Identifying fractional models from frequency and time responses

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Abstract: This paper addresses methods to identify a fractional model from a frequency response or from a time response. An existing method for identifying an integer model from frequency data, developed to be used with second-generation Crone controllers, is adapted to identify fractional order plants. An example is given showing how this method can also be used when a time response is available.

Keywords: Fractional control; Identification algorithm; Discrete-time systems; Parameter identification; Phase response.

1. INTRODUCTION

Data from which we may have to identify a model may be of two types: a frequency response or a time response. Several well-established methods exist for the case when the model desired is a differential equation employing integer orders only. Fewer exist for the case of commensurate fractional orders\(^1\). This paper addresses the latter problem. Section 2 concerns the problem of identification from a frequency response. Possible methods are listed; an existing method for the integer case, developed to be used with second-generation Crone controllers, is adapted to the fractional case, and an example comparing its performance with Levy’s method is given. Section 3 concerns the problem of identification from a time response. Possible methods are mentioned; an example is given showing how the method developed for frequency data can also be used when a time response is available.

2. USE OF A FREQUENCY RESPONSE

There are three main ways to identify a fractional model from a frequency response.

The first is to find zeros and poles from the asymptotic behaviour of the Bode diagram. This can be easily done (both in the integer and fractional cases) when the data corresponds to a low order model.

The second is to use Levy’s method, or one of its variations. Reviews of how this method can be applied to the fractional case are given by Valerio and Sá da Costa (2007); Valério et al. (2008). These references include the use of weights, the iterative approach of Sanathanan and Koerner (developed by Santhanathan and Koerner (1963) for the integer case), the iterative approach of Lawrence and Rogers (developed by Lawrence and Rogers (1979) for the integer case), and the use of summed or stacked matrices. This method requires fixing in advance the orders of the numerator, denominator, and commensurability. Because of this, it may be useful to begin with a visual inspection of the Bode diagram, to estimate what orders are reasonable, and try several combinations around the most likely values.

The third is the method described below. Its original version, for the integer case, was developed to be used when conceiving second-generation Crone controllers, and is given by Oustaloup (1991); Oustaloup et al. (2002). It is based upon the trigonometric formula of Lemma 1, in Appendix A.

2.1 Original (integer case) identification model

We know the frequency behaviour of a plant \(G(j\omega)\) at \(f\) frequencies, \(\omega_i, i = 1 \ldots f\). Thus the gain is \(g_i = |G(j\omega_i)|\) and the phase is \(\phi_i = \text{arg}[G(j\omega_i)]\). We want to fit to this data an integer order frequency-domain model:

\[
\hat{G}(s) = G_0 \prod_{k=1}^{n} \frac{1 + \frac{s}{b_k}}{1 + \frac{s}{a_k}}, \quad G_0 > 0, \quad m + n = M \tag{1}
\]

The phase of its frequency behaviour, at each frequency \(\omega_1\), is

\[
\text{arg} \hat{G}(j\omega_1) = \text{arg} \left( \prod_{k=1}^{n} \frac{1 + \frac{j\omega_1}{b_k}}{1 + \frac{j\omega_1}{a_k}} \right) = \sum_{k=1}^{m} \text{arctan} \frac{\omega_1}{b_k} - \sum_{k=1}^{n} \text{arctan} \frac{\omega_1}{a_k} \tag{2}
\]
= - \sum_{k=1}^{M} \arctan \frac{\omega_i}{c_k}

\begin{align*}
c_k &= \begin{cases} 
a_k & \text{if } k \leq n \\
-b_k & \text{if } k \geq n + 1
\end{cases}
\end{align*}

