ZIEGLER-NICHOLS TYPE TUNING RULES FOR FRACTIONAL PID CONTROLLERS

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ABSTRACT
This paper presents two sets of tuning rules for fractional PIDs that rely solely on the same plant time-response data used by the first Ziegler-Nichols tuning rule for (usual, integer) PIDs. Thus no model for the plant to control is needed; only an S-shaped step response is. These rules are quadratic and their results compare well with those obtained with rule-tuned integer PIDs.

INTRODUCTION
Controllers whose output is a linear combination of the input, the derivative of the input and the integral of the input, known as PID (proportional—derivative—integrative) controllers, are widely used and enjoy significant popularity, because they are simple, effective and robust.

One of the reasons why this is so is the existence of tuning rules for finding suitable parameters for PIDs, rules that do not require any model of the plant to control. All that is needed to apply such rules is to have a certain time response of the plant. Examples of such sets of rules are those due to Ziegler and Nichols, those due to Cohen and Coon, and the Kappa-Tau rules [1]. It is true that PIDs tuned with such rules often perform in a non-optimal way. But even though further fine-tuning be possible and sometimes necessary, rules provide a good starting point. Their usefulness is obvious when no model of the plant is available, and thus no analytic means of tuning a controller exists, but rules may also be used when a model is known.

Fractional PIDs are generalisations of PIDs: their output is a linear combination of the input, a fractional derivative of the input and a fractional integral of the input [2]. Fractional PIDs are also known as PI^λD^μ controllers, where λ and μ are the integration and differentiation orders; if both values are 1, the result is a usual PID (henceforth called “integer” PID as opposed to a fractional PID). They have been increasingly used over the last years, but methods proposed to tune them always require a model of the plant to control [3, 4]. (An exception is [5], but the proposed method is far from the simplicity of tuning rules for integer PIDs.) This paper addresses this issue proposing sets of tuning rules for fractional PIDs. Proposed rules bear similarities to the first rule proposed by Ziegler and Nichols for integer PIDs, making use of the same plant time response data.

The paper is organised as follows. Next section sums up the fundamentals of fractional calculus needed to understand fractional PIDs. Then two analytical methods for tuning fractional PIDs when a plant model is available are addressed; these are used as basis for deriving the tuning rules. The last two sections give some examples of application and draw some conclusions.

FRACTIONAL ORDER SYSTEMS
Definitions
Fractional calculus is a generalisation of ordinary calculus. The main idea is to develop a functional operator D, associated
to an order $\nu$ not restricted to integer numbers, that generalises
the usual notions of derivatives (for a positive $\nu$) and integrals
(for a negative $\nu$). The most usual definition of $D$ is due to Rie-
mann and Liouville (although there are others) and generalises the
equalities

$$
\begin{align*}
\mathcal{D}_t^{-\nu} f(t) &= \int_t^\infty \frac{f(\tau)}{(\tau-t)^{\nu+1}} d\tau, \quad \nu \in \mathbb{N} \\
D^\nu D^m f(x) &= D^{\nu+m} f(x), \quad m \in \mathbb{Z}_0 \lor n, m \in \mathbb{N}_0
\end{align*}
$$

(1) (2)

which are easily proved for integer orders. The full definition of $D$
becomes

$$
\begin{align*}
\mathcal{D}_t^{-\nu} f(t) &= \left\{
\begin{array}{ll}
\int_t^\infty \frac{f(\tau)}{(\tau-t)^{\nu+1}} d\tau, & \text{if } \nu < 0 \\
\mathcal{D}_t^\nu [D^{-\nu} f(t)], & \text{if } \nu > 0, \quad n = \min \{k \in \mathbb{N}\}
\end{array}
\right.
\end{align*}
$$

(3)

It is worth noticing that, when $\nu$ is positive but not integer, oper-
ator $D$ still needs integration limits $c$ and $\nu$; in other words, $D$
is a local operator for natural values of $\nu$ (usual derivatives) only.

