the x-axis. The distances light travels up to instant 1 are depicted in the figure as well. Note that the distance between the origin of S and point 2 is $c\ \mathrm{tic}_1$, whereas the distance between the origins of S and S' is $v\ \mathrm{tic}_1$. The figures drawn correspond to $v/c \simeq 0.81$.

Since in S the one-way speed of light is the same in all directions, the light ray sent in the x-direction has covered exactly the same distance as the other ray. Thus, its position at instant 2 can be easily identified. It is also plotted in figure 1 and identified as label 2 on the x-axis. Clearly, tic_0 occurs before tic_1, which can be used to address time dilation, as emphasized below. This concludes the analysis of figure 1.

Another notable instant corresponds to the moment when the two light rays recombine at the origin of the moving frame and make tac, denoted by instant 4. The situation is completely symmetric in what concerns the ray travelling along the y'-axis in the up and down trips. It is then immediate to identify the position of S' at this instant by drawing a circle with the centre at the top mirror in instant 2. This position is represented in figure 2 by the vertical line with two 4s. Note that, in order to keep the figures less cluttered, the auxiliary lines used to build figure 1 were removed. The first point 4 along x identifies the origin of the moving frame; point 4 along the y'-axis which corresponds to the position light would have reached if it had not been reflected by the top mirror; the second point 4 along x then determines the position of the horizontal light ray, if it had not been reflected by the front mirror.

At this point, one already has everything necessary to understand and introduce time dilation. This is still classic and is brilliantly discussed by Feynman [12]. Readers unfamiliar with the argument are advised to refer to section 15-4 from [12]. Nevertheless, it is worth to note that complete tic-tacs can be seen as clock cycles, each of them defining a time unit

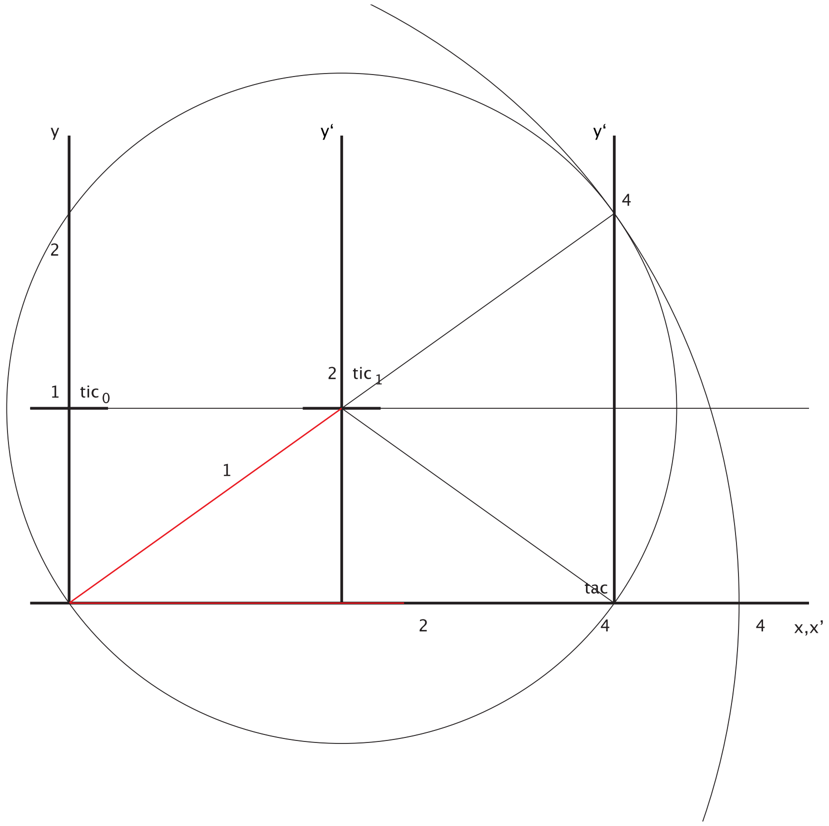

Figure 2. The recombination of both rays.