In other words, we can replace all zeros of \( \hat{G} \) by poles with a symmetric real part, and still obtain the same frequency behaviour. (Notice that (3) is valid not only if all the \( c_k \) are real, but also if there are pairs of complex conjugates. See Appendix B for details.) Since we want

\[ \arg \hat{G}(j\omega_i) = \phi_i, \quad i = 1 \ldots f \]

we make

\[ y_{i,k} = \frac{\omega_i}{c_k} \quad (4) \]

and apply Lemma 1 to obtain

\[ \tan \sum_{k=1}^{M} \arctan y_{i,k} = -\tan \phi_i \Leftrightarrow \]

\[ \frac{1}{1 + \sum_{k \in \mathcal{M}} (-1)^{[k/2]} S_{i,k}} = -\tan \phi_i \Leftrightarrow \]

\[ \sum_{k=1}^{M} (-1)^{[k/2]} \Phi_{i,k} S_{i,k} = -\Phi_{i,0}, \quad i = 1 \ldots f \quad (5) \]

where

\[ \Phi_{i,k} = \begin{cases} \tan \phi_i, & \text{if } k \text{ is even, } \forall i \\ 1, & \text{if } k \text{ is odd, } \forall i \end{cases} \quad (6) \]

We have now transformed (3), a non-linear set of equations, into a linear set of equations, but each of the \( f \) equations in (5) has a different set of variables. Fortunately, all equations can be rewritten using only one set of variables, say, those in the first equation (\( S_{1,k}, \quad k = 1 \ldots M \)), because

\[ y_{i,k} = \frac{\omega_i}{c_k} = \frac{\omega_1 \omega_i}{c_k \omega_1} = y_{1,k} A_i \quad (7) \]

\[ A_i = \frac{\omega_i}{\omega_1} \quad i = 1 \ldots f \]

and thus

\[ S_{1,1} = S_{1,1} A_1 \]

\[ S_{1,2} = S_{1,2} A_1^2 \]

\[ \vdots \]

\[ S_{1,k} = S_{1,k} A_1^k \quad (8) \]

Hence (5) becomes

\[ \sum_{k=1}^{M} (-1)^{[k/2]} \Phi_{1,k} S_{1,k} A_1^k = -\Phi_{1,0}, \quad i = 1 \ldots f \quad (9) \]

Once this system is solved, the values of \( y_{1,k}, \quad k = 1 \ldots M \) are found as the \( M \) roots of the polynomial equation

\[ y_1^M + \sum_{k=1}^{M} (-1)^k y_1^{M-k} S_{1,k} = 0 \quad (10) \]

and finally the values of \( c_k, \quad k = 1 \ldots M \) are found from (4). Actually each \( c_k \) can correspond to a zero or a pole (we do not know in which order they were found), so what we have is a set of \( 2^M \) possible transfer functions. The gains of each of them must be found and the combination of zeros and poles that leads to gains closer to \( g_i \) is chosen.

When developing a second-generation Crone controller, all we know is the desired phase behaviour of the controller. So each \( c_k \) will be made into a stable pole or a minimum phase zero, as the case may be. No data on gain is necessary.

### 2.2 Adaptation for discrete transfer functions

This identification method can be adapted to the case where \( \hat{G}(z^{-1}) \) is a discrete transfer function (Valério and Sá da Costa, 2004), with an important difference, however. It is impossible to replace zeros by poles with the same effect in the phase; so, if there are only poles or if there are only zeros, it is possible to proceed in a manner similar to that above (albeit the expression corresponding to (8) becomes much more complicated), but, if there are both poles and zeros, the non-linear system (3) must be solved numerically, because there is no expression corresponding to (8) (that is to say, variables \( S_{i,k} \), corresponding to different frequencies, cannot be rewritten as linear combinations of the \( S_{1,k} \) or of the variables of any other single frequency).

### 2.3 Adaptation for fractional transfer functions

This identification method can also be adapted to the case where \( \hat{G}(s) \) is a fractional transfer function, with a commensurate order \( \alpha \). This adaptation will have much in common with that for the discrete case. We will have

\[ \hat{G}(s) = G_0 \prod_{k=1}^{m} 1 + \frac{s^\alpha}{b_k} \quad G_0 > 0, \quad m + n = M \quad (11) \]

\[ \arg \hat{G}(j\omega_i) = \frac{m}{k=1} \frac{1 + (j\omega_i)^\alpha}{b_k} = \frac{\prod_{k=1}^{m} 1 + \frac{(j\omega_i)^\alpha}{a_k}}{b_k} \]

