Plasmoid and Kelvin-Helmholtz Instabilities in Sweet-Parker Current Sheets

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Introduction

- Plasmoids have become popular in magnetic reconnection (enhance reconnection rate, particle acceleration, etc)

- Original linear theory for the instability of Sweet-Parker current sheets (Loureiro, Schekochihin & Cowley, *Phys. Plasmas* ’07) predicts

$$\gamma_{\text{max}} \tau_A \sim S^{1/4}$$

$$k_{\text{max}} L_{CS} \sim S^{3/8}$$

These scalings have been confirmed numerically (Samtaney et al, *PRL* ’09)

- *LSC* ’07 assumed a very simple equilibrium, designed to retain the key features of a reconnecting SP sheet:

$$\mathbf{B}_{eq} = (0, B_y(x)) \quad \phi_{eq} = xyV_A/L_{CS}$$

- *This work generalizes previous calculation to a more realistic, two-dimensional equilibrium.*

- **Key findings:** 1) plasmoid growth rate and wave-number *increase* with distance from the center of the sheet; 2) *transition to KH instability*
Kelvin-Helmholtz instability of the current sheet

There are two shear layers, at $x \sim \pm \delta_{CS}$

\[
\frac{du_y}{dx} \sim \frac{V_A}{\delta_{CS}} \frac{y}{L_{CS}}
\]

These layers would be KH unstable were it not for the stabilizing effect of the upstream magnetic field, $B_y$. Stability criterion:

$$|B_y| > |u_y|$$

But $B_y$ is not constant along the sheet! E.g.: $B_y = B_0 \sqrt{1 - y^2/L_{CS}^2}$

Since the outflow speed is Alfvénic (based on $B_0$), there is a position along the sheet where

$$|B_y| < |u_y| \quad \rightarrow \quad \text{KH unstable!}$$
KH instability: heuristic derivation

From standard KH linear theory:

\[ \gamma_{\text{max}}^{\text{KH}} \sim \frac{d u_y}{d x} \sim \frac{V_A}{a}, \quad k_{\text{max}}^{\text{KH}} a \sim 1 \]

where ‘a’ is the characteristic scale-length of the sheared flow profile.

Now, rescale ‘a’ to the Sweet-Parker current sheet width:

\[ a \rightarrow \delta_{CS} \sim L_{CS} S^{-1/2} \]

Obtain:

\[ \gamma_{\text{max}}^{\text{KH}} \tau_A \sim S^{1/2} \]
\[ k_{\text{max}}^{\text{KH}} L_{CS} \sim S^{1/2} \]

This is even faster than the plasmoid instability
Rigorous derivation: outline

Solve RMHD eqs.
\[ \partial_t \nabla_\perp^2 \phi + \{ \phi, \nabla_\perp^2 \phi \} = \{ \psi, \nabla_\perp^2 \psi \} + \nu \nabla_\perp^4 \phi, \]
\[ \partial_t \psi + \{ \phi, \psi \} = \eta \nabla_\perp^2 \psi - E_0. \]

Introduce asymptotic expansion parameter:
\[ \epsilon = \frac{\delta_{CS}}{L_{CS}} \sim S^{-1/2} \ll 1 \]

Equilibrium expansion:
\[ \psi_{eq}(x, y)|_{y=y_0} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{y - y_0}{L_{CS}} \right)^n \psi_n(x, y_0), \]
\[ \phi_{eq}(x, y)|_{y=y_0} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{y - y_0}{L_{CS}} \right)^n \phi_n(x, y_0). \]

Substitute these expressions into the linearized RMHD equations. Eqs. simplify if:
\[ (\bar{y} - \bar{y}_0) \ll \gamma/\kappa. \]
\[ \kappa (\bar{y} - \bar{y}_0) \gg 1. \]

The characteristic scale of the plasmoid instability is much shorter than the equilibrium variation scale, so a local analysis is valid.
Analytical Dispersion Relation

\[-\frac{\pi}{8} (\kappa f_0')^{1/3} \Lambda^{5/4} \frac{\Gamma[(\Lambda^{3/2} - 1)/4]}{\Gamma[(\Lambda^{3/2} + 5)/4]} = \Delta' = \frac{2}{\kappa \epsilon} \frac{f_0'^2}{f_\infty^2 - \bar{y}_0^2 v_0^2}\]

Solve \( \frac{d\gamma}{d\kappa} = 0 \)

\[\kappa_{\text{max}} = \left( \frac{1}{\sqrt{\pi} f_\infty^2 - \bar{y}_0^2 v_0^2} \right)^{3/4} \epsilon^{-3/4},\]

\[\gamma_{\text{max}} = \frac{2}{3\pi^{1/4}} \sqrt{\frac{f_0'^3}{f_\infty^2 - \bar{y}_0^2 v_0^2}} \epsilon^{-1/2}.

