integral. Combining the hidden conditions with (*1*), (*2*), and (*3*), we obtain the mathematical program

$$\begin{aligned} \text{minimize:} \quad & z = 20x_1 + 22x_2 + 18x_3 \\ \text{subject to:} \quad & 4x_1 + 6x_2 + x_3 \geq 54 \\ & 4x_1 + 4x_2 + 6x_3 \geq 65 \\ & x_1 \leq 7 \\ & x_2 \leq 7 \\ & x_3 \leq 7 \\ \text{with:} \quad & \text{all variables nonnegative and integral} \end{aligned} \tag{4}$$

System (*4*) is an integer program; its solution is determined in Problem 7.4.

1.6 A manufacturer is beginning the last week of production of four different models of wooden television consoles, labeled I, II, III, and IV, each of which must be assembled and then decorated. The models require 4, 5, 3, and 5 h, respectively, for assembling and 2, 1.5, 3, and 3 h, respectively, for decorating. The profits on the models are \$7, \$7, \$6, and \$9, respectively. The manufacturer has 30 000 h available for assembling these products (750 assemblers working 40 h/wk) and 20 000 h available for decorating (500 decorators working 40 h/wk). How many of each model should the manufacturer produce during this last week to maximize profit? Assume that all units made can be sold.

The objective is to maximize profit (in dollars), which we denote as z. Setting

$x_1 \equiv$ number of model I consoles to be produced in the week
$x_2 \equiv$ number of model II consoles to be produced in the week
$x_3 \equiv$ number of model III consoles to be produced in the week
$x_4 \equiv$ number of model IV consoles to be produced in the week

we can formulate the objective as

$$\text{maximize:} \quad z = 7x_1 + 7x_2 + 6x_3 + 9x_4 \tag{1}$$

There are constraints on the total time available for assembling and the total time available for decorating. These are, respectively, modeled by

$$4x_1 + 5x_2 + 3x_3 + 5x_4 \leq 30\,000 \tag{2}$$

$$2x_1 + 1.5x_2 + 3x_3 + 3x_4 \leq 20\,000 \tag{3}$$

As negative quantities may not be produced, four hidden constraints are $x_i \geq 0$ $(i = 1, 2, 3, 4)$. Additionally, since this is the last week of production, partially completed models at the week's end would remain unfinished and so would generate no profit. To avoid such possibilities, we require an integral value for each variable. Combining the hidden conditions with (*1*), (*2*), and (*3*), we obtain the mathematical program

$$\begin{aligned} \text{maximize:} \quad & z = 7x_1 + 7x_2 + 6x_3 + 9x_4 \\ \text{subject to:} \quad & 4x_1 + 5x_2 + 3x_3 + 5x_4 \leq 30\,000 \\ & 2x_1 + 1.5x_2 + 3x_3 + 3x_4 \leq 20\,000 \\ \text{with:} \quad & \text{all variables nonnegative and integral} \end{aligned} \tag{4}$$

System (*4*) is an integer program; its solution is determined in Problem 6.4.

1.7 The Aztec Refining Company produces two types of unleaded gasoline, regular and premium, which it sells to its chain of service stations for \$12 and \$14 per barrel, respectively. Both types are blended from Aztec's inventory of refined domestic oil and refined foreign oil, and must meet the following specifications:

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One always branches from that program which appears most nearly optimal. When there are a number of candidates for further branching, one chooses that having the largest z-value, if the objective function is to be maximized, or that having the smallest z-value, if the objective function is to be minimized.

Additional constraints are added one at a time. If a first approximation involves more than one nonintegral variable, the new constraints are imposed on that variable which is furthest from being an integer; i.e., that variable whose fractional part is closest to 0.5. In case of a tie, the solver arbitrarily chooses one of the variables.

Finally, it is possible for an integer program or an associated linear program to have more than one optimal solution. In both cases, we adhere to the convention adopted in Chapter 1, arbitrarily designating one of the solutions as the optimal one and disregarding the rest.

