Bronson & Naadimuthu, 1997, pp 56–57

Chapter 4

Linear Programming: Duality and Sensitivity Analysis

Every linear program in the variables $x_1, x_2, \ldots, x_n$ has associated with it another linear program in the variables $w_1, w_2, \ldots, w_m$ (where m is the number of constraints in the original program), known as its *dual*. The original program, called the *primal*, completely determines the form of its dual.

SYMMETRIC DUALS

The dual of a (primal) linear program in the (nonstandard) matrix form

$$\begin{aligned} \text{minimize:} \quad & z = \mathbf{C}^T\mathbf{X} \\ \text{subject to:} \quad & \mathbf{AX} \geq \mathbf{B} \\ \text{with:} \quad & \mathbf{X} \geq \mathbf{0} \end{aligned} \tag{4.1}$$

is the linear program

$$\begin{aligned} \text{maximize:} \quad & z = \mathbf{B}^T\mathbf{W} \\ \text{subject to:} \quad & \mathbf{A}^T\mathbf{W} \leq \mathbf{C} \\ \text{with:} \quad & \mathbf{W} \geq \mathbf{0} \end{aligned} \tag{4.2}$$

Conversely, the dual of program (*4.2*) is program (*4.1*). (See Problems 4.1 and 4.2.)

Programs (*4.1*) and (*4.2*) are symmetrical in that both involve nonnegative variables and inequality constraints; they are known as the *symmetric duals* of each other. The dual variables $w_1, w_2, \ldots, w_m$ are sometimes called *shadow costs*.

DUAL SOLUTIONS

Theorem 4.1 (Duality Theorem): If an optimal solution exists to either the primal or symmetric dual program, then the other program also has an optimal solution and the two objective functions have the same optimal value.

In such situations, the optimal solution to the primal (dual) is found in the last row of the final simplex tableau for the dual (primal), in those columns associated with the slack or surplus variables (see Problem 4.3). Since the solutions to both programs are obtained by solving either one, it may be computationally advantageous to solve a program's dual rather than the program itself. (See Problem 4.4.)

Theorem 4.2 (Complementary Slackness Principle): Given that the pair of symmetric duals have optimal solutions, then if the kth constraint of one system holds as an inequality—i.e., the associated slack or surplus variable is positive—the kth component of the optimal solution of its symmetric dual is zero.

(See Problems 4.11 and 4.12.)

UNSYMMETRIC DUALS

For primal programs in standard matrix form, duals may be defined as follows:

Primal			***Dual***		
minimize:	$z = \mathbf{C}^T\mathbf{X}$		maximize:	$z = \mathbf{B}^T\mathbf{W}$	*(4.4)*
subject to:	$\mathbf{AX} = \mathbf{B}$	*(4.3)*	subject to:	$\mathbf{A}^T\mathbf{W} \le \mathbf{C}$	
with:	$\mathbf{X} \ge \mathbf{0}$				
maximize:	$z = \mathbf{C}^T\mathbf{X}$		minimize:	$z = \mathbf{B}^T\mathbf{W}$	*(4.6)*
subject to:	$\mathbf{AX} = \mathbf{B}$	*(4.5)*	subject to:	$\mathbf{A}^T\mathbf{W} \ge \mathbf{C}$	
with:	$\mathbf{X} \ge \mathbf{0}$				

(See Problems 4.5 and 4.6.) Conversely, the duals of programs *(4.4)* and *(4.6)* are defined as programs *(4.3)* and *(4.5)*, respectively. Since the dual of a program in standard form is not itself in standard form, these duals are *unsymmetric*. Their forms are consistent with and a direct consequence of the definition of symmetric duals (see Problem 4.8).

Theorem 4.1 is valid for unsymmetric duals too. However, the solution to an unsymmetric dual is not, in general, immediately apparent from the solution to the primal; the relationships are

$$\mathbf{W}^{*T} = \mathbf{C}_0^T\mathbf{A}_0^{-1} \quad \text{or} \quad \mathbf{W}^* = (\mathbf{A}_0^T)^{-1}\mathbf{C}_0 \tag{4.7}$$

$$\mathbf{X}^{*T} = \mathbf{B}_0^T(\mathbf{A}_0^T)^{-1} \quad \text{or} \quad \mathbf{X}^* = \mathbf{A}_0^{-1}\mathbf{B}_0 \tag{4.8}$$

In *(4.7)*, $\mathbf{C}_0$ and $\mathbf{A}_0$ are made up of those elements of $\mathbf{C}$ and $\mathbf{A}$, in either program *(4.3)* or *(4.5)*, that correspond to the *basic variables* in $\mathbf{X}^*$; in *(4.8)*, $\mathbf{B}_0$ and $\mathbf{A}_0$ are made up of those elements of $\mathbf{B}$ and $\mathbf{A}$, in either program *(4.4)* or *(4.6)*, that correspond to the *basic variables* in $\mathbf{W}^*$. (See Problem 4.7.)