Real Estate and Real Options – A Case Study*

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Abstract

Real estate investments in emerging economies are characterized by low liquidity, slow payback and high sunk costs; enduring uncertainties about demand, price/m^2 and land costs. The introduction of the real options methodology in their analysis considers a housing development as an investment opportunity encompassing several options regarding information acquisition, deferral and abandonment.

The model proposed values these managerial flexibilities and shows improved risk management, identifying the optimal strategy (simultaneous vs. sequential) and timing for the construction phases. The maximum rent to pay for the exclusive rights on the land is also determined, a less capital intensive alternative to land ownership.

JEL classifications: G13; R20; D81

Keywords: Real Options; Real Estate; Sequential Investments; Risk Management

1 Introduction

Real estate developments in emerging economies present tight working capital, low liquidity, slow payback, capital-intensive outflows that are not immediately recovered, and short to medium construction times. For the long run, these investments are attracting the interest of a banking sector, searching for more attractive returns and the diversification if its portfolio.

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There are also several uncertainties related to demand, sale prices, land costs, unsold inventories, and regulatory and local government risks (authorizations, occupancy permits, etc.), which increase the investors' perceived risk. It is necessary to have good expertise of a constantly changing regulations on rent, taxes, project licenses, etc., which increases the administrative costs of projects[1]. Examples of buildings with their occupancy permits revoked even after being already issued are frequent in the sector.

Simultaneous and sequential investments are common in the real estate market. The first strategy is usually implemented during periods of increasing demand and implies lower construction costs but, in turn, carries more uncertain returns. Bitter experiences with residential housing developments and mega entertainment resorts that started simultaneously, have generated profits only after five or more years of construction.

On the other hand, sequential strategies face the risks in sequence, with relatively smaller increments at every phase of the project, but at the expense of higher construction costs. However, in order to take full advantage of the sequential strategy, real estate enterprises must own the land for future developments or possess the exclusive rights on the serviceable land[2] (a less capital intensive alternative).

Sequentiality of investment introduces several characteristics common in option pricing, i.e., decisions that can or cannot be exercised by the housing developer in the future. The most relevant *real options* found in this kind of projects are:

- Information Option. How the success/failure of the first construction phase (first launch) will affect the performance and expectations of the next development phases.
- Waiting Option. For the next phase of the construction if the market does not positively receive the previous launch.
- Abandonment Option. In case of high cost/benefit ratio.

Real option theory provides a methodology to better value investment projects in the presence of these managerial flexibilities. A detailed description of the different types and methodologies can be found in Dixit and Pindyck (1994) and Trigeorgis (1999). Also, Schwartz and Trigeorgis (2004) include classical readings where real options have been applied in several investment projects to account for the value of flexibility where traditional net present value (NPV) is unable to. Trigeorgis (1993) studied the interaction among several real options embedded in a single project,

[1]This is called regulatory risk, which is common in emerging economies.

[2]Exclusive rights are considered as the amount to pay to the landowner in order to assure the developer the exclusive availability of the lot to be used in the next phase of the construction during a certain period of time.

showing the non-additivity principle of their individual values.
Lander and Pinches (1998) identified the lack of mathematical skills, restrictive modeling assumptions, and increasing complexity as the main obstacles to the practical implementation of the real options approach.

Past failures and increasing uncertainties have also led real estate management to intuitively apply real options concepts. This essay develops a real options model for investment analysis in real estate that determines both the optimal investment strategy (simultaneous vs. sequential), by identifying the critical cost/m^2 whence there is no incentive for a housing development in stages, and the optimal investment timing. The methodology improves the risk management of the project and quantifies the maximum price to be paid for the exclusive rights on the serviceable land.

During the last years it has been common to see that developments are not constructed simultaneously at once but sequentially in several phases in order to reduce the risk exposure. Thus, Quigg (1993) has provided empirical evidence of the descriptive value of the option to wait based on actual real estate transactions in the US. Titman (1985) argued that the existence of several empty lots, but of high sale price, in West Los Angeles was due to the presence of an option to wait, i.e., the future potential of the lot was more valuable than its immediate use for construction. Capozza and Sick (1994) showed that agricultural landowners have the option to convert their property into urban land suitable for real estate developments, and the optimal conversion rule depends on the distance to urban areas. Williams (1991) studied the optimal timing for development and abandonment of the property as well as the optimal density in the presence of uncertainties about price/m^2 and cost/m^2. Capozza and Li (1994) focused on how the density and capital intensity choices interact with the timing and value of residential or commercial developments. Grenadier (1995) determined the intertemporal optimal mix of tenants in shopping centers, where the landlord has both options to increase or decrease the current mix with exercise prices being the cost of mix adjustments. He showed that the difference between the dynamic and static strategies was the value added by the embedded options. Finally, Grenadier (1996) introduced the option game concepts to explain the behavior of real estate markets, linking the investment timing in strategic equilibrium to the boost or slowdown in development activity.