Solved Problems

6.1 Draw a schematic diagram (tree) depicting the results of Examples 6.1 through 6.3.

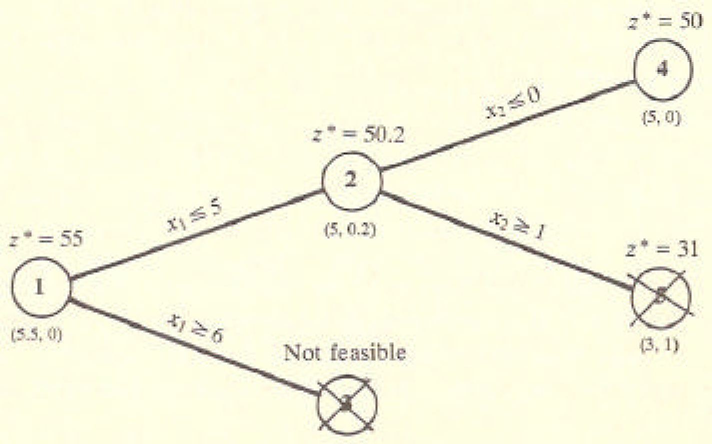

Fig. 6-1

See Fig. 6-1. The original integer program, here *(6.1)*, is designated by a circled 1, and all other programs formed through branching are designated in the order of their creation by circled successive integers. Thus, programs *(6.2)* through *(6.5)* are designated by circled 2 through 5, respectively. The first approximate solution to each program is written by the circle designating the program. Each circle (program) is then connected by a line to that circle (program) which generated it via the branching process. The new constraint that defined the branch is written above the line. Finally, a large cross is drawn through a circle if the corresponding program has been eliminated from further consideration. Hence, branch 3 was eliminated because it was not feasible; branch 5 was eliminated by bounding in Example 6.3. Since there are no nonintegral branches left to consider, the schematic diagram indicates that program 1 is solved with $x_1^* = 5$, $x_2^* = 0$, and $z^* = 50$.

6.2

$$\begin{aligned} \text{maximize:} \quad & z = 3x_1 + 4x_2 \\ \text{subject to:} \quad & 2x_1 + x_2 \le 6 \\ & 2x_1 + 3x_2 \le 9 \\ \text{with:} \quad & x_1, x_2 \text{ nonnegative and integral} \end{aligned}$$

Neglecting the integer requirement, we obtain $x_1^* = 2.25$, $x_2^* = 1.5$, with $z^* = 12.75$, as the solution to the associated linear program. Since x_2^* is further from an integral value than x_1^*, we use it to generate the branches $x_2 \leq 1$ and $x_2 \geq 2$.

Program 2

maximize: $z = 3x_1 + 4x_2$

subject to: $2x_1 + x_2 \leq 6$

$2x_1 + 3x_2 \leq 9$

$x_2 \leq 1$

with: x_1, x_2 nonnegative and integral

Program 3

maximize: $z = 3x_1 + 4x_2$

subject to: $2x_1 + x_2 \leq 6$

$2x_1 + 3x_2 \leq 9$

$x_2 \geq 2$

with: x_1, x_2 nonnegative and integral

The first approximation to Program 2 is $x_1^* = 2.5$, $x_2^* = 1$, with $z^* = 11.5$; the first approximation to Program 3 is $x_1^* = 1.5$, $x_2^* = 2$, with $z^* = 12.5$. These results are shown in Fig. 6-2. Since Programs 2 and 3 both have nonintegral first approximations, we could branch from either one; we choose Program 3 because it has the larger (more nearly optimal) value of the objective function. Here $1 < x_1^* < 2$, so the new programs are

Program 4

maximize: $z = 3x_1 + 4x_2$

subject to: $2x_1 + x_2 \leq 6$

$2x_1 + 3x_2 \leq 9$

$x_2 \geq 2$

$x_1 \leq 1$

with: x_1, x_2 nonnegative and integral

Program 5

maximize: $z = 3x_1 + 4x_2$

subject to: $2x_1 + x_2 \leq 6$

$2x_1 + 3x_2 \leq 9$

$x_2 \geq 2$

$x_1 \geq 2$

with: x_1, x_2 nonnegative and integral

There is no solution to Program 5 (it is infeasible), while the solution to Program 4 with the integer constraints ignored is $x_1^* = 1$, $x_2^* = 7/3$, with $z^* = 12.33$. See Fig. 6-2. The branching can continue from either Program 2 or Program 4; we choose Program 4 since it has the greater z-value.

