

CHAPTER 6

PURPOSIVE CONTRADICTION INTERVENING DUALITY AND THE NATURE OF PROBABILITY

1. Introduction

Consider a matching pennies game (Owen 1982, Wang, 1988, Hart 1992) with the following payoff matrix:

		PLAYER 2	
		H	T
PLAYER 1	H	1	-1
	T	-1	1

Table 1

According to one interpretation, this table represents conditional payoffs to Player 1 in a coin tossing game against nature where that player wins if correctly predicting the outcome of a toss of a coin, and loses otherwise. According to another it represents conditional payoffs to the first of two persons engaged in a coin tossing game against each other. By considering these interpretations together it is evident that, whereas nature plays an explicit role in the determination of outcomes in the first, in the second that role is not explicit at all.

The incorporation of these points into linear programming extensions of constant sum games leads directly to the intervening duality idea which is considered in the context of applications to tosses of fair coins in Sections 3 and 4.

One way of understanding intervening duality is by recognizing in it an emphasis on potentiating gains to initially relatively unknown outcomes. From that perspective what follows is related to decision-making under uncertainty with recourse (Charnes et al 1967, White 1992) except that here the emphasis is on *gains* stemming from opportunities relatively outside an initial constraint set rather than on potential *losses* due to expenses stemming from resource requirements beyond an initial set of resource constraints. Thus the emphasis here is on opportunities for individuals to gain by initiating *divergence* from relatively stable initial states - in contrast, too, to the emphasis in the literature

on evolutionary games (Mailath 1992), in *convergence* to a relatively stable stationary state. The structure of the paper is as follows: after some preliminaries in the following Section, the intervening duality idea is introduced in Section 3. In Section 4 it is shown how an intervening duality formulation can allow two rival players each to perceive a coin as fair relative to a system and yet strictly favourable relative to self. This leads to interpretations in relation to luck and to distinctions between relatively subjective and relatively objective judgements, including distinctions between axiomatic and subjective probabilities and ex ante and ex post behaviours. Such relations and distinctions are then considered more generally by extending earlier goal programming formulations to explicitly include processes potentially determining relatively prior restrictions on individual probabilities as well as on the ranges of distributions.

2. A preliminary note on linear programming representations of two person constant sum games.

Consider a two person constant sum game in which the contingent payoffs to player 1 playing strategies $j=1,2,...,J$ are π_{kj} if player 2 plays strategies k , $k=1,2,...,K$. If player 1 adopts the objective of maximizing the minimum expected payoff to self and player 2 adopts the objective of minimizing the maximum payoff to player 1, the respectively optimal pure and mixed strategies for each of the two players can be represented as optimal solutions to the following dual pair of linear programmes:

$$\begin{array}{ll}
 \text{Maximize } \rho & \text{Minimize } \mu \\
 \text{st } \sum_j \pi_{kj} q_j \geq \rho & \text{st } \sum_k \pi_{kj} p_k \leq \mu \\
 \sum_j q_j \leq 1 \quad (I) & \sum_k p_k \leq 1 \quad (I') \\
 q_j \geq 0 & p_k \geq 0
 \end{array}$$

Formally, since a feasible solution to (I) always exists - consider $q_j=0$ for all $j=1,2,...,J$ - by the dual theorem optimal solutions to both (I) and (I') exist

with $q_j^* > 0$, $\sum q_j^* = 1$ for some $j=1,2..J$, $p_k^* > 0$, $\sum p_k^* = 1$ for some $k=1,2..K$ and $\rho^* = \mu^*$. In this case the quantities q_j^* , p_k^* become defined as (optimal) probabilities with which the two players adopt their pure or mixed strategies with the properties that, *inter alia*, $\sum q_j^* = 1$, $\sum p_k^* = 1$, *only at the optimum*.

An alternative approach, and one which leads naturally to ideas which are developed further in the following sections, is to make the latter conservation conditions not only implicit in the solution to the problem but explicit in its initial specification. This is achieved by extending the specification (I),(I)' so that it pre-emptively requires relations $\sum q_j^* = 1$, $\sum p_k^* = 1$ *a priori* as follows:

$$\begin{aligned} &\text{Maximize } \rho - Mq^+ - Mq^- \\ &\text{st } \sum_j \pi_{kj} q_j \geq \rho \\ &\sum_j q_j + q^+ - q^- = 1 \quad (\text{II}) \\ &-M \leq \rho \leq M \\ &q_j, q^+, q^- \geq 0 \end{aligned}$$

$$\begin{aligned} &\text{Minimize } \mu + Mp^+ + Mp^- \\ &\text{st } \sum_k \pi_{kj} p_k \leq \mu \\ &\sum_k p_k + p^+ - p^- = 1 \quad (\text{II}') \\ &-M \leq \mu \leq M \\ &p_k, p^+, p^- \geq 0 \end{aligned}$$