\[ = \sum_{k=1}^{m} \arctan \frac{\omega_0 \sin \frac{\omega_0}{2}}{b_k + \omega_0 \cos \frac{\omega_0}{2}} - \sum_{k=1}^{n} \arctan \frac{\omega_0 \sin \frac{\omega_0}{2}}{a_k + \omega_0 \cos \frac{\omega_0}{2}} \quad (12) \]

and it is not possible to replace zeros by poles of the opposite sign, keeping the phase unchanged. (Again, (12) is valid even if there are pairs of complex conjugate zeros or poles; see Appendix B.)

### 2.4 Fractional transfer function without zeros

Suppose that there are no zeros (\( M = n, \quad m = 0 \)). We can make
and we want that

\[ y_{i,k} = \frac{a_k + \omega_i^\alpha \cos \frac{\pi \alpha}{2}}{\omega_i^\alpha \sin \frac{\pi \alpha}{2}} = a_k \frac{1}{\omega_i^\alpha \sin \frac{\pi \alpha}{2}} + \cot \frac{\pi \alpha}{2} \]  

(13)

Consequently, (14) becomes

\[ \arg \hat{G}(j\omega_i) = \phi_i \Leftrightarrow \]

\[ \sum_{k=1}^{M} - \arctan \frac{1}{y_{i,k}} = \phi_i \Leftrightarrow \]

\[ \sum_{k=1}^{M} \frac{\pi}{2} - \arctan y_{i,k} = -\phi_i \Leftrightarrow \]

\[ \tan \sum_{k=1}^{M} \arctan y_{i,k} = \tan \left( \frac{M \pi}{2} + \phi_i \right) \Leftrightarrow \]

\[ \sum_{k \in \Phi_{i}} (-1)^{\lfloor k/2 \rfloor} S_{i,k} = -\tan \left( \frac{M \pi}{2} - \phi_i \right) \Leftrightarrow \]

\[ \sum_{k=1}^{M} (-1)^{\lfloor k/2 \rfloor} \Phi_{i,k} S_{i,k} = -\Phi_{i,0}, \quad i = 1 \ldots f \]  

(14)

where now we define

\[ \Phi_{i,k} = \begin{cases} \tan \left( -\frac{M \pi}{2} - \phi_i \right), & \text{if } k \text{ is even, } \forall i \\ 1, & \text{if } k \text{ is odd, } \forall i \end{cases} \]  

(15)

Notice that save for the definition of \( \Phi \) we have a system of equations equal to (5). To rewrite all equations using only one set of variables, we notice that

\[ y_{i,k} = y_{1,k} A_i + B_i, \]

\[ A_i = \left( \frac{\omega_i}{\omega_1} \right)^\alpha, \quad B_i = \left[ 1 - \left( \frac{\omega_i}{\omega_1} \right)^\alpha \right] \cot \frac{\pi \alpha}{2}, \quad i = 1 \ldots f \]

and thus

\[ S_{i,1} = S_{1,1} A_i + M B_i, \]

\[ S_{i,2} = S_{1,2} A_i^2 + (M - 1) S_{1,1} A_i B_i + B_i^2 \sum_{i=1}^{M-1} i \]

\[ S_{i,3} = S_{1,3} A_i^3 + (M - 2) S_{1,2} A_i^2 B_i + S_{1,1} A_i B_i^2 \sum_{i=1}^{M-2} i + B_i^3 \sum_{j=1}^{M-2} \sum_{i=1}^{j} i \]  

(17)

\[ \vdots \]

\[ S_{i,k} = \sum_{r=0}^{k} \pi_{k-r-1}^{M-k+1} S_{1,1} A_i^r B_i^{k-r} \]

where \( S_{1,0} = 1 \) and

\[ \pi_0 = 1 \]

\[ \pi_0 = a \]

\[ \pi_1 = a \sum_{i=1}^{a} i \]

\[ \pi_2 = a \sum_{j=1}^{a} \sum_{i=1}^{j} i \]  

(18)

Otherwise the algorithm is exactly the same.

