Convolution integrals of Normal distribution functions

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1 Convolution

1.1 Definition

The convolution $f * g$ of two functions $f(x)$ and $g(x)$ defined in $\mathbb{R}$ is given by:

$$f * g(z) = \int_{\mathbb{R}} f(x)g(z-x)dx \tag{1}$$

1.2 Properties

Some of the properties of $f * g$ are described below.

1. $f * g = g * f$ (commutative)
2. $f * (g * h) = (f * g) * h$ (associative)

3. $f * (g + h) = (f * g) + (f * h)$

4. $\frac{d(f*g)}{dx} = \frac{df}{dx} * g = f * \frac{dg}{dx}$

5. $\int f * g = \int f \cdot \int g$

6. laplace transform[1] $\mathcal{L}\left[f * g\right] = \mathcal{L}(f)\mathcal{L}(g)$

7. in probability theory, the convolution of two functions has a special relation with the distribution of the sum of two independent random variables. If the two random variables X and Y are independent, with pdf's f and g respectively, the distribution $h(z)$ of $Z = X + Y$ is given by $h(z) = f * g$. This result is obtained below.

$$\begin{aligned}
H(z) &= P(Z \leq z) = P(X + Y \leq z) \\
&= \int P(X + Y \leq z | Y = y) \cdot g(y)dy \\
&= \int P(X \leq z - y) \cdot g(y)dy \\
&= \int F_X(z - y) \cdot g(y)dy \\
h(z) &= \frac{dH(z)}{dz} = \frac{d(\int F_X(z - y) \cdot g(y)dy)}{dz} \\
&= \int \frac{d(F_X(z - y))}{dz} \cdot g(y)dy \\
&= \int f(z - y) \cdot g(y)dy \\
&= f * g
\end{aligned}$$

2 Normal distribution function

The Gaussian or Normal p-dimensional distribution with mean μ and covariance matrix Σ is given by the following equation 2, where $x \in \mathbb{R}^p$ is a p-dimensional random vector, x^T is the transpose vector of x and $|\Sigma|$ is the determinant of Σ:

$$g_p(x; \mu, \Sigma) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x - \mu)^T \Sigma^{-1} (x - \mu)\right) \tag{2}$$

When a random variable X, taking values in $\mathbb{R}^p$, has a probability density function (pdf) given by the former equation we say that $X \sim N_p(\mu, \Sigma)$.

[1] Laplace transform of function f(t) is defined as $\mathcal{L}[f(t)](s) = \int_0^\infty f(t)e^{-st}dt$

3 Convolution of Normal distribution functions

Given two p-dimensional normal probability density functions $G_1 \equiv g_p(x; a, A)$ and $G_2 \equiv g_p(x; b, B)$ (see eq. 2) we will prove that the convolution of these two functions is a normal probability density distribution function with mean $a + b$ and variance $A + B$, i.e. $g_p(x; a + b, A + B)$:

$$G_1 * G_2(z) = g_p(z; a + b, A + B)$$

The next sections demonstrate this result by first presenting an algebraic simplification of integrals using some properties of determinants and the factorization of quadratic forms.

3.1 Integral simplification

The following deduction represents the simplification of the integral $\int G_1 G_2 dx$ where G_1 and G_2 are the pdf of the normal distribution described above.