Formal equivalence between optimal solutions to (II),(II)' and those of (I),(I)' is evident once it is noted that, if the weights M are assumed to be arbitrarily large, conditions $\sum q_j^* = 1$, $\sum p_k^* = 1$ hold as if pre-emptively or, with reference to (I),(I)', as if *a priori*.

More interesting here is an interpretation of such an implicit preliminary and preemptive process as a *conditioning* process corresponding to a prior and conditional restriction of the collective ranges of the measures q_j (resp p_k) such that they equate exactly to predetermined and relatively normalized measures “1”.

From that perspective with the restrictive context of a heads-tails game such a normalization *both requires one of the two outcomes heads or tails and precludes any other outcome*. Conversely, the measures q^-, p^- (resp q^+, p^+) point toward

conditionally respectively larger and smaller ranges, and thence potentially relatively larger or smaller sample spaces, for the measures q_j, p_k .

For example, with a context of a single toss of a coin: i) conditions as if optimally $q^- = 0$, $p^- = 0$ might be interpreted as corresponding to prior restrictions on the coin tossing experiment to preclude other *non coin related* outcomes, while generating potentially mixed prior heads-tails strategies; ii) conditions as if optimally $q^+ = 0$, $p^+ = 0$ might be interpreted as corresponding to the prior exclusion of other *coin related* outcomes, including, for instance, a finite ex ante probability that the coin would land on its side.

3. Linear programming representation of the coin tossing game

It has been noted that the payoff matrix of Section 1 may variously represent payoffs to an individual engaged in a coin tossing game against nature, or payoffs to the first of two persons engaged in a coin tossing game against each other.

Assuming that in each case the first player chooses to maximize the minimum expected payoff on the assumption that the second will minimize the maximum expected payoff, the game can be represented by specializations of (II) and (II)' to this case as follows:

$$\begin{aligned} &\text{Maximize } \rho - Mq^+ - Mq^- \\ &\text{st } 1q_1 - 1q_2 \geq \rho \\ &\quad -1q_1 + 1q_2 \geq \rho \\ &q_1 + q_2 + q^+ - q^- = 1 \quad (\text{IIa}) \\ &-M \leq \rho \leq M \\ &q_j, q^+, q^- \geq 0 \end{aligned}$$

$$\begin{aligned} &\text{Minimize } \mu + Mp^+ + Mp^- \\ &\text{st } 1p_1 - 1p_2 \leq \mu \\ &\quad -1p_1 + 1p_2 \leq \mu \\ &p_1 + p_2 + p^+ - p^- = 1 \quad (\text{IIa}') \\ &-M \leq \mu \leq M \\ &p_k, p^+, p^- \geq 0 \end{aligned}$$

For the first class of interpretations the first player would be associated with problem (IIa) and nature with (IIa)', whereas in the second a player different from “nature” would be associated with problem (IIa)'.

In either case the optimal solution corresponds to an implicitly fair solution in the sense of equiprobability of heads and tails outcomes as follows:

$$\begin{array}{lll}
\rho^*=0 & \mu^*=0 & \\
q_1=1/2 & p_1=1/2 & (A) \\
q_2=1/2 & p_2=1/2 &
\end{array}$$

Notice that this solution emerges even though no a priori *assumption* of fairness in the sense of ex ante equiprobability of heads and tails outcomes has been made and that, if such an assumption were to be made explicitly, then (I)' would need to be modified accordingly.

More generally, given the context of the first class of single player versus nature interpretations, the inherent symmetry of the two person case suggests that an appropriate approach to the modelling of the second class of two player games would be by first considering each player as engaged in a coin tossing game against nature, with a structure similar to that in (I),(I)' and then introducing the *intervening duality* idea by considering these games as dually interrelated through the coin. This idea is developed in stages in the next Section.