**Surprise 1:** for a Syrovatskii-like upstream magnetic field profile, \( f_\infty = \sqrt{1 - \bar{y}_0^2} \) both \( \kappa_{\text{max}} \) and \( \gamma_{\text{max}} \) increase with \( y_0 \).

**Surprise 2:** Blow-up when \( f_\infty = \bar{y}_0 v_0 \) \( \rightarrow \) **Alfvén Mach point:** transition to **Kelvin-Helmholtz**
Dispersion Relation

Comparison of the analytical dispersion relation and the direct numerical simulation of the linearised equations

\[ S = 10^{12} \]

\[ S = 10^{16} \]
Scaling with $S$

The plasmoid scalings, valid at the center of the sheet, are replaced with the KH scalings at the edge.
Dependence on the position along the sheet

Plasmoid growth rate and wave number gently increase until there is a transition to the KH instability
Large Prandtl number limit

- In some plasmas, the ratio of the viscosity to the resistivity can be large.
  \[ Pm = \frac{\nu}{\eta} = \sqrt{\frac{m_i}{m_e}} \beta \]

- Heuristic derivation of the plasmoid instability scalings using
  - known results for the visco-tearing mode and visco-resistive kink (Porcelli ’87)
  - Sweet-Parker scalings for large Pm (W. Park et al., 84)

\[ k_{\max} L_{CS} \sim S^{3/8} Pm^{-3/16}, \]
\[ \gamma_{\max} \tau_A \sim S^{1/4} Pm^{-5/8}, \]
\[ \delta_{\text{inner}} / \delta_{CS} \sim S^{-1/8} Pm^{1/16}. \]
A Sweet-Parker-like analytical equilibrium (I)

To lowest order in $(y - y_0)/L_{CS}$

$\xi = 0$

$f'_0 = \bar{E}_0 - \bar{y}_0^2 \nu_0 g_0$

$u_\infty = \frac{\bar{E}_0}{f_\infty}$

Introduce

$g(\xi) = \frac{u_\infty}{f_\infty} v(\xi) - s(\xi)$

$s''(\xi) = s(\xi) \frac{f''(\xi)}{f(\xi)}$

Then, from Eq. (A1) we obtain

$u(\xi) f(\xi) - \bar{y}_0^2 v(\xi) g(\xi) = f'(\xi) - \bar{E}_0,$ \hspace{1cm} (A1)

$v(\xi) f''(\xi) - v(\xi) f''(\xi) + g(\xi) f''(\xi) - f(\xi) g''(\xi) + Pm v'''(\xi),$ \hspace{1cm} (A2)

$v(\xi) = \frac{f_\infty}{u_\infty} \frac{s(\xi)}{2} \pm \sqrt{\frac{f_\infty^2}{u_\infty^2} \frac{s^2(\xi)}{4} + \frac{f_\infty}{u_\infty} u(\xi) f(\xi) - f'(\xi) + \bar{E}_0}.$

$s(\xi) = C_1 f(\xi) + C_2 f(\xi) \int^{\xi} \frac{d\xi'}{f^2(\xi')}.$
A Sweet-Parker-like analytical equilibrium (II)

So far, everything is exact. However, we have only 2 equations for 4 unknowns. Will now introduce \textit{ad hoc} choices for upstream magnetic field profile and inflow velocity:

\[
\begin{align*}
  f(\xi) &= f_\infty \tanh \left( \frac{f_0'}{f_\infty} \xi \right) \\
  u(\xi) &= -u_\infty \frac{f(\xi)}{f_\infty}
\end{align*}
\]

with \( f_\infty = \sqrt{1 - \bar{y}_0^2} \) (Syrovatskii)

\[
s(\xi) = \left( g_0 - \frac{u_\infty}{f_\infty} v_0 \right) \left[ \frac{f_0'}{f_\infty} \xi \tanh \left( \frac{f_0'}{f_\infty} \xi \right) - 1 \right]
\]

![Graphs showing $y_0/L_{CS}$ values 0.4 and 0.8](image.png)
Discussion

- Increase in $\gamma$ along the sheet not observed – why?
- KH never observed in SP-like current sheets (?); why?

$(S=10^8$, from Samtaney et al., PRL ’09)

In Samtaney et al., the simulation domain was not long enough to include the Alfvén Mach point. Also, the imposed upstream magnetic field was constant.