Artificial variables in Linear Programming

Adapted from H&L [2005] and Taha [1992]

Equality constraints [H&L, p 125]

Suppose a modification to the original *Wyndor* problem, as follows ({1}).

$$\begin{array}{llll} [\max] z = & 3x_1 & +5x_2 & \\ \text{s.to} & x_1 & & \le 4 \\ & & 2x_2 & \le 12 \\ & 3x_1 & +2x_2 & =18 \end{array} \qquad \{1\}$$

with $\boldsymbol{x} \ge 0$. Thus, the third constraint is now an equality. This can become

$$\begin{array}{llllll} (0) & z & -3x_1 & -5x_2 & & & =0 \\ (1) & & x_1 & & +x_3 & & =4 \\ (2) & & & 2x_2 & & +x_4 & =12 \\ (3) & & 3x_1 & +2x_2 & & & =18 \end{array} \qquad \{2\}$$

However, these equations **do not have** an obvious initial (basic feasible) solution. So, the **artificial variable technique** is applied. With M a very high number ($+\infty$) —this is the **Big M method**[*]—, we can ***augment*** the system {2} to obtain[†]

$$\begin{array}{llllllll} (0) & z & -3x_1 & -5x_2 & & & -M\,\bar{x}_5 & =0 \\ (1) & & x_1 & & +x_3 & & & =4 \\ (2) & & & 2x_2 & & +x_4 & & =12 \\ (3) & & 3x_1 & +2x_2 & & & +\bar{x}_5 & =18 \end{array} \qquad \{3\}$$

Converting equation 0 to proper form

In {3}, the (obvious) initial basic variables are x_3, x_4 and $\bar{x}_5$ (non-basic $x_1 = 0$ and $x_2 = 0$). However, this system is not yet in proper form for Gaussian elimination because a basic variable ($\bar{x}_5$) has a **non-zero coefficient** in Eq. 0. Indeed, all the basic variables must be (algebraically) eliminated from Eq. 0 before the simplex method can find the entering basic variable. (This elimination is necessary so that the negative of the coefficient of each non-basic variable will give the rate at which z would increase if that non-basic variable were to be increased from 0 while adjusting the values of the basic variables accordingly.)

To eliminate $\bar{x}_5$ from Eq. 0, we need to subtract from Eq. 0 the product M times Eq. 3:

$$\begin{array}{lllll} z & -3x_1 & -5x_2 & +M\,\bar{x}_5 & =0 \\ & -M(3x_1 & +2x_2 & +\bar{x}_5 & =18) \\ \hline z & -(3M+3)x_1 & -(2M+5)x_2 & & =-18M \end{array} \qquad \{4\}$$

[*] Another method to solve this matter is the "two-phase method".
[†] To obtain a ***different*** problem !

In this example, there is only one equation with an artificial variable. If there were **several** equations with artificial variables, we would have to subtract accordingly.

Application of the simplex method

The new Eq. 0 gives z in terms of just the non-basic variables (x_1, x_2):

$$z = -18M + (3M + 3)x_1 + (2M + 5)x_2 \qquad \{5\}$$

Since the coefficient of x_1 is the **best** (greatest), this variable is chosen as the *entering* variable.

The leaving variable, as always, will correspond to the smallest "positive" (non-negative) ratio (from the so-called "minimum ratio test").

Another (more general) example (Taha [1992], p 72)

$$\begin{array}{rllll} [\min] z = & 4x_1 & + x_2 & & \\ \text{s.to} & 3x_1 & + x_2 & = 3 & \\ & 4x_1 & + 3x_2 & \geq 6 & \\ & x_1 & + 2x_2 & \leq 4 & \end{array} \qquad \{6\}$$

with $\boldsymbol{x} \geq 0$. The *augmented* standard form is

$$\begin{array}{rllllllll} [\min] z = & 4x_1 & + x_2 & + 0x_3 & + 0x_4 & + Ma_1 & + Ma_2 & \\ \text{s.to} & 3x_1 & + x_2 & & & + a_1 & & = 3 \\ & 4x_1 & + 3x_2 & - x_3 & & & + a_2 & = 6 \\ & x_1 & + 2x_2 & & + x_4 & & & = 4 \end{array} \qquad \{7\}$$

References

- HILLIER, Frederick S., and Gerald J. LIEBERMAN, 2005, "Introduction to Operations Research", 8.th ed., McGraw-Hill
- TAHA, Hamdy, 1992, "Operations Research: an introduction", 5.th ed., MacMillan Publishing Company

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