4 “Fair” coin tossing games and the intervening duality idea

Stage 1. Interpreting (IIa) and (IIa)' as problems for a player and nature respectively then, with data as in the payoff matrix in (II),(IIa)', these programmes yield the *implicitly* fair optimal solutions indicated in (A) above. Now make the fairness of the coin explicit by assuming that player 1 appends equiprobability conditions with associated dual variables R_k to (IIa)' to obtain the modified *constrained game* (Charnes 1953) form (IIIa)' and its dual (IIIa):

$$\begin{array}{ll}
\text{Maximize } \rho - Mq^+ - Mq^- & \\
\text{st } 1q_1 - 1q_2 - R_1 \geq \rho & \\
-1q_1 + 1q_2 - R_2 \geq \rho & \\
q_1 + q_2 + q^+ - q^- = 1 & (IIIa) \\
-M \leq \rho \leq M & \\
q_j, q^+, q^- \geq 0, R_k \text{ unrestricted} & \\
\\
\text{Minimize } \mu' + Mp^{+'} + Mp^{-'} & \\
\text{st } 1p_1' - 1p_2' \leq \mu & \\
-1p_1' + 1p_2' \leq \mu & \\
p_1' + p_2' + p^{+'} - p^{-'} = 1 & (IIIa)' \\
p_1' = 1/2 & \\
p_2' = 1/2 & \\
-M \leq \mu' \leq M & \\
p_k', p^{+'}, p^{-'} \geq 0 &
\end{array}$$

It is easy to verify that the “fair” solution to (IIIa),(IIIa)' is one class of optimal solution to (IIIa),(IIIa)'. Formally $p_1'=1/2$ and $p_2'=1/2 \Rightarrow \mu'=0$, which is consistent with $R_1=R_2=0$, $q_1=1/2$, $q_2=1/2 \Rightarrow \rho^*=0$. Going further, conditions with $R_k^*=0$ are equivalent to the redundancy of the final constraints of (IIIa)'. So, at that optimum, optimal specifications to (IIIa),(IIIa)' become formally equivalent to those of (IIa),(IIa)', except that what is formally implicit in the first pair becomes explicit in the second and vice versa.

Another class of solutions to (IIIa),(IIIa)' is one according to which player 1 plays against nature and acknowledges the coin as fair via explicit equiprobability constraints as in (IIIa)', but plays the game as if the anticipated heads or tails probabilities will be otherwise relative to him/her. In particular these systems can be consistent with an optimally pure strategy $q_1^*=1$, $q_2^*=0$ with $R_1=1$, $R_2=-1$ for player 1, even though he/she imputes and optimally mixed strategy $p_j^*=1/2$, $j=1,2$ to nature. For example player 1 might perceive himself/herself as “lucky” and use measures R_1, R_2 as if to negate the alternative outcome and call a pure heads strategy ex ante via $q_1^*=1$, $q_2^*=0$, even though he/she imputes a *mixed* “fair” ex ante strategy $p_j^*=1/2$, $j=1,2$ to nature (the coin).

Potential for such ex ante distinctions suggests potential, too, through the measures R_k , for distinctions between particular instances and general expectations in general, and between ex ante expectations and ex post realizations in particular. For example, with data as in their payoff matrices, (IIIa) and (IIIa)' are consistent with optimal ex ante solutions with mixed strategies and optimal ex post strategies for player 1 with pure strategies as follows:

Ex ante player 1	Ex post player 1	Nature	
$\rho^*=0$	$\rho^*=0$	$\mu^*=0$	
$q_1^*=1/2$	$q_1^*=1$	$p_1^*=1/2$	(B)
$q_2^*=1/2$	$q_2^*=0$	$p_2^*=1/2$	
$R_1^*=0$	$R_2^*=1$?	
$R_2^*=0$	$R_2^*=-1$?	

This in turn suggests potential for the measures R in establishing relations and distinctions between extensive and normal form characterizations of player 1's behaviour. (From this perspective (IIIa)' seems incomplete since it does not incorporate analogues of the variables R - hence the ?? under “nature” in conditions (B). I will return to this

point in the more general context of goal programming representations and potentials for gains from exchange in Sections 4,5 and 6 below.)