$$\begin{aligned}
& \int g_p(x; a, A) \cdot g_p(x; b, B) dx \\
= & \int \frac{1}{(2\pi)^{p/2} |A|^{1/2}} e^{-\frac{1}{2}(x-a)' A^{-1} (x-a)} \frac{1}{(2\pi)^{p/2} |B|^{1/2}} e^{-\frac{1}{2}(x-b)' B^{-1} (x-b)} dx \\
= & \int \frac{1}{(2\pi)^{p/2} |A|^{1/2}} \frac{1}{(2\pi)^{p/2} |B|^{1/2}} e^{-\frac{1}{2}\left((x-a)' A^{-1} (x-a) + (x-b)' B^{-1} (x-b)\right)} dx \\
= & \int \frac{1}{(2\pi)^{p/2} |A|^{1/2}} \frac{1}{(2\pi)^{p/2} |B|^{1/2}} e^{-\frac{1}{2}\left((x-c)'\left(A^{-1}+B^{-1}\right)(x-c) + (a-b)' C (a-b)\right)} dx \\
= & \frac{\left|\left(A^{-1} + B^{-1}\right)^{-1}\right|^{1/2}}{(2\pi)^{p/2} |A|^{1/2} |B|^{1/2}} e^{-\frac{1}{2}(a-b)' C (a-b)} \cdot \\
& \qquad \cdot \int \frac{1}{(2\pi)^{p/2} \left|\left(A^{-1} + B^{-1}\right)^{-1}\right|^{1/2}} e^{-\frac{1}{2}(x-c)'\left(A^{-1}+B^{-1}\right)(x-c)} dx \\
= & \frac{\left|\left(A^{-1} + B^{-1}\right)^{-1}\right|^{1/2}}{(2\pi)^{p/2} |A|^{1/2} |B|^{1/2}} e^{-\frac{1}{2}(a-b)' C (a-b)} \qquad (3) \\
= & \frac{1}{(2\pi)^{p/2} \left(|A| \, |B| \, |A^{-1} + B^{-1}|\right)^{1/2}} e^{-\frac{1}{2}(a-b)'(A+B)^{-1}(a-b)} \\
= & \frac{1}{(2\pi)^{p/2} |ABA^{-1} + ABB^{-1}|^{1/2}} e^{-\frac{1}{2}(a-b)'(A+B)^{-1}(a-b)} \\
= & \frac{1}{(2\pi)^{p/2} |ABA^{-1} + A|^{1/2}} e^{-\frac{1}{2}(a-b)'(A+B)^{-1}(a-b)} \\
= & \frac{1}{(2\pi)^{p/2} |A(B + A) A^{-1}|^{1/2}} e^{-\frac{1}{2}(a-b)'(A+B)^{-1}(a-b)}
\end{aligned}$$

$$= \frac{1}{(2\pi)^{p/2}|A+B|^{1/2}} e^{-\frac{1}{2}(a-b)'(A+B)^{-1}(a-b)}$$

3.1.1 Properties of determinants

1. $|AB| = |A|\,|B|$
2. $\left|A^{-1}\right| = \frac{1}{|A|}$
3. $|cA| = c^n\,|A|$
4. $\left|BAB^{-1}\right| = |B|\,|A|\left|B^{-1}\right| = \frac{|B||A|}{|B|} = |A|$
5. $\left|B^{-1}AB - \lambda I\right| = \left|B^{-1}AB - B^{-1}\lambda IB\right| = \left|B^{-1}(A-\lambda I)B\right| = |A-\lambda I|$

3.1.2 Factorization of quadratic forms

Give x, a and b vectors of dimension p, A and B symmetric matrices of order p positively defined such as $A+B$ is not singular, we have:

$$\begin{gathered}(x-a)'A(x-a) + (x-b)'B(x-b) = \\ (x-c)'(A+B)(x-c) + (a-b)'C(a-b)\end{gathered} \tag{4}$$

where

$$\begin{aligned} c &= (A+B)^{-1}(Aa+Bb) \\ C &= A(A+B)^{-1}B = \left(A^{-1}+B^{-1}\right)^{-1} \end{aligned}$$

3.2 Result

Let $G_1(x)$ and $G_2(x)$ be the probability density function of the p-dimensional normal distributions $N(a, A)$ and $N(b, B)$ respectively. The convolution $G_1 * G_2$ is defined as:

$$\begin{aligned} G_1 * G_2(z) &= \int G_1(x)G_2(z-x)dx \\ &= \int \frac{1}{(2\pi)^{p/2}|A|^{1/2}} e^{-\frac{1}{2}(x-a)'A^{-1}(x-a)} \cdot \\ &\qquad \cdot \frac{1}{(2\pi)^{p/2}|B|^{1/2}} e^{-\frac{1}{2}(z-x-b)'B^{-1}(z-x-b)}dx \qquad (5) \\ &= \int g_p(x; a, A) \cdot g_p(x; z-b, B)dx \\ &= \frac{1}{(2\pi)^{p/2}|A+B|^{1/2}} e^{-\frac{1}{2}(z-(a+b))'(A+B)^{-1}(z-(a+b))} \\ &= g_p(z; a+b, A+B) \end{aligned}$$

This means that the convolution $G_1 * G_2(z)$ is the pdf of the normal distribution $N(a+b, A+B)$.

References

[1] Richard A. Johnson and Dean W. Wichern. *Applied multivariate statistical analysis.* Prentice Hall, New Jersey, 4th edition, 1998.

[2] Kyle Siegrist. Virtual laboratories in probability and statistics. *http://www.math.uah.edu/stat/*, 1997–2004.

[3] Gilbert Strang. *Linear Algebra and Its Applications.* International Thomson Publishing, 3rd edition, 1988.