tabulation is helpful in clarifying thought and detecting incorrect mathematical descriptions.

Other problems

The main areas in which IP is used in practice include:

- imposition of logical conditions in LP problems (such as the either/or condition dealt with above)
- blending with a limited number of ingredients
- depot location
- job shop scheduling
- assembly line balancing
- airline crew scheduling
- timetabling

Integer programming example

Recall the blending problem dealt with before under linear programming. To remind you of it we reproduce it below.

Blending problem

Consider the example of a manufacturer of animal feed who is producing feed mix for dairy cattle. In our simple example the feed mix contains two active ingredients and a filler to provide bulk. One kg of feed mix must contain a minimum quantity of each of four nutrients as below:

Nutrient	A	B	C	D
gram	90	50	20	2

The ingredients have the following nutrient values and cost

	A	B	C	D	Cost/kg
Ingredient 1 (gram/kg)	100	80	40	10	40
Ingredient 2 (gram/kg)	200	150	20	-	60

What should be the amounts of active ingredients and filler in one kg of feed mix?

Blending problem solution

Variables

In order to solve this problem it is best to think in terms of one kilogram of feed mix. That kilogram is made up of three parts - ingredient 1, ingredient 2 and filler so: let
x_1 = amount (kg) of ingredient 1 in one kg of feed mix
x_2 = amount (kg) of ingredient 2 in one kg of feed mix
x_3 = amount (kg) of filler in one kg of feed mix
where $x_1 >= 0$, $x_2 >= 0$ and $x_3 >= 0$.

Constraints

- balancing constraint (an *implicit* constraint due to the definition of the variables)

$x_1 + x_2 + x_3 = 1$

- nutrient constraints

$100x_1 + 200x_2 >= 90$ (nutrient A)
$80x_1 + 150x_2 >= 50$ (nutrient B)
$40x_1 + 20x_2 >= 20$ (nutrient C)
$10x_1 >= 2$ (nutrient D)

Note the use of an inequality rather than an equality in these constraints, following the rule we put forward in the Two Mines example, where we assume that the nutrient levels we want are lower limits on the amount of nutrient in one kg of feed mix.

Objective

Presumably to minimise cost, i.e.

minimise $40x_1 + 60x_2$

which gives us our complete LP model for the blending problem.

Suppose now we have the additional conditions:

- if we use any of ingredient 2 we incur a fixed cost of 15
- we need not satisfy all four nutrient constraints but need only satisfy three of them (i.e. whereas before the optimal solution required all four nutrient constraints to be satisfied now the optimal solution could (if it is worthwhile to do so) only have three (any three) of these nutrient constraints satisfied and the fourth violated.

Give the complete MIP formulation of the problem with these two new conditions added.

Solution

To cope with the condition that if $x_2>=0$ we have a fixed cost of 15 incurred we have the standard trick of introducing a zero-one variable y defined by

```
y = 1 if x2>=0
  = 0 otherwise
```

and

- add a term +15y to the objective function

and add the additional constraint

- $x_2 <=$ [largest value x_2 can take]y

In this case it is easy to see that x_2 can never be greater than one and hence our additional constraint is $x_2<= y$.

To cope with condition that need only satisfy three of the four nutrient constraints we introduce four zero-one variables z_i (i=1,2,3,4) where

```
zi = 1 if nutrient constraint i (i=1,2,3,4) is satisfied
```

```
   = 0 otherwise
```

and add the constraint

- $z_1+z_2+z_3+z_4$ >= 3 (at least 3 constraints satisfied)

and alter the nutrient constraints to be

- $100x_1 + 200x_2 >= 90z_1$
- $80x_1 + 150x_2 >= 50z_2$
- $40x_1 + 20x_2 >= 20z_3$
- $10x_1 >= 2z_4$

The logic behind this change is that if a z_i=1 then the constraint becomes the original nutrient constraint which needs to be satisfied. However if a z_i=0 then the original nutrient constraint becomes

- same left-hand side >= zero

which (for the four left-hand sides dealt with above) is always true and so can be neglected - meaning the original nutrient constraint need not be satisfied. Hence the complete (MIP) formulation of the problem is given by

```
minimise 40x1 + 60x2 + 15y
subject to
x1 + x2 + x3 = 1
100x1 + 200x2 >= 90z1
80x1 + 150x2 >= 50z2
40x1 + 20x2 >= 20z3
10x1 >= 2z4
z1 + z2 + z3 + z4 >= 3
x2 <= y
zi = 0 or 1    i=1,2,3,4
y = 0 or 1
xi >= 0    i=1,2,3
```

Integer programming example

In the planning of the monthly production for the next six months a company must, in each month, operate either a normal shift or an extended shift (if it produces at all). A normal shift costs £100,000 per month and can produce up to 5,000 units per month. An extended shift costs £180,000 per month and can produce up to 7,500 units per month. Note here that, for either type of shift, the cost incurred is fixed by a union guarantee agreement and so is independent of the amount produced.

It is estimated that changing from a normal shift in one month to an extended shift in the next month costs an extra £15,000. No extra cost is incurred in changing from an extended shift in one month to a normal shift in the next month.

The cost of holding stock is estimated to be £2 per unit per month (based on the stock held at the end of each month) and the initial stock is 3,000 units (produced by a normal shift). The amount in stock at the end of month 6 should be at least 2,000 units. The demand for the company's product in each of the next six months is estimated to be as shown below:

```
Month   1     2     3     4     5     6
Demand  6,000 6,500 7,500 7,000 6,000 6,000
```