Stage 2. Now interpret programmes (IIa) and (IIa)' as respectively representing nature and a second player so that with data as in the payoff matrix these programmes, too, yield implicitly fair optima as in (A) above. Again making the fairness of the coin explicit, in this case by assuming that player 2 explicitly imputes fairness to nature's strategies as in (IVa)' below, yields modified forms (IVa),(IVa)' for his/her game against nature as follows:

$$\begin{aligned} &\text{Minimize } \mu + \sum S_j .1/2 + Mp^+ + Mp^- \\ &\text{st } 1p_1 - 1p_2 - S_1 \leq \mu \\ &\quad -1p_1 + 1p_2 - S_2 \leq \mu \\ &\quad p_1 + p_2 + p^+ - p^- = 1 \quad (\text{IVa}) \\ &\quad -M \leq \mu \leq M \\ &\quad p_k, p^+, p^- \geq 0 \quad S_j \text{ unrestricted} \end{aligned}$$

$$\begin{aligned} &\text{Maximize } \rho' - Mq^+ - Mq^- \\ &\text{st } 1q_1' - 1q_2' - R_1 \geq \rho \\ &\quad -1q_1' + 1q_2' - R_2 \geq \rho \\ &\quad q_1' + q_2' + q^+ - q^- = 1 \quad (\text{IVa}') \\ &\quad q_1' = 1/2 \\ &\quad q_2' = 1/2 \\ &\quad -M \leq \rho' \leq M \\ &\quad q_j', q^+, q^- \geq 0 \end{aligned}$$

By inspection a fair solution with $p_1^* = p_2^* = 1/2$, $S_1 = S_2 = 0$ is feasible. With that solution the informational value of the final probability related constraints of (IVa) becomes as if zero and in that sense they become as if redundant, so that again

$$\begin{aligned} &\text{Maximize } \rho + \sum R_k .1/2 - Mq^+ - Mq^- \\ &\text{st } 1q_1 - 1q_2 - R_1 \geq \rho \\ &\quad -1q_1 + 1q_2 - R_2 \geq \rho \\ &\quad q_1 + q_2 + q^+ - q^- = 1 \quad (\text{IIIa}) \\ &\quad -M \leq \rho \leq M \\ &\quad q_j, q^+, q^- \geq 0, R_k \text{ unrestricted} \end{aligned}$$

$$\begin{aligned} &\text{Minimize } \mu + \sum S_j .1/2 + Mp^+ + Mp^- \\ &\text{st } 1p_1 - 1p_2 - S_1 \leq \mu \\ &\quad -1p_1 + 1p_2 - S_2 \leq \mu \\ &\quad p_1 + p_2 + p^+ - p^- = 1 \quad (\text{IVa}) \\ &\quad -M \leq \mu \leq M \\ &\quad p_k, p^+, p^- \geq 0 \quad S_j \text{ unrestricted} \end{aligned}$$

the two programmes (IVa),(IVa)' become similar to (IIa),(IIa)'.

As for (IIIa),(IIIa)' a more interesting class of interpretations are those for which player 2 plays against nature acknowledging the game as fair - but playing otherwise - with associated classes of ex ante interpretations via the measures S_j in relation to luck and judgement, as well as in relation to single plays in general and single responses in particular.

Notice here that, respectively via measures R_k and S_j both players may regard themselves ex ante as lucky relative to the same coin, while at the same time acknowledging that coin as fair ex ante relative to a wider system. Indeed in practice such a difference of ex ante views concerning the favourability of outcomes relative to self may be the source of individuals' motivation for playing such a game. This idea and associated roles for the quantities R_k and S_j can be approached more formally in a way which introduces intervening duality as follows:

Stage 3. The intervening duality idea is evident once it is recognized that two players may in practice choose to play against each other via the same "fair" coin and that, in the present context, the system (IIIa)' then becomes dual to (IVa)' in a manner analogous to the duality of (I) to (I)'. To see this formally consider (IIIa),(IIIa)' and (IVa),(IVa)' together as follows:

$$\begin{aligned} &\text{Minimize } \mu' + Mp^+ + Mp^- \\ &\text{st } 1p_1' - 1p_2' \leq \mu \\ &\quad -1p_1' + 1p_2' \leq \mu \\ &\quad p_1' + p_2' + p^+ - p^- = 1 \quad (\text{IIIa}') \\ &\quad p_1' = 1/2 \\ &\quad p_2' = 1/2 \\ &\quad -M \leq \mu' \leq M \\ &\quad p_k', p^+, p^- \geq 0 \end{aligned}$$

$$\begin{aligned} &\text{Maximize } \rho' - Mq^+ - Mq^- \\ &\text{st } 1q_1' - 1q_2' \geq \rho \\ &\quad -1q_1' + 1q_2' \geq \rho \\ &\quad q_1' + q_2' + q^+ - q^- = 1 \quad (\text{IVa}') \\ &\quad q_1' = 1/2 \\ &\quad q_2' = 1/2 \\ &\quad -M \leq \rho' \leq M \\ &\quad q_j', q^+, q^- \geq 0 \end{aligned}$$

This specification admits the fair coin case for the intervening programmes (IIIa)',(IVa)' while also, via R_k, S_j potentially admitting differing ex ante and/or ex post perceptions, e.g. as if each individual may have relatively favourable ex ante perceptions of probabilities.