2.6 Fractional transfer function with both poles and zeros

Because fractional zeros cannot be replaced by fractional poles while leaving the phase unchanged, we are in the same situation of a discrete transfer function: (12) must be equal to \( \phi_i \), for every \( i = 1 \ldots f \), but this non-linear system of equations must be solved numerically. This is because there is no way to find relations similar to (8) or (17).

2.7 Comments

In the fractional case, this identification method can be applied, just as the original integer version was, when there is only data on the phase behaviour. Its reliance on phase data requires that it be reliable, free from noise as much as possible. When a model with both fractional zeros and poles is needed, the need for a numerical resolution makes this method computationally slower and less reliable. With the original version for integer models, it was not necessary to choose in advance the orders of the numerator and denominator separately; only its sum \( M \). Now, for fractional orders, we are in the same situation of Levy’s method, and it is necessary to choose in advance the orders of the numerator, denominator, and commensurability. So comments offered above on how that might be reasonably done also apply here.
2.8 Example

An example of fractional identification from a frequency response will be shown where this method performs better than Levy’s. Let the frequency response of

\[ G(s) = \frac{3}{s^\alpha (s^\alpha - 1 + j)(s^\alpha - 1 - j)} = \frac{3}{s - 2s^{2\alpha} + 2s^\alpha} \]  

(24)

be sampled at \( \omega = \{ .001, .01, 1, 11, 100, 1000 \} \) rad/s. We make \( \alpha = \frac{1}{6}, m = 0, n = 6 \), and identify a transfer function. The results are

\[ \hat{G}_L(s) = \frac{13.04}{s^\frac{2}{3} - 4.066s^{\frac{2}{3}} + 27.33s^{\frac{5}{6}} - 52.23s^{\frac{1}{2}} + 35.68s^{\frac{1}{2}} - 5.448s^{\frac{1}{2}} + 2.102s^{\frac{1}{2}}} \]  

(25)

for Levy’s method and

\[ \hat{G}_C(s) = \frac{3}{s - 2s^{2\alpha} + 2s^\alpha} \]  

(26)

for the method of section 2 (see Figure 1, top). Of course, to obtain this result with Levy’s method, and since it is well known that it does not cope with poles (or zeros) at the origin, it was necessary to algebraically eliminate the effect of \( \frac{1}{s^{\alpha}} \). In spite of this additional step, the result is incorrect. We might think this is because what is being identified, after the correction mentioned, is actually a transfer function with a \( \frac{1}{s^{\alpha}} \)-order denominator. But not even making \( n = 5 \) (avoiding the extra useless fractional pole) makes things better, for then we get

\[ \hat{G}_L(s) = \frac{-1.079}{s - 8.593s^{\frac{1}{2}} + 20.53s^{\frac{1}{2}} - 20.87s^{\frac{1}{2}} + 9.256s^{\frac{1}{2}} - 1.699s^{\frac{1}{2}}} \]  

(27)

(see Figure 1, bottom).

Generally speaking, it is not always clear which of those two frequency response identification methods performs better; there are other cases in which Levy’s gets a better answer. As is usually the case, having two different tools allows checking one’s results with the other; if one fails, it may happen that the other does not.

3. USE OF A TIME RESPONSE

Very often the data available to identify a model is a time response\(^2\). A review of methods available when a fractional model is desired is given by Malti et al. (2008). Both linear and nonlinear methods exist; with the latter, it is possible not to fix in advance the differentiation orders of the model, but rather to identify them together with the other parameters.

We can also, however, follow the steps normally employed with integer models:

1. if the response is not an impulse response, perform a deconvolution to obtain, from the response and from the input signal, the system’s impulse response;
2. identify a discrete transfer function from the impulse response;
3. convert the discrete transfer function into a frequency-domain transfer function.

Steps 1 and 2 are often performed together, as when prediction error methods are identified.