Unlike the previous studies, where a stochastic price was obtained by using a demand function with stochastic shocks, demand is introduced here via *sales speed*, a typical variable in the housing sector that defines how fast the project's units will be sold. Thus, the combination of simulation in sales speed with the stochastic price modeling common in real options allows the introduction of several sources uncertainty without significantly increasing complexity (which inexorably appears in the case of two or more stochastic variables).

The article is organized as follows. In Section 2, the option pricing model for the evaluation of sequential investments under several uncertainties is presented. Section 3 applies the methodology to a real estate investment, showing the main results and providing guidance for optimal decision-making. Section 4 concludes and two Appendixes provide mathematical details for the optimal investment timing and development probability.

2 Investment Analysis in Real Estate and the Real Option Approach

Simultaneous investment corresponds to a static decision making process, i.e., a now-or-never decision where all irreversible resources are compromised at once. This strategy is presented in Fig. 1.

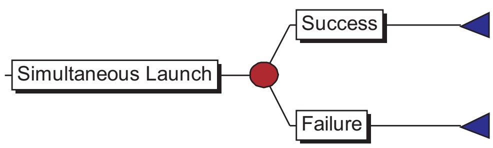


Fig. 1: Simultaneous (Static) Strategy

A residential housing development, however, is not usually built in a single phase but as a series of sequential decisions[3]. This is a way to diversify risk since uncertainty during the first launch is higher than in the later ones, and thus the corresponding market price is considerably lower in the initial phase than in subsequent ones. This corresponds to the sequential strategy, where risks are faced in sequence with relatively smaller increments at every phase of the project, but at the expense of relatively higher construction costs. In sequential strategy, the initial outflow is lower than in the simultaneous case and the expected inflows of previous phases may finance subsequent ones.

Before the first phase, the property developer analyzes diverse information relative to market potential, target consumer, revenue per-capita, price levels, empty lots, unsold units, etc. in order to maximize the project's value. But it is only after the first phase when the investor obtains relevant information about the housing investment, either for future expansions or development of the potential local market. Therefore, the first launch provides a valuable option to obtain market information about the future of the development. The cost to obtain this information is equivalent to the cost and risk borne during the first phase.

If the first launch has good market acceptance, the next construction stages will increase in value, leading to higher revenues and attractiveness

[3]For simplicity, a two phase project is consider here, though the framework can easily be generalized to investments with more than two phases.

for the region. In the opposite case, the developer will revaluate its prospects and wait for a better moment to continue with the next phases or, if necessary, abandon the project.

If the project is not well accepted in the initial stage, there is an option to defer the subsequent phases. This option is profitable only if its value exceeds the contractual cost of the exclusive rights to use the serviceable land during the time of the deferral, while always keeping the option to abandon alive. Fig. 2 shows the housing project incorporating the available options in each phase of the sequential decision-making process.

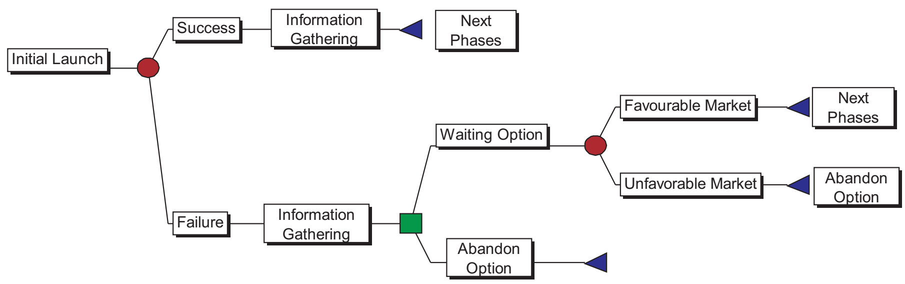


Fig. 2: Sequential (Dynamic) Strategy

Demand is modeled via sales speed, a typical variable that defines how fast the project's units will be sold. Uncertainty in demand is often dealt with sale speed scenarios (y^w):

- Before Groundbreaking, y_1^w
  $y_1^w \sim \text{Triang}[0; y_1; 100]\%$
- During Construction, y_2^w
  $y_2^w \sim \text{Triang}[0; (100 - y_1)/2; (100 - y_1)]\%$
- After Construction, y_3^w
  $y_3^w \sim \text{Triang}[0; (100 - y_1 - y_2); (100 - y_1 - y_2)]\%$

The sales speed in the different sale stages is approximated by triangular probability distributions in such a way that they sum up at most to 100 % sale at the end of construction. Fig. 3 shows the sales speed distributions for both phases, corresponding to the most common mean values of 43 % during launch, 26 % during construction and 20 % after construction, with a remaining 11 % of unsold units.