The Laplace transform of $D$ follows the usual rules

$$
\mathcal{L} [\mathcal{D}_t^{-\nu} f(t)] = \left\{
\begin{array}{ll}
s^\nu F(s), & \text{if } \nu \leq 0 \\
s^\nu F(s) - \sum_{k=0}^{n-1} s^k \mathcal{D}_t^{\nu-k-1} f(0), & \text{if } n - 1 < \nu < n \in \mathbb{N}
\end{array}
\right.
$$

(4)

and thus, if zero initial conditions are assumed, systems with a
dynamic behaviour described by differential equations involving
fractional derivatives give rise to transfer functions with frac-
tional powers of $s$.

"Fractional" calculus and "fractional" order systems are the
usual names though $\nu$ may assume irrational values in Eqn. (4).
also. Thorough expositions of these subjects may be found in
[2,6,7].

**Integer order approximations**

The most usual way of making use, both in simulations and
hardware implementations, of transfer functions involving frac-
tional powers of $s$ is to approximate them with usual (integer or-
der) transfer functions with a similar behaviour. Integer transfer
functions would require an infinite number of poles and zeros to
perfectly mimic fractional transfer functions, but it is neverthe-
less possible to obtain reasonable approximations with a finite
number of zeros and poles. One of the best-known approxima-
tions is due to Oustaloup and is given by [8]

$$
s^\nu = k \prod_{n=1}^{N} \left(1 + \frac{s}{\omega_p n}\right)^{\frac{\nu}{\omega_p}}, \quad \nu > 0
$$

(5)

The approximation is valid in the frequency range $[\omega_p; \omega_u]$; gain $k$
is adjusted so that the approximation shall have unit gain at
1 rad/s; the number of poles and zeros $N$ is chosen beforehand
(low values resulting in simpler approximations but also causing
the appearance of a ripple in both gain and phase behaviours);
frequencies of poles and zeros are given by

$$
\begin{align*}
\omega_{p,1} &= \omega_p \sqrt{N} \\
\omega_{p,n} &= \omega_p \sqrt{n}, n = 1 \ldots N \\
\omega_{p,n+1} &= \omega_p \sqrt{n}, n = 1 \ldots N - 1 \\
\alpha &= (\omega_p/\omega_o)^{\frac{\nu}{\omega_p}} \\
\eta &= (\omega_p/\omega_o)^{\frac{1}{2\nu}}
\end{align*}
$$

(6) (7) (8) (9) (10)

The case $\nu < 0$ may be dealt with inverting Eqn. (5). But if $|\nu| > 1$
approximations become unsatisfactory; for that reason, it is
usual to make

$$
s^\nu = s^{\nu} \delta, \quad n + \delta = \nu \lor n \in \mathbb{Z} \land \delta \in [0;1]
$$

(11)

and then approximate only the latter term.

If a discrete transfer function approximation is desired, the
above approximation (or any other alternative approximation)
may be discretised using any usual method (Tustin, Simpson...). But
there are also formulas that directly provide discrete approxi-
mations. None shall be needed in what follows. See for in-
stance [9] for more on this subject.

**ANALYTICAL TUNING METHODS**

A fractional PID has a transfer function given by

$$
C(s) = P + \frac{1}{s^\nu} + D \mu^\nu
$$

(12)

In this section two methods published in the literature for analyt-
ically tuning the five parameters of such controllers are given.