The identification of fractional models from time responses is closely related to another issue, that of finding discrete approximations of fractional derivatives. Unfortunately, there is no closed-form \( \mathcal{Z} \)-transform of a fractional derivative, and thus approximations consist of truncated series or truncated continued fractions, requiring a great number of terms to ensure some accuracy, or having low accuracies for lower numbers of terms (Vinagre et al., 2000; Chen and Vinagre, 2003; Valério and Sá da Costa, 2005). A consequence of this is that the impulse response obtained with task 1 must be known for a great number of instants, significantly more than would be needed for an integer order model of similar complexity. (This makes sense because fractional derivatives are operators with memory.) To embody such a response, the discrete transfer function found in step 2 must have a great number of zeros and poles (again more than would be needed for an integer model). Finally, task 3 is very difficult to perform. Approximations without more than one delay can be inverted; for instance, a Grünewald-Letnikoff approximation gives

\[ s^\alpha = \frac{1}{T_{\alpha}} - \frac{\alpha}{T_{\alpha}} z^{-1} + \ldots \Rightarrow z^{-1} \approx \frac{1}{\alpha} - \frac{T_{\alpha}^{\alpha} \alpha}{\alpha} \]  

(28)

but a) since approximations with few delays are poor, their inverses must necessarily be poor as well; and b) since the discrete model obtained in step 2 must have a very high order, this type of inversion will lead to a fractional model with very high orders too, in all probability wholly unnecessary.

\(^2\) All time responses in this paper are assumed to be sampled with a constant sampling time \( T_s \).
A way out is to perform a frequency-domain identification. A frequency response may be obtained directly after step 1, convolving the impulse response with sinusoidal signals, or after identifying the discrete transfer function in step 2. In any case, a method from section 2 is then used to obtain the desired fractional model.

### 3.1 Example

We know that (Podlubny, 1999)

$$\mathcal{L}(E_{\alpha,1}(-\lambda^t)) = \frac{s^{\alpha-1}}{s^\alpha + \lambda}$$

where $E$ is the Mittag-Leffler function:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

Suppose we make $\alpha = \frac{1}{2}$, $\lambda = \frac{1}{2}$, $T_s = 0.01$, and evaluate $E_{\frac{1}{2},1}(-\frac{1}{2}t)$ up to $t = 100$. From this collection of 10000 points, we want to recover transfer function

$$G(s) = \frac{1}{s + \frac{1}{2} s^{\frac{3}{2}}}$$

Since this is an impulse response, we can skip step 1. In step 2 we identify a discrete model with 100 poles and 100 zeros. For step 3, we try using both Levy’s method and the method of section 2. In both cases, we use the frequency response of the identified discrete model reckoned at five logarithmically-spaced frequencies in the [0.1,1] rad/s range. The correct orders of the numerator, denominator and commensurability are used. The results are

$$\hat{G}_L(s) = \frac{0.717}{s + 0.005 s^2 + 0.086}$$

for Levy’s method and

$$\hat{G}_C(s) = \frac{0.999}{s + 0.339 s^2 + 0.001}$$

for the method of section 2 (see Figure 2).

The method of section 2, by giving phase greater importance, achieved a better result.

The use of the correct orders was necessary for obtaining good results; different values resulted in incorrect identification results. This shows that it is a good policy to systematically verify what results are obtained with all reasonable orders. Also notice that the order of the discrete transfer function employed is far above that which would have been necessary for an integer order transfer function of similar complexity (e.g. $\frac{1}{s^3 + 0.5}$).

### 4. CONCLUSIONS

The method given by Oustaloup (1991); Oustaloup et al. (2002) to identify an integer order model from a frequency

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3 Actually this is the same as finding the frequency response of a finite impulse response (FIR) model with coefficients given by the values of the impulse response.

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Fig. 2. Bode diagrams of the time response identification example; top: fractional model obtained using Levy’s method; bottom: fractional model obtained using the method from section 2.