[1, 12]. For instance, tic_0 followed by the corresponding tac_0 (the moment light would reach the origin of S after reflection at instant 1) defines the time unit in the rest system and can be associated with a stationary clock. Similarly, tic_1–tac corresponds to the time unit in S' and can be associated with a moving clock. As put by Feynman [12], since tic_0 occurs before tic_1, 'it takes a *longer* time for light to go from end to end in the moving clock than in the stationary clock. Therefore, the apparent time between clicks is longer for the moving clock', which means that time passes slower in the moving frame. An animation showing a tic-tac in S (light travelling along y, i.e. in the vertical) and a tic-tac in S' (light travelling along y', i.e. in the diagonal) furnishes a direct and unforgettable visualization of the effect of time dilation and the consequent slower aging of moving observers.

The dilation factor, $\gamma = 1/\sqrt{1 - v^2/c^2}$, is easily obtained from figure 2 [12]. One direct way to obtain it is by noting that, on the one hand, $\text{tic}_0 = L'/c$; on the other hand, using the Pythagorean theorem for the diagonal path of light, and the fact that while light in the diagonal path travels a distance $c\ \text{tic}_1$ the moving frame covers a distance $v\ \text{tic}_1$, $(c\ \text{tic}_1)^2 = (v\ \text{tic}_1)^2 + (L')^2$. Hence, $\text{tic}_1 = \text{tic}_0/\sqrt{1 - v^2/c^2} = \gamma\ \text{tic}_0$.

Now going back to the geometric problem, as the position where both light rays recombine is already known, and since in S light travels with the same speed in all directions, light has to hit the front mirror exactly at the midpoint of the two points of the x-axis identified as label 4, making tic_2 and defining instant 3. This determines the position of the front mirror at instant 3,

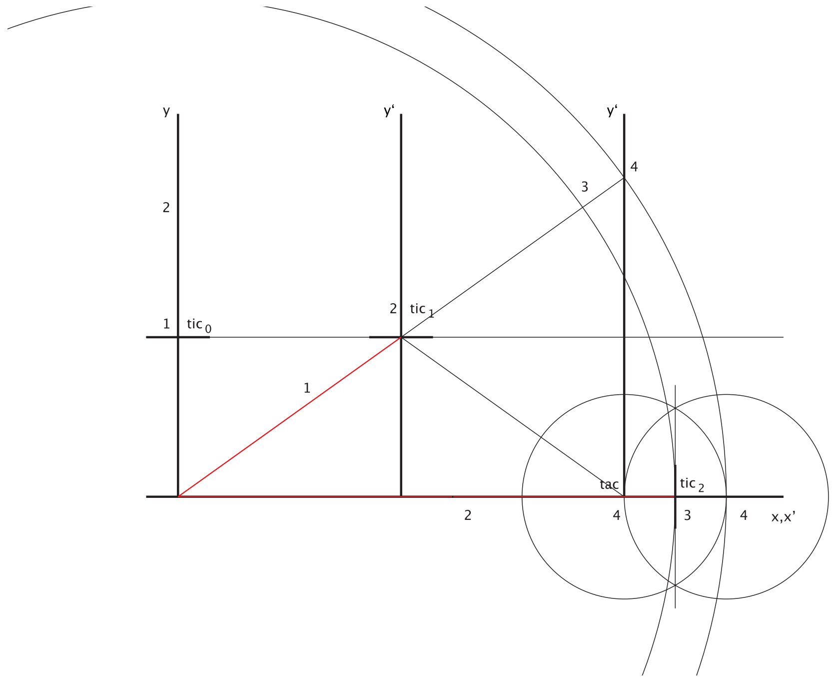

Figure 3. Light hits the front mirror.

as shown in figure 3, from which the auxiliary lines previously used were again deleted. Also shown is the distance light would have travelled in the diagonal, if it had not been reflected by the top mirror.

What is still missing is just to determine the position of S' at instant 3. It is immediately obtained by simply noting that the distance light travels in the diagonal between moments 2 and 3 after reflection is the same it would travel if it had not been reflected. A circle centred at the top mirror at instant 2 does the trick, as shown in figure 4. As before, the auxiliary lines from the previous constructions were removed. It is obvious that by marking a circle with the centre at the first point 4 along x and radius up to the position of the front mirror at instant 3, the solution can also be obtained.

