"An intuitive algebraic approach for solving Linear Programming problems"

Source: Zionts [1974] (or many others).

$$\begin{array}{lllll} [\max]z = & 0{,}56x_1 & +0{,}42x_2 & & \\ \text{s.to} & x_1 & +2x_2 & \leq & 240 \\ & 1{,}5x_1 & +x_2 & \leq & 180 \\ & x_1 & & \leq & 110 \end{array} \qquad \{1\}$$

A [1]

$$\begin{array}{llllllll} [\max]z = & 0{,}56x_1 & +0{,}42x_2 & +0x_3 & +0x_4 & +0x_5 & & \\ & x_1 & +2x_2 & +\{x_3\} & & & = & 240 \\ & 1{,}5x_1 & +x_2 & & +\{x_4\} & & = & 180 \\ & x_1 & & & & +\{x_5\} & = & 110 \end{array} \qquad \{2\}$$

This has (always) an obvious, sure solution. Let

$$x_1, x_2 = 0 \qquad \{3\}$$

Then

$$\begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 240 \\ 180 \\ 110 \end{bmatrix} \qquad \{4\}$$

$$z = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 240 \\ 180 \\ 110 \end{bmatrix} = 0 \qquad \{5\}$$

Is this optimal ? How to improve ?

There does not appear (Dantzig) to be a systematic way of setting *all* the nonbasic variables simultaneously to optimal values —hence, an *iterative*[2] method.

Choose the variable that increases the objective function *most* per unit (this choice is arbitrary), in the example, x_1, because its coefficient (0,56) is the largest.

According to the constraints, x_1 can be increased till:

B

$$\begin{array}{lllll} x_1 & = 240 & & x_1 & = 240 \\ 1{,}5x_1 & = 180 & \rightarrow & x_1 & = 120 \\ x_1 & = 110 & & x_1 & = 110 \end{array} \qquad \{6\}$$

The *third* equation (why ?) in {2} leads to x_1 = 110 and x_5 = 0. The variable x_1 will be the *entering* variable and x_5 the *leaving* variable:

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[1] A, B, C identify the iteration, as summarized below.

[2] *Iterative:* involving repetition; relating to *iteration*. *Iterate* (from Latin *iterare*), to say or do again (and again). Not to be confused with *interactive*.

C $$x_1 = 110 - x_5 \qquad \{7\}$$

Substituting for x_1 everywhere (except in its own constraint), we have

$$\begin{array}{llllllll} [\max] z = & 0{,}56(110 - x_5) & + 0{,}42x_2 & & & & & \\ & (110 - x_5) & + 2x_2 & + x_3 & & & = & 240 \\ & 1{,}5(110 - x_5) & + x_2 & & + x_4 & & = & 180 \\ & x_1 & & & & + x_5 & = & 110 \end{array} \qquad \{8\}$$

A $$\begin{array}{lllllll} [\max] z = & & 0{,}42x_2 & & & - 0{,}56x_5 & + 61{,}6 \\ & & + 2x_2 & + \{x_3\} & & - x_5 & = 130 \\ & & x_2 & & + \{x_4\} & - 1{,}5x_5 & = 15 \\ & \{x_1\} & & & & + x_5 & = 110 \end{array} \qquad \{9\}$$

which is of course equivalent to Eq. {2}.

We now have a **new** (equivalent) LP problem, **to be treated as the original was.** The process can continue *iteratively*.

$$\begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 110 \\ 130 \\ 15 \end{bmatrix} \qquad \{10\}$$

From Eq. {2} or Eq. {9}, respectively,

$$z = [0{,}56 \quad 0 \quad 0] \begin{bmatrix} 110 \\ 130 \\ 15 \end{bmatrix} = 61{,}6 \qquad \{11\}$$

$$z = [0 \quad 0 \quad 0] \begin{bmatrix} 110 \\ 130 \\ 15 \end{bmatrix} + 61{,}6 = 61{,}6 \qquad \{12\}$$

Now, x_2 is the new entering variable. According to the constraints, it can be increased till:

B $$\begin{array}{llcll} 2x_2 & = 130 & & x_2 & = 65 \\ x_2 & = 15 & \to & x_2 & = 15 \\ 0x_2 & = 110 & & x_2 & = \infty \end{array} \qquad \{13\}$$

C $$x_2 = 15 - x_4 + 1{,}5x_5 \qquad \{14\}$$

Substituting for x_2 everywhere (except its own constraint), we have

$$\begin{array}{lllllll} [\max]z = & 0{,}42(15 - x_4 + 1{,}5x_5) & & & -0{,}56x_5 & & +61{,}6 \\ & +2(15 - x_4 + 1{,}5x_5) & +x_3 & & -x_5 & = & 130 \\ & x_2 & & +x_4 & -1{,}5x_5 & = & 15 \\ x_1 & & & & +x_5 & = & 110 \end{array} \qquad \{15\}$$

A

$$\begin{array}{lllllll} [\max]z = & & & -0{,}42x_4 & +0{,}07x_5 & & +67{,}9 \\ & & \{x_3\} & -2x_4 & +2x_5 & = & 100 \\ & \{x_2\} & & +x_4 & -1{,}5x_5 & = & 15 \\ \{x_1\} & & & & +x_5 & = & 110 \end{array} \qquad \{16\}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 110 \\ 15 \\ 100 \end{bmatrix} \qquad \{17\}$$

Now, x_5 is the new entering variable. According to the constraints, it can be increased till:

B

$$\begin{array}{lllll} 2x_5 & = 100 & & x_5 & = 50 \\ -1{,}5x_5 & = 15 & \rightarrow & x_5 & = \ldots \\ x_5 & = 110 & & x_5 & = 110 \end{array} \qquad \{18\}$$

C

$$x_5 = 50 - \frac{1}{2}x_3 + x_4 \qquad \{19\}$$

Substituting for x_5 everywhere (except its own constraint), we have

$$\begin{array}{llllllll} [\max]z = & & & -0{,}42x_4 & +0{,}07\left(50 - \frac{1}{2}x_3 + x_4\right) & & +67{,}9 \\ & & x_3 & -x_4 & +x_5 & = & 50 \\ & x_2 & & +x_4 & -1{,}5\left(50 - \frac{1}{2}x_3 + x_4\right) & = & 15 \\ x_1 & & & & +\left(50 - \frac{1}{2}x_3 + x_4\right) & = & 110 \end{array} \qquad \{20\}$$

A

$$\begin{array}{lllllll} [\max]z = & & -0{,}035x_3 & -0{,}35x_4 & & & +71{,}4 \\ & & x_3 & -x_4 & +\{x_5\} & = & 50 \\ & \{x_2\} & +0{,}75x_3 & -0{,}5x_4 & & = & 90 \\ \{x_1\} & & -0{,}5x_3 & +x_4 & & = & 60 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 60 \\ 90 \\ 50 \end{bmatrix} \qquad \{21\}$$

Now, no variable produces an increase. So, this is a *maximum.*

In sum:

A In the system of equations, find the identity matrix (immediate solution).
B search for an *entering* variable (or finish)
C consequently, find a *leaving* variable (if wrongly chosen, negative values will appear).

References:

– ZIONTS, Stanley, 1974, "Linear and integer programming", Prentice-Hall, Englewood Cliffs, NJ (USA), p 5. (IST Library.) ISBN 0-13-536763-8.

– See others on the course webpage *(http://web.tecnico.ulisboa.pt/mcasquilho)*.

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