Let $S_1(P, \theta, y_1^w, y_2^w, y_3^w)$ and $S_2(P, \theta, y_1^w, y_2^w, y_3^w)$ be the present value of the revenues of the first and second phases, respectively. These are functions of the price/m^2 P, the sales speed iterations y^w, and the characteristics of the housing development θ (level of financing, amortization table and construction time). Let X_1 and X_2 be the present value of the corresponding construction costs.

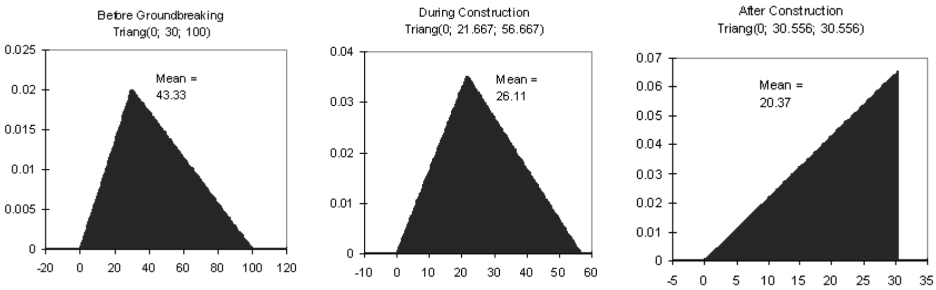


Fig. 3: Sales Speed, %

The net present value of the simultaneous strategy (NPV_1) is given by the sum of both the expected value of the first and second phases as presented in Eq. (1), having the same construction cost X_1 in both phases:

$$NPV_1 = \mathbb{E}\big[S_1(P_0, \theta, y^w) - X_1\big] + \mathbb{E}\big[S_2(P_0, \theta, y^w) - X_1\big] \qquad (1)$$

The net present value of the sequential strategy (NPV_2) is shown in Eq. (2), with the last term being the investment opportunity of the second phase (the option premium) that incorporates the information, waiting and abandon options described previously. Note that the value of this option is obtained numerically as the average of the N sales-speed iterations.

$$NPV_2 = \mathbb{E}\big[S_1(P_0, \theta, y^w) - X_1\big] + \frac{1}{N}\sum_{w=1}^{N} F\big[S_2(P_0, \theta, y^w), X_2, T\big] \qquad (2)$$

The difference between the two strategies is the value added by the option to the project. The sequential strategy becomes optimal if $NPV_2 > NPV_1$, i.e., if the following inequality holds:

$$\frac{1}{N}\sum_{w=1}^{N} F\big[S_2(P_0, \theta, y^w), X_2, T\big] > \mathbb{E}\big[S_2(P_0, \theta, y^w) - X_1\big] \qquad (3)$$

If the construction cost is the same for both strategies, the sequential investment would always have a higher or equal value than the simultaneous investment. The critical construction cost X_2^* over which the simultaneous strategy outperforms the sequential one can be obtained by numerically solving Eq. (3).

The function $F(\cdot)$ is the premium of an American call option[4] with expiration time T, underlying asset $S_2(\cdot)$, and exercise price X_2. Following the usual hypothesis in option pricing, we assume that the sale price/m^2

[4]Unlike the European type, an American option can be exercised at any time before maturity.

follows a Brownian Motion presented in Eq. (4), where dZ is the Wiener increment, σ the volatility, and μ the drift.

$$\frac{dP}{P} = \mu dt + \sigma dZ \tag{4}$$

Because of the proportionality between the revenues and the price/m^2, it is easy to prove by Itô's Lemma that the revenue $S_2(\cdot)$ also follows the same stochastic process in Eq. (4). The option premium can be obtained using the analytic approximation of Barone-Adesi and Whaley (1987) for the American call option, presented in Appendix A.