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**REFERENCES**


**Appendix A. The Tangent of a Sum of Arches of Tangent**

**Lemma 1.** The tangent of a sum of arcs of tangents is given by

\[
\tan \left( \sum_{i=1}^{a} \arctan x_i \right) = \frac{\sum_{k \in \mathcal{E}_a} (-1)^{\lfloor k/2 \rfloor} S_k}{1 + \sum_{k \in \mathcal{E}_a} (-1)^{\lfloor k/2 \rfloor} S_k} \tag{A.1}
\]

where \( \mathcal{E}_a \) is the set of even naturals smaller than or equal to \( a \), \( \mathcal{D}_a \) is the set of odd naturals smaller than or equal to \( a \), \( \lfloor y \rfloor \) is the floor of \( y \) (the largest natural smaller than or equal to \( y \)), and sums \( S_k \) are given by

\[
S_1 = x_1 + x_2 + \ldots + x_a = \sum_{1 < k < a} x_k \\
S_2 = x_1 x_2 + x_1 x_3 + \ldots + x_1 x_a + x_2 x_3 + \ldots + x_2 x_a + \ldots \tag{A.2}
\]

\[
S_{a-1} = \sum_{1 < k_1 < k_2 < a} x_{k_1} x_{k_2} \\
S_a = x_1 x_2 x_3 + x_1 x_2 x_4 + \ldots + x_1 x_2 x_a + x_1 x_3 x_4 + \ldots + x_1 x_3 x_a + \ldots \tag{A.3}
\]

**Proof.** For \( a = 2 \) we obtain the usual formula for the tangent of a sum, \( \tan(\arctan x_1 + \arctan x_2) = \frac{x_1 + x_2}{1 - x_1 x_2} \).

The proof proceeds by mathematical induction, but the details of the inductive step will be omitted here.

**Appendix B. Complex Zeros and Poles**

In this appendix it is shown that the phase behaviour of a model with a pair of complex conjugate zeros can be reckoned using the formula for the case when all zeros are real. A pair of complex conjugate poles would lead to the same result. Only the fractional case is shown, the integer being a particular case thereof, with \( a = 1 \).

The frequency behaviour of (11), when \( G_0 = 1 \) (for simplicity), \( m = 2 \), \( n = 0 \) and \( b_1 = z \), \( b_2 = z^* \), is

\[
\dot{G}(j\omega) = \left( 1 + \frac{(j\omega)^a}{z} \right) \left( 1 + \frac{j\omega}{z^*} \right) = 1 + j^2\omega^2 z^2 + \omega^2 (z + z^*) = 1 + \frac{(\cos \pi \alpha + \sin \pi \alpha) \omega^2 + (\cos \frac{\pi \alpha}{2} + \sin \frac{\pi \alpha}{2}) \omega^2 (z + z^*)}{z^2} \tag{B.1}
\]

The phase is

\[
\arg \dot{G}(j\omega) = \arctan \frac{\omega^2 \sin \frac{\pi \alpha}{2} + \omega^2 \sin \frac{\pi \alpha}{2}}{z^2 + \omega^2 \cos \frac{\pi \alpha}{2} + \omega^2 \cos \frac{\pi \alpha}{2}} \tag{B.2}
\]

If we had applied (12), we would get

\[
\arg \dot{G}(j\omega) = \arctan \frac{\omega^2 \sin \frac{\pi \alpha}{2}}{z^2 + \omega^2 \cos \frac{\pi \alpha}{2}} + \arctan \frac{\omega^2 \sin \frac{\pi \alpha}{2}}{z^2 + \omega^2 \cos \frac{\pi \alpha}{2}} \tag{B.3}
\]

To this we now apply (A.1), for the case \( a = 2 \), as follows:

\[
\arctan \left( \arg \dot{G}(j\omega) \right) = \arctan \frac{\omega^2 \sin \frac{\pi \alpha}{2}}{z^2 + \omega^2 \cos \frac{\pi \alpha}{2}} + \arctan \frac{\omega^2 \sin \frac{\pi \alpha}{2}}{z^2 + \omega^2 \cos \frac{\pi \alpha}{2}} \tag{B.4}
\]

Applying the expressions for the sine and the cosine of a double-angle, this is seen to be the same as (B.2).