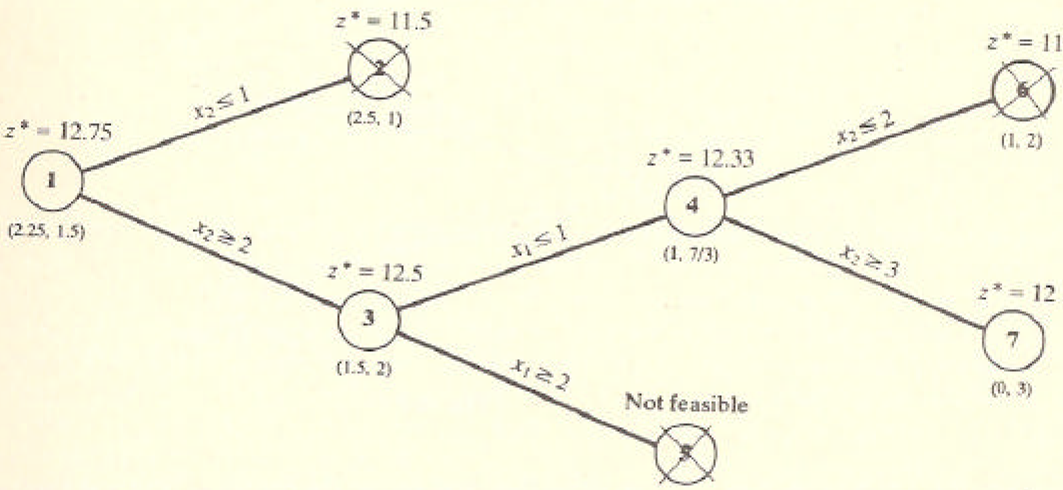

Fig. 6-2

Here $2 < x_2^* < 3$, so the new programs are

Program 6

$$\begin{aligned} \text{maximize:}\quad & z = 3x_1 + 4x_2 \\ \text{subject to:}\quad & 2x_1 + x_2 \le 6 \\ & 2x_1 + 3x_2 \le 9 \\ & x_2 \ge 2 \\ & x_1 \le 1 \\ & x_2 \le 2 \\ \text{with:}\quad & x_1, x_2 \text{ nonnegative and integral} \end{aligned}$$

Program 7

$$\begin{aligned} \text{maximize:}\quad & z = 3x_1 + 4x_2 \\ \text{subject to:}\quad & 2x_1 + x_2 \le 6 \\ & 2x_1 + 3x_2 \le 9 \\ & x_2 \ge 2 \\ & x_1 \le 1 \\ & x_2 \ge 3 \\ \text{with:}\quad & x_1, x_2 \text{ nonnegative and integral} \end{aligned}$$

The solution to Program 6 with the integer constraints ignored is $x_1^* = 1$, $x_2^* = 2$, with $z^* = 11$. Since this is an integral solution, $z = 11$ becomes a lower bound for the problem; any program yielding a z-value smaller than 11 will henceforth be eliminated. The first approximation to Problem 7 is $x_1^* = 0$, $x_2^* = 3$, with $z^* = 12$. Since this is an integral solution with a z-value greater than the current lower bound, $z = 12$ becomes the new lower bound, and the program that generated the old lower bound, Program 6, is eliminated from further consideration, as is Program 2. Figure 6-2 now shows no branches left to consider other than the one corresponding to the current lower bound. Consequently, this branch gives the optimal solution to Program 1: $x_1^* = 0$, $x_2^* = 3$, with $z^* = 12$.

6.3 Solve Problem 1.9.

Dropping the integer requirements from program (*1*) of Problem 1.9, we solve the associated linear program first, to find (see Problem 5.4): $x_1^* = 2$, $x_2^* = 18$, $x_3^* = 0$, $x_4^* = 20$, $x_5^* = 0$, $x_6^* = 5$, with $z^* = 45$. This is the first approximation. Since it is integral, however, it is also the optimal solution to the original integer program.

6.4 Solve Problem 1.6.

Ignoring the integer requirements in program (*4*) of Problem 1.6, we obtain $x_1^* = x_2^* = 0$, $x_3^* = 1666.67$, $x_4^* = 5000$, with $z^* = 55\,000$, as the first approximation. Since x_3^* is not integral, we branch to two new programs, and solve each with the integer constraints ignored. The results are indicated in Fig. 6-3. Program 3 possesses an integral solution with a z-value greater than the z-value of Program 2. Consequently, we eliminate Program 2 and accept the solution to Program 3 as the optimal one: $x_1^* = 1$, $x_2^* = 0$, $x_3^* = 1667$, $x_4^* = 4999$, with $z^* = \$55\,000$.

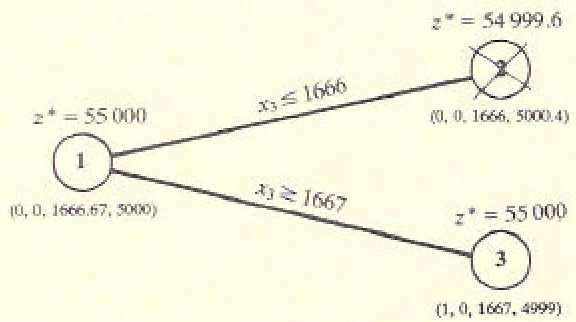

Fig. 6-3

6.5 Discuss the errors involved in rounding the first approximations to the original programs in Problems 6.2 and 6.4 to integers and then taking these answers as the optimal ones.