More exactly, via the systems (IIIa),(IVa) two different individuals may each anticipate net gains relative to himself/herself, even though acknowledging the game (IIIa)',(IVa)' as preemptively fair and zero sum relative to the system. Indeed, as was noted above, their opportunity to experience such essentially subjective individual differences of perception within a conditionally objective experimental environment may in practice constitute a major part of the purpose of playing such a game.

A more subtle point is that, contrasting (IIIa) with (IVa) and (IVa) with (IIIa)' apparently the recognition of elements of the subsystems (IIIa)' and (IVa)' as if objectively fair will in general necessitate the negation by the players of relatively subjective measures R_k, S_j if they are to play such a fair game. That is, an individual axiomatizing a coin as fair relative to a system will in general need to negate the possibility that it might be otherwise relative to himself/herself.

So, in the context of the systems (IIIa),(IIIa)',(IVa),(IVa)', relative certainty can be understood as potentially negating elements of relative uncertainty and conversely. This point is developed further in the context of extensions of these goal programming representations in the next section.

5. Axiomatic probability and preemptive goals

The imputation by player 1 of equal prior probabilities to nature's strategies (e.g. on grounds of fairness, of axiomatization and/or of personal belief) may be represented formally by modifying (IIIa)' to extend it so that it comprehends explicitly goal programming (Charnes and Cooper 1961) formulations for the ranges of the individual probabilities p_k' as well as for their sum. This can be done in a way which, in turn, paves the way for the introduction of lexicographic orderings on such prior conditioning processes by introducing individual probability goals $p_k^*=1/2$ with weights c_k^+, c_k^- associated with potential deviations q_k^+, q_k^- from them to generate (Va),(Va)' from (IIIa),(IIIa)' as follows:

$$\begin{aligned} & \text{Maximize } \rho + \sum R_k \cdot 1/2 - Mq^+ - Mq^- \\ & \text{st } \sum \pi_{kj} q_j - R_k \geq \rho \\ & \quad \sum q_j + q^+ - q^- = 1 \quad (\text{Va}) \\ & \quad -c_k^+ \leq R_k \leq c_k^+ \\ & \quad -M \leq \rho \leq M \\ & \quad j=1,2, q_j, q^+, q^- \geq 0, R_k \text{ unrestricted} \end{aligned}$$

$$\begin{aligned} & \text{Minimize } \mu' + \sum c_k^+ p_k^+ + \sum c_k^- p_k^- + M p^+ + M p^- \\ & \text{st } \sum \pi_{kj} p_k' \leq \mu \\ & \quad \sum p_k' + p^+ - p^- = 1 \quad (\text{Va}') \\ & \quad p_k^+ + p_k^- - p_k^* = 1/2 \\ & \quad -M \leq \mu' \leq M \\ & \quad k=1,2, p_k', p^+, p^-, p_k^+, p_k^- \geq 0 \end{aligned}$$

There is a point here concerning hierarchical or lexicographic preorderings. The specification (Va)' is open to interpretation as if: *first* the conditional range $\sum p_k'=1$ is set pre-emptively via arbitrarily large penalties M associated with any deviations from that range; *then* the ranges of individual probabilities p are conditionally restricted via the penalties c_k^+, c_k^- associated with potential deviations from specified prior values and; *finally*, optimizing values p_k^* and q_j^* are selected conditional on these preconditions. (Note that, because $p_k^*=1/2$, $k=1,2$ in the present as-if-complete -information example, the overall range restriction $\sum p_k'=1$ is apparently redundant. But such a condition would not be redundant in more generally specified incomplete information cases, i.e. cases for which prior information took the form only of upper bounds on subsets of probabilities.)

In any case it is evident from (Va),(Va)' that the measures R_k again have potential interpretations as measures of relative fairness/unfairness. And, the form of these systems also indicates a more general point to the effect that there is apparently no purpose in verifying a statement as true relative to self - e.g. the fairness of the a coin relative to player 1 via r and the specification (Va) - unless the statement is perceived as potentially otherwise. (If an individual axiomatizes a statement as true, then, among other things, it is as if he/she has purposively negated potentially relatively untrue alternatives.)