The geometric construction is now finished and the problem is solved. Before moving on to less customary issues, one should still call attention to the Lorentz–Fitzgerald contraction, as shown in figure 4. Evidently, the length of the horizontal arm is the distance between the position of the origin of the moving frame and the position of the front mirror. In S, this length can be readily obtained from the last figure, using instant 3. It is marked in figure 4, where it is denoted by L. However, in S' both mirrors are placed at the same distance L' from the origin. This other length is also identified and marked in figure 4 (recall that distances along the y-direction are not affected by motion). The figure demonstrates that there is a space contraction along the direction of motion in the moving frame: the length corresponding to the distance between the two points with label 3 in the xx'-axis in figure 4 is L in S and $L' > L$ in S', which can be interpreted by saying that 'the moving metres have became shorter'. It is even possible to use the figure to obtain quantitatively the contraction factor, $1/\gamma$. To do so,

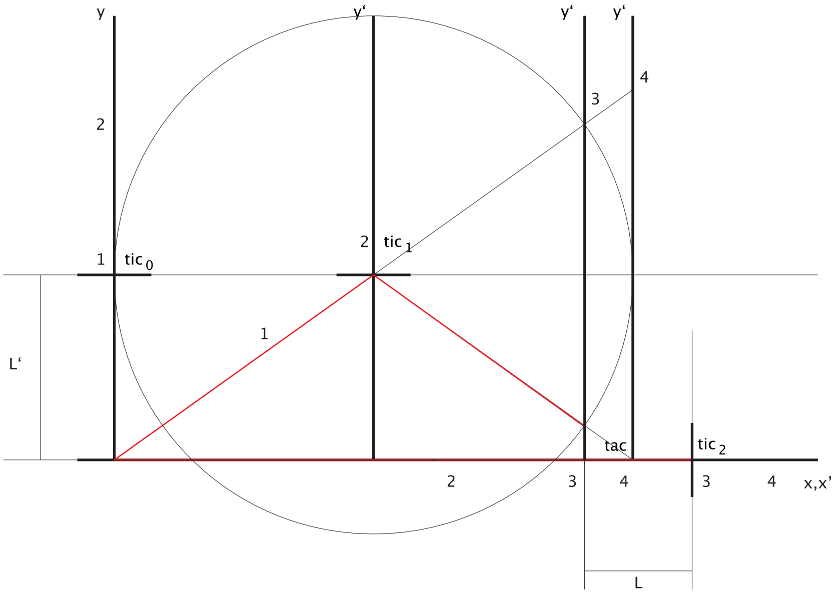

Figure 4. Light hits the front mirror and the Lorentz contraction.

start by drawing two circles of radii L' and L centred at the origin of S, then project on y the intersection of the former with the diagonal line and verify that the projection on the y-axis exactly coincides with the circle of radius L. Note that the circle with radius L' was already drawn, defining instant 1 (figure 1), although it was not really required to find the solution to the problem.

Finally, since in practice the auxiliary lines can be made thin but cannot be deleted neither on a blackboard nor on a pencil and paper drawing, for completeness figure 5 shows the resulting image with all the construction lines included.

4. Further discussion

This section suggests some additional insight that can be provided and motivated with the problem that has just been solved, which can be used as a conducting line helping to address other aspects of special relativity. It can be decided to skip the subsequent remarks in an introductory relativity course; nevertheless, we find it worth pointing them out and opening new possible approaches for teachers.

Following the proposals from [1–3], we consider that it would now be interesting to briefly address the possible constancy of the one-way speed of light. This is done without any further assumption at first, in a relatively unorthodox way.

The argument is as follows. From figure 5, the light ray travelling along y' clearly divides the time unit into two equal time intervals, 0–tic$_1$ and tic$_1$–tac. In contrast, for the ray going along (x, x'), $\Delta t_1 =$ 0–tic$_2$ is much longer than $\Delta t_2 =$tic$_2$–tac. The corresponding time intervals in S' are simply related to Δt_1 and Δt_2 by the time dilation factor γ, so that $\Delta t'_1 > \Delta t'_2$. Since, in S', the distances covered in the two time intervals are the same (L'), then, in S', light goes slower in the forward direction, c_v^+, than in the backward one, c_v^-.

