Chapter 9 - Number Systems

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CEFET-RJ

1. Motivation
2. Positional Number Systems
3. Binary System

4 Converting between Decimal and Binary

5 Hexadecimal Notation

6 References

Motivation

Lets start the semester with an easy subject:

In everyday life how do you count numbers? Any ideas?

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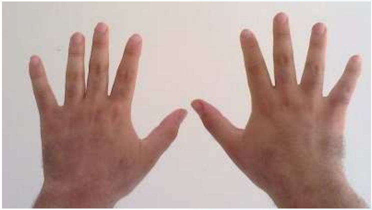

;)

- 1 Fingers;
- 2 Fingers;
- 3 Fingers;
- . . .
- 10 Fingers;

Decimal system is used to represent numbers:

- Digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9;

Consider the number 83:

What does this mean using the decimal system? Any ideas?

Example

Consider the number 83:

What does this mean using the decimal system? Any ideas?

- Number 10 was counted 8 times:
  - $8 \times 10 = 8 \times 10^1$
- Number 1 was counted 3 times:
  - $3 = 3 \times 10^0$
- Combining these elements:
  - *I.e.*: $83 = 8 \times 10 + 3 = 8 \times 10^1 + 3 \times 10^0$

Example

Consider the number 4728:

What does this mean using the decimal system? Any ideas?

Exercise

Consider the number 4728:

> What does this mean using the decimal system? Any ideas?

- Number 1000 was counted X times:
  -
- Number 100 was counted Y times:
  -
- Number 10 was counted Z times:
  -
- Number 1 was counted D times:
  -
- Combining these elements:
  -

Exercise

Consider the number 4728:

What does this mean using the decimal system? Any ideas?

- Number 1000 was counted 4 times:
  - $4 \times 1000 = 4 \times 10^3$
- Number 100 was counted 7 times:
  - $7 \times 100 = 7 \times 10^2$
- Number 10 was counted 2 times:
  - $2 \times 10 = 2 \times 10^1$
- Number 1 was counted 8 times:
  - $8 = 8 \times 10^0$
- Combining these elements:
  - *I.e.*: $4728 = 4 \times 10^3 + 7 \times 10^2 + 2 \times 10^1 + 8 \times 10^0$

But what if we have decimal fractions? Any ideas?

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- *E.g.:* how do we represent the number 0.256 using the decimal system?

Example

Consider the number 0.256:

What does this mean using the decimal system? Any ideas?

- Number 0.1 was counted 2 times:
  - $2 \times 0.1 = 2 \times 10^{-1}$
- Number 0.01 was counted 5 times:
  - $5 \times 0.01 = 5 \times 10^{-2}$
- Number 0.001 was counted 6 times:
  - $6 \times 10^{-3} = 6 \times 10^{-3}$
- Combining these elements:
  - *I.e.:* $0.256 = 2 \times 10^{-1} + 5 \times 10^{-2} + 6 \times 10^{-3}$

But wait:

What if we have an **integer** part and a **fractional** part? Any ideas?

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What if we have an **integer** part and a **fractional** part? Any ideas?

- *E.g.:* how do we represent the number 442.256 using the decimal system?

Example

Consider the number 442.256:

What does this mean using the decimal system? Any ideas?

Exercise

Consider the number 442.256:

What does this mean using the decimal system? Any ideas?

- Number 100 was counted X times:
  -
- Number 10 was counted Y times:
  -
- Number 1 was counted Z times:
  -
- Number 0.1 was counted Q times:
  -
- Number 0.01 was counted W times:
  -
- Number 0.001 was counted E times:
  -

Combining these elements:

- *I.e.:* $442.256 =$

Exercise

Consider the number 442.256:

> What does this mean using the decimal system? Any ideas?

- Number 100 was counted 4 times:
  - $4 \times 100 = 4 \times 10^2$
- Number 10 was counted 4 times:
  - $4 \times 10 = 4 \times 10^1$
- Number 1 was counted 2 times:
  - $2 \times 1 = 2 \times 10^0$
- Number 0.1 was counted 2 times:
  - $2 \times 0.1 = 2 \times 10^{-1}$
- Number 0.01 was counted 5 times:
  - $5 \times 0.01 = 5 \times 10^{-2}$
- Number 0.001 was counted 6 times:
  - $6 \times 0.001 = 6 \times 10^{-3}$

Combining these elements:

- *I.e.:* $442.256 = 4 \times 10^2 + 4 \times 10^1 + 2 \times 10^0 + 2 \times 10^{-1} + 5 \times 10^{-2} + 6 \times 10^{-3}$

Some important observations:

- Decimal system is said to have a **base**, or **radix**, of 10;
- In any number:
    - Leftmost digit is referred to as the most significant digit (MSD);
    - Rightmost digit is called the least significant digit (LSD);

In conclusion:

4	7	2	2	5	6
100s	10s	1s	tenths	hundredths	thousandths
10^2	10^1	10^9	10^{-1}	10^{-2}	10^{-3}
position 2	position 1	position 0	position –1	position –2	position –3

Figure: Positional interpretation of decimal number: 472, 256 (Source: (Stallings, 2015))

TYPO:

- Position 0 of the table should read 10^0

In general, X where:

- $X = \{\cdots d_2 d_1 d_0 . d_{-1} d_{-2} d_{-3} \cdots\}$
- $X = \sum_i (d_i \times 10^i)$

Positional Number Systems

Decimal system illustrates a positional number system (1/2):

- Each number is represented by a string of digits;
- Each digit position i has an associated weight r^i:
  - r is the radix / base of the system;
- General form of a number in such a system with radix r is:

$$(\cdots a_3 a_2 a_1 a_0 . a_{-1} a_{-2} a_{-3} \cdots)_r$$

  - Where $a_i : 0 \leq a_i < r$

Positional Number Systems

Decimal system illustrates a positional number system (2/2):

- Number is defined to have the value:

$$\cdots a_3 r^3 + a_2 r^2 + a_1 r^1 + a_0 r^0 + a_{-1} r^{-1} \cdots$$

The question is:

Do we really need to use the decimal system? Any ideas?

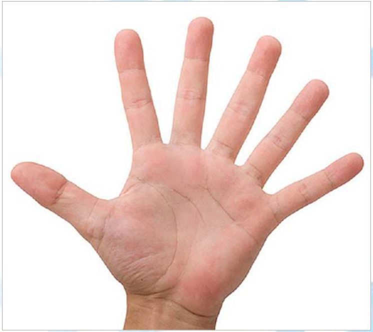
Decimal system:

- Radix 10;
- Digits in the range 0 through 9

What if human beings had 12 fingers?

What if human beings had 12 fingers?

- Radix 12;
- Digits in the range 0 through 11;

Fun fact

You think 12 fingers is weird?

Fun fact

You think 12 fingers is weird?

Have a look at Polydactyly:

Binary System

Binary system only uses two digits:

- Radix / base 2;
- Binary digits 1 and 0 have the same meaning as in decimal notation:

$$0_2 = 0_{10}$$
$$1_2 = 1_{10}$$

- Also a positional number system:
    - Each binary digit in a number has a value;

Exercise

$$10_2 = ?_{10}$$
$$11_2 = ?_{10}$$
$$100_2 = ?_{10}$$

Exercise

$$10_2 = (1 \times 2^1) + (0 \times 2^0) = 2_{10}$$
$$11_2 = (1 \times 2^1) + (1 \times 2^0) = 3_{10}$$
$$100_2 = (1 \times 2^2) + (0 \times 2^1) + (0 \times 2^0) = 4_{10}$$

But what if are trying to represent a binary number with a fractional part? Any ideas?

But what if are trying to represent a binary number with a fractional part? Any ideas?

- Binary number 1001.101_2 converts to what decimal number?

Exercise

$$1001.101_2 = ?_{10}$$

Exercise

$$
\begin{aligned}
1001.101 &= 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 1 \times 2^0 + 1 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3} \\
&= 9.62510
\end{aligned}
$$

Remember this formula for the decimal system:

- $X = \{\cdots d_2 d_1 d_0 . d_{-1} d_{-2} d_{-3} \cdots\}$
- $X = \sum_i (d_i \times 10^i)$

What do you think would be the necessary changes for a **binary** system? Any ideas?

Remember this formula for the **decimal** system:

- $X = \{\cdots d_2 d_1 d_0 . d_{-1} d_{-2} d_{-3} \cdots\}$
- $X = \sum_i (d_i \times 10^i)$

What do you think would be the necessary changes for a **binary** system? Any ideas?

- Radix / base has value 2;
- *I.e.:* $X = \sum_i (d_i \times 2^i)$

Converting between Decimal and Binary

How can we convert between decimal and binary numbers? Any ideas?

Converting between Decimal and Binary

How can we convert between decimal and binary numbers? Any ideas?