The analytic approximation also provides information regarding the optimal stopping threshold s_t^*, i.e., the critical revenue from where the investment in the second phase becomes optimal. The stopping threshold is obtained from the numerical solution of Eq. (5), which parameters are explained in Appendix A.

$$s_t^* - X_2 = \text{call}(s_t^*, t) + \left[1 - e^{-\delta(T-t)} N[d_1(s_t^*)]\right] \frac{s_t^*}{\gamma} \tag{5}$$

The proposed model is also useful for estimating the investment probability in the second phase, i.e., the probability of exercising the American call option $F(\cdot)$. Formally, the expression shown in Eq. (6) and discussed in Appendix B is the expected value of the first-hitting time distribution function for the different sales-speed iterations, i.e., the mean value of the probability that the stochastic process $S_2(\cdot)$ will reach the optimal investment curve before expiring.

$$\frac{1}{N} \sum_{w=1}^{N} \int_{t=0}^{T} \frac{\left| \ln\left(\frac{S_2(\cdot)}{s_t^*}\right) \right|}{\sigma\sqrt{2\pi t^3}} e^{-\frac{\left(\left|\ln\left(\frac{S_2(\cdot)}{s_t^*}\right)\right| + \left(\mu - \sigma^2/2\right)t\right)^2}{2\sigma^2 t}} dt \tag{6}$$

3 Case Study

The case study presented here consists in a two-phase residential housing in the West zone of Rio de Janeiro that can be constructed simultaneously or sequentially. New real estate investments in Rio are currently concentrated in this area due to the lack of serviceable land in other parts of the city and the increasingly high land prices in the South zone. The lack of space makes the estimation of the maximum price for the exclusive rights on the serviceable land (in the case of optimal sequential strategy) a quite important factor for the price negotiations and commercial viability of the real estate projects.

Due to the lack of financing and high interest rates, the exclusive rights alternative rather than a portfolio of proprietary lots has become increasingly popular. During the last years, however, more resources have been made available (World Bank (2005)) and the Brazilian government has recently (2004) introduced reforms intended to spur growth in the sector and increase access to housing loans[5].

In this context, the choice of the optimal density is not considered as an option since zoning regulations limit the maximum height of buildings in most developed regions. Moreover, regulation places further restrictions in Rio de Janeiro such as a minimum distance to the neighboring building, minimum distance to the beach in order to avoid shadow areas, etc. Therefore, the maximum height allowed is chosen in most cases because of the high price/m^2 and economies of scale.

The project's revenues depend on several parameters that are specific of the sector and the project, which will define the development's success or failure. The usual indicators used in the evaluation of a housing project, in present value, are presented in Appendix C.

If the first phase is a success, the second one is built immediately. Otherwise, the second phase can be deferred for a period of up to five years, which is equivalent to an American call option that the housing developer can exercise at any moment by paying the total construction cost.

Both phases of the project have an equivalent area of 20,736 m^2 and habitable area of 16,173 m^2. The selling price is USD 962/m^2 and the cost/m^2 is X_1 = USD 308 and X_2 = USD 338. The operational expenses are estimated at 15 % of the revenues and the land cost is agreed as an exchange for 30 % of the revenues and , both estimates considered usual in the sector[6].

The project is considered 100 % self-financing, with 6/24/1 months for the groundbreaking/construction/move-in respectively, 224 units and a credit to the buyer during the previous periods of: down payment (10/15/20 %); monthly installments during construction (30/25/0 %); monthly installments after moving in (60/60/80 %) of the value of the apartment.

Fig. 4 presents the average residential sale prices (adjusted to prices of USD January 2006[7]) in the West zone of Rio de Janeiro.

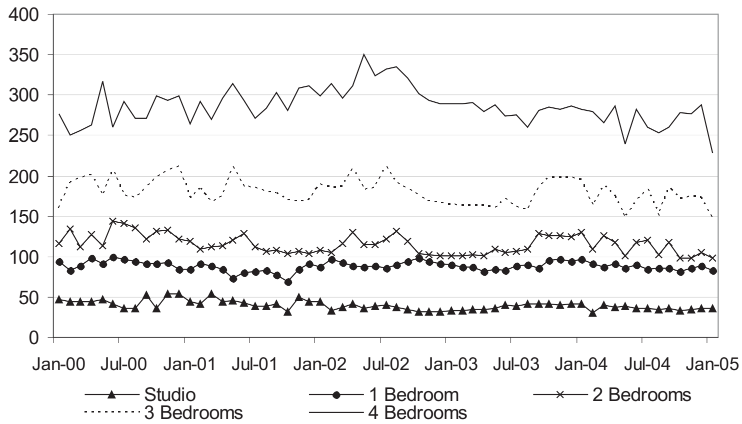
The parameters for the drift and volatility of the annual real returns are estimated from Fig. 4 via ordinary mean squares and presented in Table 1. We set the annual volatility at 15 % for the base case.