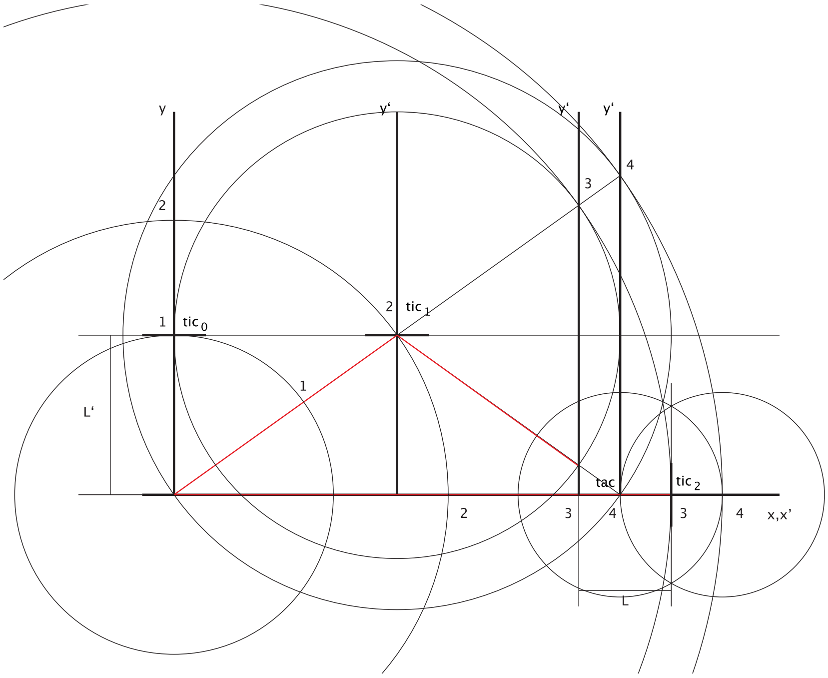

Figure 5. The complete geometric construction.

It is beyond doubt that this fact can be shocking for students with a relatively poor grasp of special relativity, as they simply learn that 'the speed of light is the same in all inertial frames' and, accordingly, one has to have $\Delta t_1' = \Delta t_2'$ (we will sort out this apparent contradiction below). However, it is quite natural for students being just introduced to the subject. Indeed, from classical mechanics they would expect to find $c_v^{\pm} = (c \mp v)$. Furthermore, by taking into account the corrections in lengths and in rhythms (obtained through this example or otherwise), the expression has to be rectified to

$$c_v^{\pm} = \gamma^2 (c \mp v), \tag{1}$$

with $\gamma = 1/\sqrt{1 - v^2/c^2}$. One factor γ accounts for time dilation; the second one holds for space contraction. Note that many students actually believe that the two γ factors cancel each other out, which would justify the constant value of the one-way speed of light they use in relativity. This is *not* the case.

Expression (1), obtained here from quite simple and intuitive arguments, can also be derived directly from the IST transformation as shown in our previous works [2, 3]. This being so, it is possible to work it the other way round, and obtain directly the IST transformation from time dilation and length contraction first and then verify that it is consistent with equation (1). Note that c_v^- goes to infinity when v approaches c. This is also evident from figure 5, as the time interval tic$_2$–tac tends to zero, whereas in S' the distance to be covered is always L'. Moreover, it is not difficult to verify that the two-way speed of light in S' is c, as it should be.

For the situation is presented here, there is no reciprocity between frames, and Lorentzian philosophy [4] is being used. How to proceed from the IST transformation to the Lorentz transformation, to the principle of relativity and to Einsteinian philosophy is explained in detail in [2, 3]. Let us simply note that we can *decide* that we want to work in S' as if the one-way speed of light were c, although we do not know if this is so. The reasons to operationally proceed in this way can be more or less justified and elucidated, or this decision can simply be taken as a new definition of a particular notion of 'speed' [2, 3].