From a related perspective, problems (Va) and (Va)' illustrate a particular kind of significance for apparent redundancy since in these problems potentials for degeneracy and overdeterminacy can be conditions for verifiability. For instance if the "truth" of the fairness of the coin is explicitly

stated by means of an appropriate specification and solution of a goal programme of the form of (Va)' then that solution is necessarily degenerate and overdetermined; i.e. at that optimum both the elements p_k^+, p_k^- potentiating that optimum and the overall probability condition $\sum p_k = 1$ become as if redundant.

More evocatively from the perspectives of empirical verification and of empirical experimentation and experience, apparently there are positive roles even in the apparently full information coin tossing context here for elements of redundancy and linear dependence as *essential preconditions for* appropriately optimizing decision theoretic specifications. This point seems fundamental and is an example of a Heisenberg-like principle according to which in general any form of experiment or experience interacting with any body will necessarily alter that body relative to the experimenter.

$$\begin{aligned} &\text{Maximize } \rho + \sum R_k . 1/2 - Mq^+ - Mq^- \\ &\text{st } \sum \pi_{kj} q_j - R_k \geq \rho \\ &\quad \sum q_j + q^+ - q^- = 1 \quad (Va) \\ &\quad -c_k^+ \leq R_k \leq +c_k^+ \\ &\quad -M \leq \rho \leq M \\ &j=1,2, q_j, q^+, q^- \geq 0, R_k \text{ unrestricted} \end{aligned}$$

$$\begin{aligned} &\text{Minimize } \mu + \sum S_j . 1/2 + Mp^+ + Mp^- \\ &\text{st } \sum \pi_{kj} p_k - S_j \leq \mu \\ &\quad \sum p_k + p^+ - p^- = 1 \quad (Va') \\ &\quad -d_k^+ \leq S_j \leq +d_k^+ \\ &\quad -M \leq \mu \leq M \\ &k=1,2, p_k, p^+, p^- \geq 0 S_j \text{ unrestricted} \end{aligned}$$

Comparisons and contrasts between the individually related problems (Va) and (VIa) and their system related duals suggest potential for individuals to gain by exploiting relative self indeterminacy and self inconsistency principles and processes. Specifically, there is potential for net gains/losses via R_k, S_j if the individuals choose to perturb *both* the relatively primal specification (Va) and the relatively dual specification (VIa) away from the preemptively optimal values associated with solutions to then dually intervening programmes (Va)' and (VIa)'

6. A concluding remark

While here the emphasis has been on the introduction of the intervening duality idea and its application to a relatively simple class of coin

Here, with the context of coin tossing related exchanges between persons it suggests how, for example via measures p_k^+, p_k^- , R potential "errors" relative to an apparently conditionally agreed ex ante specification and solution may be as if purposively induced relative to one individual in order, among other things, to generate otherwise unattainable opportunities relative to another. (Notice here that a coin toss is frequently used as an apparently impartial means of selecting between otherwise potentially unagreed - and in that sense unattainable - outcomes.)

With such possibilities in mind now consider relations and distinctions between the structures of an intervening dual pair of programmes (Va)', (VIa)' and individually related programmes (V) and (VI) as follows. (Compare the four programmes (IIIa), (IIIa)', (IVa), (IVa)' considered together in Stage 3 above.):

$$\begin{aligned} &\text{Minimize } \mu' + \sum c_k^+ p_k^+ + \sum c_k^- p_k^- + Mp^+ + Mp^- \\ &\text{st } \sum \pi_{kj} p_j \leq \mu \\ &\quad \sum p_k + p^+ - p^- = 1 \quad (Va') \\ &\quad p_k^+ + p_k^- - p_k = 1/2 \\ &\quad -M \leq \mu \leq M \\ &k=1,2, p_k, p^+, p^-, p_k^+, p_k^- \geq 0 \end{aligned}$$

$$\begin{aligned} &\text{Maximize } \rho' - \sum d_j^+ q_j^+ - \sum d_j^- q_j^- - Mq^+ - Mq^- \\ &\text{st } \sum \pi_{kj} q_j \geq \rho \\ &\quad \sum q_j + q^+ - q^- = 1 \\ &\quad q_j^+ + q_j^- - q_j = 1/2 \quad (IVa') \\ &\quad -M \leq \rho' \leq M \\ &j=1,2, q_j, q^+, q^-, q_j^+, q_j^- \geq 0 \end{aligned}$$

tossing applications, in further work I expect to show how it is possible to extend intervening duality based analyses to include more generally applicable classes of barter and trade related games between persons.

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