The opportunity cost or carrying cost of the option corresponds to the potential cash flows that would be generated by the investment if

[5]The tendency in Brazil is to pay cash for dwellings or to buy on relatively short-term financing from the developer rather than to get long-term mortgages, a consequence of the period of high inflation, general lack of funding for mortgages, and legal difficulties to foreclose.

[6]A common practice in real estate is to pay for the land as a percentage of the revenue or as an exchange for a certain number of units (apartments) of the condominium.

[7]Exchange rate: 1 USD = 2.6 BRL (average of month).



Source: SECOVI-RJ (2004)[8]

Fig. 4: Sale Prices in USD, thousand

Table 1: Annual Drift/Volatility of Price Returns

Studio	1 Bedroom	2 Bedrooms	3 Bedrooms	4 Bedrooms	Mean
-5.05 %	-2.56 %	-3.40 %	-1.69 %	-3.79 %	-3.30 %
14.62 %	13.05 %	17.50 %	16.39 %	13.39 %	15.00 %

implemented but lost by the holder of the option. This opportunity cost is equivalent to the estimated rental revenues the real estate would provide as a percentage of its value, which usually lies between 4 % and 12 % per annum. A rate of 10 % is selected for the base case. The weighted average cost of capital (WACC) of the property developer is 15 % per year and the Brazilian estimated risk free interest rate is 10 % per annum.

The optimal decision between the simultaneous and sequential strategies considering different construction costs for the second phase (X_2) is presented in Fig. 5. Notice that the sequential investment strategy is optimal below a cost/m^2 of USD 356. Also, the option premium equals the maximum payment to be made for the exclusive rights to the serviceable land.

Fig. 6 shows the payoff for immediate implementation of the second phase (optional). Considering the base case scenario where the construction cost is USD 5.46 million[9], the second phase should only be immediately implemented if the expected revenues are higher than USD 7.48 million. As it can be seen in Fig. 7, the value of USD 7.48 corresponds to the stopping

[8]Rio de Janeiro State Association of Real Estate Agents and Brokers.

[9]Present value of the construction cost flows when the cost/m^2 is USD 308.

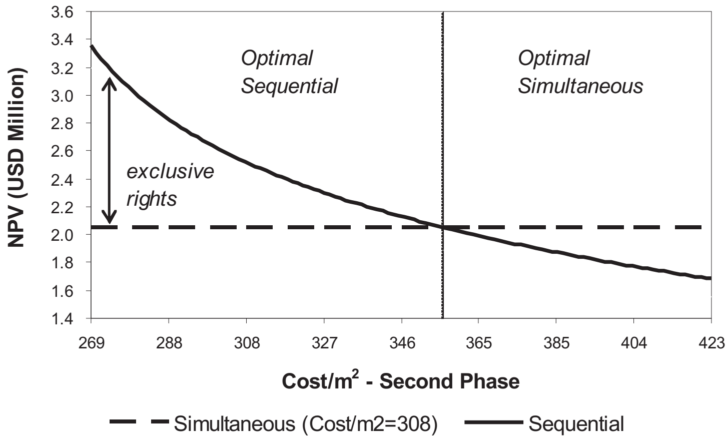


Fig. 5: Simultaneous vs. Sequential Strategies

curve when the time to expiration is five years. At this point the smooth-pasting and value-matching conditions apply and the value of the waiting option equals the value of the immediate exercise, leading to a NPV of USD 7.48 − 5.46 = 2.02 million. Note that the classic result of option theory holds, i.e., the option premium, calculated via Eq. (A–3) in Appendix A, is always higher or equal to the value of immediate exercise and, at the threshold point, both are equivalent and the NPV rule applies.