Suppose we need to convert N from decimal into binary form (1/3):

- If we divide N by 2 we obtain a quotient N_1 and a remainder R_0;
- We then may write:
  - $N = 2 \times N_1 + R_0$
  - $R_0 = 0$ or 1

Suppose we need to convert N from decimal into binary form (2/3):

- N_1 can also be divided by 2, then:
    - $N_1 = 2 \times N_2 + R_1$
    - $R_1 = 0$ or 1

Suppose we need to convert N from decimal into binary form (3/3):

- N_2 can also be divided by 2, then:
  - $N_2 = 2 \times N_3 + R_2$
  - $R_2 = 0$ or 1

Continuing this sequence will eventually produce:

- a quotient $N_{m-1} = 1$
- a remainder R_{m-2} which is 0 or 1;

We are now able to obtain the **binary form**:

$$N = (R_{m-1} \times 2^{m-1}) + (R_{m-2} \times 2^{m-2}) + \cdots + (R_2 \times 2^2) + (R_1 \times 2^1) + R_0$$

Example

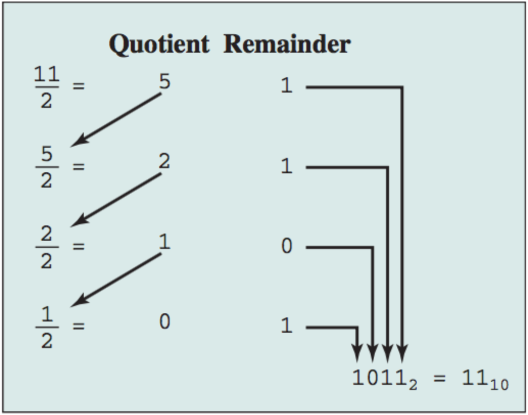


Figure: (Source: (Stallings, 2015))

Example

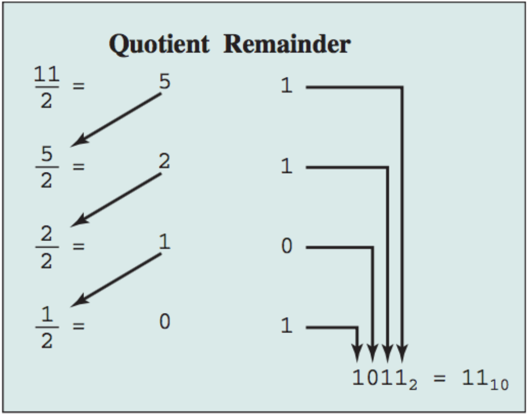


Figure: (Source: (Stallings, 2015))

Exercise

Converting the following decimal numbers into binary:

- $8_{10} = ?_2$
- $9_{10} = ?_2$
- $1024_{10} = ?_2$
- $1025_{10} = ?_2$
- $1027_{10} = ?_2$
- $2049_{10} = ?_2$
- $4096_{10} = ?_2$
- $4106_{10} = ?_2$

Hexadecimal Notation

In computation:

- All forms of data is represented in a binary fashion;
- Very cumbersome for human beings =(
- Most computer professionals prefer a more compact notation:
    - Hexadecimal notation FTW! =)
        - Binary digits are grouped into sets of four bits (nibble);
        - Each possible combination of four binary digits is given a symbol;

This is the hexadecimal table:

$0000 = 0$	$0100 = 4$	$1000 = 8$	$1100 = C$
$0001 = 1$	$0101 = 5$	$1001 = 9$	$1101 = D$
$0010 = 2$	$0110 = 6$	$1010 = A$	$1110 = E$
$0011 = 3$	$0111 = 7$	$1011 = B$	$1111 = F$

Figure: (Source: (Stallings, 2015))

In general:

$$Z = \sum_i (h_i \times 16^i)$$

- Radix / base has value 16;
- Each hexadecimal digit h_i is in the decimal range $0 \leq h_i < 15$;

Decimal (base 10)	Binary (base 2)	Hexadecimal (base 16)
0	0000	0
1	0001	1
2	0010	2
3	0011	3
4	0100	4
5	0101	5
6	0110	6
7	0111	7
8	1000	8
9	1001	9
10	1010	A
11	1011	B
12	1100	C
13	1101	D
14	1110	E
15	1111	F
16	0001 0000	10
17	0001 0001	11
18	0001 0010	12
31	0001 1111	1F
100	0110 0100	64
255	1111 1111	FF
256	0001 0000 0000	100

Figure: (Source: (Stallings, 2015))

Example

$$2C_{16} = ?_{10}$$

Example

$$
\begin{aligned}
2C_{16} &= (2_{16} \times 16^1) + (C_{16} \times 16^0) \\
&= (2_{10} \times 16^1) + (12_{10} \times 16^0) \\
&= 44
\end{aligned}
$$

This concludes this lession:

- Thank you for your time =)

References I

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