Solutions for Questions 1

Solution 1.1:

a) Smaller rolls can be created using a number of methods. In method M_1, the roll is cut into two equal 9 ft. rolls with no waste. In method M_2 the roll is cut into a 9 ft. and a 7 ft. roll with 2 ft. of waste. In method M_3 the roll is cut into a 9 ft. and a 5 ft. roll with 4 ft. of waste. In method M_4 the roll is cut into two 7 ft. rolls with 4 ft. of waste. In method M_5 the roll is cut into a 7 ft. and two 5 ft. rolls with 1 ft. of waste. Finally in method M_6 the roll is cut into three 5 ft. rolls with 3 ft. of waste.

b) Let x_i be the number of large rolls that are cut using method M_i. Then there are $2x_1 + x_2 + x_3$ rolls available that are 9 ft long, $x_2 + 2x_4 + x_5$ rolls available that are 7 ft long and $x_3 + 2x_5 + 3x_6$ rolls available that are 5 ft long. Thus the linear programming problem becomes:

$$\text{minimise } x_1 + x_2 + x_3 + x_4 + x_5 + x_6, \text{ subject to } x_i \geq 0 \quad (1 \leq i \leq 6) \text{ and}$$

$$\begin{aligned} 2x_1 + x_2 + x_3 &\geq 10, \\ x_2 + 2x_4 + x_5 &\geq 20, \\ x_3 + 2x_5 + 3x_6 &\geq 50. \end{aligned}$$

There are at least two problems that could occur in practice. One is the possibility that x_i may not be integers; clearly it would be necessary to use the next largest integer number of rolls. The second is the difficulty in making the cut using method M_1, which involves no waste at all; in practice the resulting rolls may be too short. It may be that all three constraints above are tight, so we create exactly the right the number of smaller rolls of each size; if not some account should be taken of that waste.

c) There are a total of $x_1 + 2(x_2 + x_3 + x_4) + 3(x_5 + x_6)$ cuts to be made, while the amount of waste in feet is $2x_2 + 4(x_3 + x_4) + x_5 + 3x_6$. Thus the total net cost N in pounds is

$$\begin{aligned} N = (x_1 + x_2 + x_3 + x_4 + x_5 + x_6)P \\ + (x_1 + 2x_2 + 2x_3 + 2x_4 + 3x_5 + 3x_6)C \\ - (2x_2 + 4x_3 + 4x_4 + x_5 + 3x_6)k. \end{aligned}$$

and this gives the objective function to minimise subject to the same constraints as above.

Solution 1.2:

a) Let the factory produce b copies of the ``Bashful'' sculpture, $d$ copies of the ``Dozy'' sculpture and h copies of the ``Happy'' sculpture each week. The ``reality'' condition insists that $b \geq 0$, $d \geq 0$ and $h \geq 0$ -- and to be really formal, that each of b, d and h is integral. The total profit P in £'s is given by

$$P = 2b + 4d + 3h.$$

We have constraints based on the availability of the machines. From the given table, we will use $2b + d$ hours on machine A, $b + 3h$ hours on machine B and $2b + 3d + 2h$ hours on machine C. Thus our availability constraints are:-

$$2b + d \leq 43,$$

$$b + 3h \leq 37,$$

$$2b + 3d + 2h \leq 42.$$

All the constraints are linear; hence the formulation as a linear programming problem is simply to maximise P subject to these three constraints and the reality constraint.

Solution 1.3: Let b, d and f be the number of kilograms of binder, disintegrant and filler in each 100 kilograms of the formulation. Then since there will be 14 kilograms of active ingredient in each 100 kilograms of the formulation, $b + d + f = 86$. The binder - filler constraint is that $10b \geq f$, while the constraint on the disintegrant gives $4d \leq b + 14$. These, together with the reality requirement, that $b \geq 0$, $d \geq 0$ and $f \geq 0$ are all the constraints, and the problem is to minimise the total cost $C = 50b + 15d + 2f$ subject to these constraints.

Solution 1.4: Let a, b and c be the number of cars of each type that are to be made. The reality constraint, that $a \geq 0$, $b \geq 0$ and $c \geq 0$ is clearly essential. The labour availabilty in the two factories gives:

$$8a + 8b + 9c \leq 10120,$$

$$8a + 9b + 11c \leq 11000$$

An aditional constraint might be the need for a, b and c to be integers, although since this is a monthly figure, it would be natural to hold uncompleted cars until the following month. The total profit made P, in pounds, is then $P = 1100a + 1200b + 1450c$.

The mathematical model is unrealistic in many respects. Some factors are:

- it is very unlikely that the assumption of constant profit per vehicle is true; there are probably some fixed costs involved and also capacity problems.
- there is no reflection of market demand; it *is* plausible that no cars of type B are made, a result which would be unacceptable in practice; and
- there is an unlikely simplicity in the product range; I would expect there to be many more options with varying nett profits in a real situation.

Solution 1.5: Let r_1, r_2 and r_3 be the number of kilos of cereal, dried fruit and nuts respectively which are mixed to make the ``Rich'' blend, and define $h_1$, $h_2$ and $h_3$ and $c_1$, $c_2$ and $c_3$ to be the corresponding weights for the ``Healthy'' and ``Crunchy'' mixes.

The total costs of the cereals is

$$C = 1.5(c_2 + h_2 + r_2) + 1.0(c_3 + h_3 + r_3) + 0.8(r_1 + h_1 + c_1)$$

while the total sales income is

$$S = 2.0(r_1 + r_2 + r_3) + 1.6(c_1 + c_2 + c_3) + 1.2(h_1 + h_2 + h_2)$$

and the difference $S - C$ between these two figures gives the profit which is to be maximised.

The constraint that the ``Crunchy'' blend must contain at least 60% nuts becomes

$$c_3 \geq 0.6(c_1 + c_2 + c_3).$$

The other constraints are given in the same way: for the ``Healthy'' mix,

$$h_1 \geq 0.6(h_1 + h_2 + h_3) \quad \text{and} \quad h_3 \leq 0.2(h_1 + h_2 + h_3)$$

and for the ``Rich'' blend,

$$r_1 \leq 0.2(c_1 + r_2 + r_3) \quad \text{and} \quad r_2 \geq 0.6(c_1 + r_2 + r_3).$$

We have three supply constraints

$$\begin{aligned} c_1 + h_1 + r_1 &\leq 100, \\ c_2 + h_2 + r_2 &\leq 80, \\ c_3 + h_3 + r_3 &\leq 60. \end{aligned}$$

In addition of course we have the reality constraints that $h_i \geq 0$, $c_i \geq 0$ and $r_i \geq 0$.