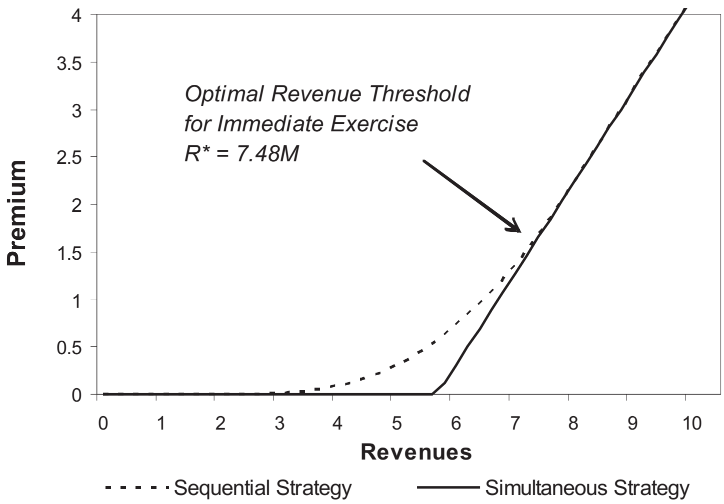


Fig. 6: Option Premium - Second Phase, USD million

Fig. 7 illustrates the optimal timing for developing the second phase, i.e., commit to invest when the expected revenue surpasses the optimal threshold.

The stopping curve is obtained by solving Eq. (5) with the parameters of the problem for every period of time until expiration, and remains unchanged for all sales speed simulations. The graph shows an example of two sales speed events of the simulation, $[y_1^w; y_2^w; y_3^w]$, which generate revenues of USD 5.46 (v_1) and USD 3.08 (v_2) millions. These values become the initial point for the real stochastic variation of the revenues.
Thus, in the case of $v_1 = [30\%; 30\%; 30\%]$, the second investment phase should be implemented since it hits the optimal stopping curve, while the stochastic paths of $v_2 = [20\%; 20\%; 5\%]$ never attain the required level to justify the investment.

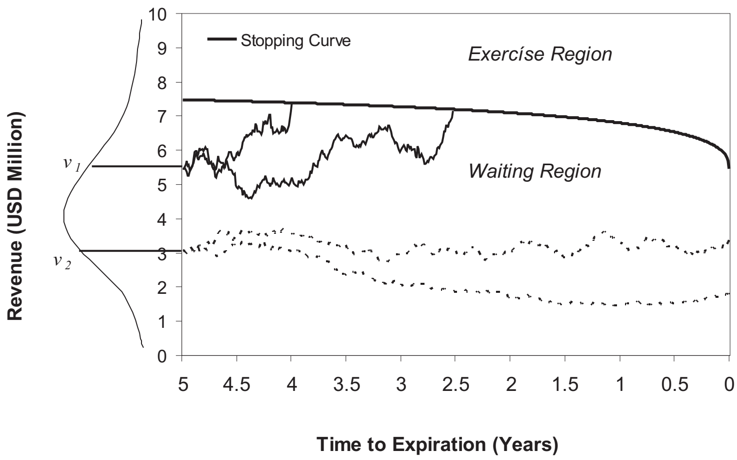


Fig. 7: Optimal Investment Strategy - Second Phase

In the event of success, if the price/m^2 (and thus the revenue) follows the first path, the option will be exercised one year after the conclusion of the first phase. In the second path the option is exercised only after two years and a half. In the event of failure, the third and fourth paths lead to abandonment of the option at the end of the fifth year because the revenue is not enough to justify the second phase of the project.

The proposed model can be used to estimate the probability of investing in the second phase (exercise of the option) by simply applying Eq. (6). Considering all possible runs of sales speed, the overall probability of exercising the option is 62 %.

Regarding the project risk management, Fig. 8 shows the NPV distribution considering the simultaneous strategy (static decision making), totaling USD 2.14 million with a 10.59 % probability of loss.

The NPV distribution of the housing development considering the sequential strategy (dynamic decision making) is shown in Fig. 9. The NPV rose to USD 2.36 millions, a 10 % increase, with a probability of loss of

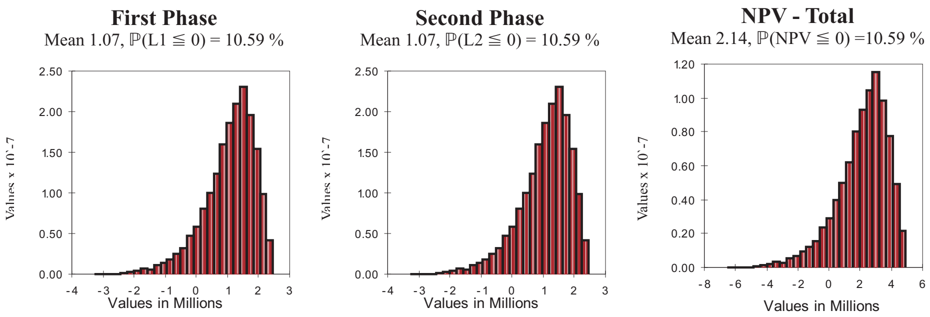


Fig. 8: NPV_1 Distribution - Simultaneous Strategy

3.59 %, a 66 % decrease. This is due to the characteristics of the option instrument as a risk management tool, since it is only exercised in the favorable scenarios (upside risk) while disregarding the unfavorable ones (downside risk).