In any case, if we decide that we want to work with $\Delta t'_1 = \Delta t'_2$ as read by clocks in S' or, equivalently, if we want the time readings of two moving clocks co-punctual with both mirrors to be the same at tic_1 and tic_2, the only option is to delay the clock from S' co-punctual with the front mirror, so that when light hits it (instant 3) it reads the same number (time coordinate) which the clock co-punctual with the top mirror has marked at instant 2. Though no alteration in the rhythm of the clocks is made, only their initial setting is changed. Note that physics can even be studied using two sets of clocks at the same time, one delayed and the other not, since physics and its laws are not changed by the way we decide to set our clocks. It is precisely the same physics describing the same reality, in spite of the fact that from the readings of one set of clocks the one-way speed of light in the moving frame is given by (1) and from the readings of the other one it is c. There is no contradiction in these assertions, as the definition of 'speed' and the time coordinates used are different in each of them [2, 3].

Evidently, at this point everything is already set to address the conventionality of simultaneity thesis [3, 13, 14], independence of physics from coordinates [3, 15–17] and compatibility between the Lorentz–Poincaré and the Einstein–Minkowski philosophies [2, 3, 4, 18], if the teacher wants to do so. A more conservative approach would consist in merely introducing relativity of simultaneity with the remark that in S instants 2 and 3 are distinct, whereas, under the assumption of invariant one-way speed of light, they are the same in S'.

Finally, let us conclude with the observation that the geometric construction presented in this work is a particular case of a more general problem, namely to know where and when does light emitted at a certain point hits a moving object. This is the key point of the question of aberration of light, which will be treated in a similar way elsewhere.

References

[1] Guerra V and de Abreu R 2005 The conceptualization of time and the constancy of the speed of light *Eur. J. Phys.* **26** S117–23
[2] Guerra V and de Abreu R 2006 On the consistency between the assumption of a special system of reference and special relativity *Found. Phys.* **36** 1826–45
[3] de Abreu R and Guerra V 2008 The principle of relativity and the indeterminacy of special relativity *Eur. J. Phys.* **29** 33–52
[4] Bell J S 1988 *Speakable and Unspeakable in Quantum Mechanics* (Cambridge: Cambridge University Press)
[5] Selleri F 1996 Noninvariant one-way velocity of light *Found. Phys.* **26** 641–64
[6] Selleri F 2005 The inertial transformations and the relativity principle *Found. Phys. Lett.* **18** 325–39
[7] de Abreu R 2002 The physical meaning of synchronization and simultaneity in special relativity (arXiv:physics/0212020)
[8] Tangherlini F R 1961 On energy–momentum tensor of gravitational field *Nuovo Cimento Suppl.* **20** 351
[9] Guerra V and de Abreu R 2004 Comment on 'From classical to modern ether-drift experiments: the narrow window for a preferred frame' *Phys. Lett.* A **333** 355
Guerra V and de Abreu R 2007 *Phys. Lett.* A **361** 509–12
[10] Spavieri G, Guerra V, de Abreu R and Gillies G T 2008 Ether drift experiments and electromagnetic momentum *Eur. Phys. J.* D **47** 457–63
[11] Iyer C 2008 Comment on the paper by de Abreu and Guerra 'The principle of relativity and the indeterminacy of special relativity' *Eur. J. Phys.* **29** L13–7
[12] Feynman R P, Leighton R B and Sands M 1979 *The Feynman Lectures on Physics* 13th edn (Reading, MA: Addison-Wesley)
[13] Capria M M 2001 On the conventionality of simultaneity in special relativity *Found. Phys.* **5** 775–818

[14] Martínez A 2005 Conventions and inertial reference frames *Am. J. Phys.* **73** 452–4
[15] Leubner C, Aufinger K and Krumm P 1992 Elementary relativity with 'everyday' clock synchronization *Eur. J. Phys.* **13** 170–7
[16] Ivezić T 2001 True transformations relativity and electrodynamics *Found. Phys.* **31** 1139–83
[17] Oziewicz Z 2006 Electric and magnetic fields, within groupoid relativity *Electromagn. Phenom.* **6** 159–2005
[18] Duffy M C 2004 Geometrized space-time and the relativistic world ether *Proc. Physical Interpretations of Relativity Theory IX* vol 1 (Liverpool: PD Publications) pp 158–202