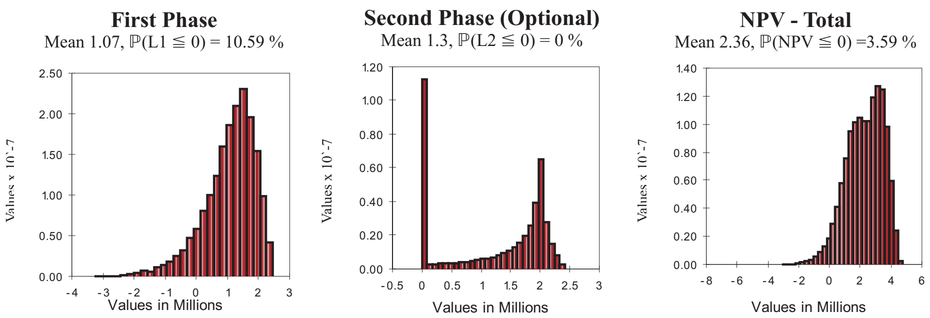


Fig. 9: NPV_2 Distribution - Sequential Strategy

The maximum payment for the exclusive rights in the serviceable land is the value added by the sequential investment strategy, i.e., the difference between the total NPV of the sequential and simultaneous strategies, which in this case equals USD 220,000. Therefore, if the landowner requests a higher amount, the property developer should exercise the option to abandon over the second phase.

4 Summary and Conclusions

Real estate investments are characterized as being capital intensive, low in liquidity and slow in payback while suffering from several uncertainties regarding demand, sale price/m^2, land cost, etc. that increase the risk perceived by investors. Several options such as information acquisition, deferral and abandonment of the project are usual in the sector that, if properly managed, may increase the value of the investment and reduce its risk exposure.

This study shows how the real options methodology can improve the economic analysis of real estate investments and support the decision-making process by managing the different options and uncertainties embedded in the project. The model developed here identifies the optimal strategy and timing for simultaneous or sequential investments, discusses issues related to risk management of the project and determines the maximum amount to be paid for the exclusive rights to the serviceable land.

The methodology is applied to a housing investment in Rio de Janeiro, where the sequential strategy increased the value of the project by 10 % while reducing the risk exposure by more than half compared to the traditional discounted cash flow methodology. These values cannot be neglected in a market that involves high sunk costs, high economic uncertainty and falling margins.

In practice, many real estate enterprises have intuitively already implemented the concept of options in their investment appraisals. However, it is important to establish a managerial culture in order to quantify the value of these options objectively, to enhance the value of the project and to provide effective management and risk assessment.

Appendix A Barone-Adesi & Whaley Analytical Approximation

Let S be the risk-neutral stochastic process of the underlying asset represented in Eq. (A–1), where dZ is the Wiener increment, r is the risk-free interest rate, δ the carrying costs (similar to a dividend rate) and σ the volatility.

$$\frac{dS}{S} = (r-\delta)dt + \sigma dZ \tag{A–1}$$

Then, the Black-Scholes-Merton formula, Black and Scholes (1973) and Merton (1973), for the calculation of the premium of a European call option, where $N(\cdot)$ is the cumulative normal distribution, S_0 the current value of the underlying asset, X the exercise price and T the time to maturity, is given by:

$$\begin{aligned} \text{call}(S_0,t) &= S_0 e^{-\delta(T-t)}N(d_1) - Xe^{-r(T-t)}N(d_2) \qquad \text{(A–2)} \\ d_1 &= \frac{\ln\left(S_0/X\right) + (r-\delta+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

The American call option premium is given by the following equation:

$$\text{Call}(S_0,t) = \begin{cases} \text{call}(S_0,t) + A\left(\frac{S_0}{s_t^*}\right)^\gamma & \text{if } S_0 < s_t^* \\ S_0 - X & \text{if } S_0 \geq s_t^* \end{cases} \tag{A–3}$$

whose parameters are defined below:

$$A = \left(\frac{s_t^*}{\gamma}\right)\left[1 - e^{-\delta t} N[d_1(s_t^*)]\right]; \;\; \gamma = \frac{-(\beta-1) + \sqrt{(\beta-1)^2 + \frac{4\nu}{h}}}{2} \tag{A–4}$$
$$\nu = \frac{2r}{\sigma^2}; \;\; \beta = \frac{2(r-\delta)}{\sigma^2}; \;\; h = 1 - e^{-r(T-t)}$$

and s_t^* is the solution of:

$$s_t^* - X_2 = \text{call}(s_t^*, t) + \left[1 - e^{-\delta(T-t)} N[d_1(s_t^*)]\right] \frac{s_t^*}{\gamma} \tag{A–5}$$

Appendix B Exercise Probability of an American Option

Let x be the stochastic process defined in Eq. (B–1), where μ is the real drift, σ is the volatility and dZ is the Wiener increment.

$$dx = \mu dt + \sigma dZ \tag{B–1}$$

According to Karatzas and Shreve (1991), the density of the first-hitting of x in t, i.e., $t = \inf\{t \geq 0, x(t) > 0\}$, is given by:

$$\psi[t|x_0, \mu, \sigma] = \frac{|x_0|}{\sigma\sqrt{2\pi t^3}} e^{-\frac{(x_0+\mu t)^2}{2\sigma^2 t}} \tag{B–2}$$

If S follows a geometric process with the same parameters as Eq. (B–1), it can be proved by Itô's Lemma that if $x = \ln(S/s_t^*)$, then x follows the following stochastic differential equation:

$$dx = \left(\mu - 0.5\sigma^2\right)dt + \sigma dZ \tag{B–3}$$

Therefore, the density of the first-hitting time of S in t, $t = \inf\{t \geq 0, S(t) > s_t^*\}$, is given by:

$$\varphi[t|S_0, \mu, \sigma, s_t^*] = \frac{\left|\ln\left(S_0/s_t^*\right)\right|}{\sigma\sqrt{2\pi t^3}} e^{-\frac{\left(\ln\left(S_0/s_t^*\right) + \left(\mu - \sigma^2/2\right)t\right)^2}{2\sigma^2 t}} dt \tag{B–4}$$

The probability to exercise an American option, i.e., the probability that variable S crosses the threshold s_t^* (defined in Appendix A) at any time before maturity is given by:

$$\int_{t=0}^{T} \frac{\left|\ln\left(S_0/s_t^*\right)\right|}{\sigma\sqrt{2\pi t^3}} e^{-\frac{\left(\left|\ln\left(S_0/s_t^*\right)\right| + \left(\mu - \sigma^2/2\right)t\right)^2}{2\sigma^2 t}} dt \tag{B–5}$$

Appendix C Static Indicators in Real Estate

The usual indicators applied in the evaluation of a housing project, in present value, are presented in Table C–1.

Table C–1: Static Ratios of Success

Land Cost / Revenue	$\leq$ 35%
Construction Cost / Revenue	$\leq$ 50%
(Revenue – Total Expenses) / Revenue	$\geq$ 20%
Net Income / Land Cost	$\geq$ 80%
Net Income / Total Expenses	$\geq$ 15%
Habitable Area / Equivalent Area	$\geq$ 60%

The different concepts included in Table C–1 are defined as:

- Equivalent Area. Constructed area used to calculate the construction cost.
- Habitable Area. Area for sale, corresponding to the square meters of floor space that the developer is selling.
- Revenues (General Sales Value, GSV). The present value of the revenue flows as the Habitable Area multiplied by the sale price/m^2, calculated from the cash flows according to the sales speed and the amortization table (*price* method). The amortization table assumes that customers will buy with loans, and different amortization conditions apply for the different sale stages.
- Construction Cost. The present value of the total cost of construction, i.e., the equivalent area multiplied by the construction cost/m^2 considering the time to build. An administration fee for the construction company and expenses related to the architecture project are also considered.
- Land Cost. The land can be bought in cash, through a loan, exchange contract for other realty, exchange contract for units in the completed project or other buildings, and by exchange contract on the GSV. To the base value, which is the taxable price of the land, we have to add some inherent expenses such as taxes (transfer and property taxes), commissions, demolition and infrastructure expenses, etc.
- Total Expenses. They are the construction costs plus the rest of the expenses (commissions, taxes, marketing campaigns, legal expenses, etc.).
- Net Income. Net revenue after taxes, i.e., the GSV – Total Expenses, income taxes already deducted.

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