**Internal model control**

The internal model control methodology may, in some cases,
be used to obtain PID or fractional PID controllers. It makes use
of the control scheme of Fig. 1, where $G$ is the plant to control,
$G^*$ is an inverse of $G$ or at least a plant as close as possible to
the inverse of $G$, $G^*$ is a model of $G$ and $F$ is some judiciously
chosen filter. If $G^*$ were exact, the error $e$ would be equal to
disturbance $d$. If, additionally, $G^*$ were the exact inverse of $G$ and $F$ were
unity, control would be perfect. Since no models are perfect, $e$
will not be exactly the disturbance. That is also exactly why $F$
exists and is usually a low-pass filter: to reduce the influence
of high-frequency modelling errors. It also helps ensuring that product $FG^*$ is realisable. The interconnections of Fig. 1 are equivalent to those of Fig. 2 if

$$C = \frac{FG^*}{1 - FG^*G'}$$  \hspace{1cm} (13)

Controller $C$ is not, in the general case, a PID or a fractional PID, but it will if

$$G = \frac{K}{1 + s^{\mu} T} e^{-Ls}$$  \hspace{1cm} (14)

Let

$$F = \frac{1}{1 + sTF}, \quad G^* = \frac{1 + s^{\mu} T}{K}, \quad G' = \frac{K}{1 + s^{\mu} T} (1 - sL)$$  \hspace{1cm} (15)

Notice that the delay of $G$ was neglected in $G^*$ but not in $G'$. Then Eqn. (13) becomes

$$C = \frac{\frac{1}{K}\frac{1}{(T + L)s} + \frac{T}{s^{1-\mu}}}{\frac{1}{K}\frac{1}{(T + L)s} + \frac{T}{s^{1-\mu}}}$$  \hspace{1cm} (16)

that can be viewed as a fractional PID controller with the proportional part equal to zero. And if the model of $G'$ in Eqn. (15) is improved to

$$G' = \frac{K}{1 + s^{\mu} T} \frac{1 - sL/2}{1 + sL/2}$$  \hspace{1cm} (17)

then Eqn. (13) becomes

$$C = \frac{\frac{L}{2K(T + L)} + \frac{1}{s^{1-\mu}} + \frac{T}{s^{1-\mu}}}{\frac{L}{2K(T + L)} + \frac{1}{s^{1-\mu}} + \frac{T}{s^{1-\mu}}}$$  \hspace{1cm} (18)

If one of the two integral parts is neglectable, Eqn. (18) will be a fractional PID controller. Obviously, should $\mu \in \mathbb{Z}$, both Eqn. (16) and Eqn. (18) become usual PIDs.

**Tuning by minimisation**

Monje et al. [4] proposed that fractional PIDs be tuned by requiring them to satisfy the following conditions ($C$ being the controller and $G$ the plant):

1. The gain-crossover frequency $\omega_{cg}$ is to have some specified value:

$$\left| C(\omega_{cg}) G(\omega_{cg}) \right| = 0 \text{dB}$$  \hspace{1cm} (19)

2. The phase margin $\varphi_m$ is to have some specified value:

$$-\pi + \varphi_m = \arg \left[ C(\omega_{cg}) G(\omega_{cg}) \right]$$  \hspace{1cm} (20)

3. So as to reject high-frequency noise, the closed loop transfer function must have a small magnitude at high frequencies; thus it is required that at some specified frequency $\omega_h$ its magnitude be less than some specified gain:

$$\left| \frac{C(\omega_h) G(\omega_h)}{1 + C(\omega_h) G(\omega_h)} \right| < H$$  \hspace{1cm} (21)

4. So as to reject output disturbances and closely follow references, the sensitivity function must have a small magnitude at low frequencies; thus it is required that at some specified frequency $\omega_l$ its magnitude be less than some specified gain:

$$\left| \frac{1}{1 + C(\omega_l) G(\omega_l)} \right| < N$$  \hspace{1cm} (22)

5. So as to be robust in face of gain variations of the plant, the phase of the open-loop transfer function must be (at least roughly) constant around the gain-crossover frequency:

$$\frac{d}{d\omega} \arg \left[ C(\omega) G(\omega) \right]_{\omega = \omega_{cg}} = 0$$  \hspace{1cm} (23)

Conditions are five because five are the parameters to tune. To satisfy them all those authors proposed the use of numerical optimisation algorithms, namely those implemented in Matlab’s function *fmincon* (the condition in Eqn. (19) is assumed as the condition to minimise; conditions in Eqns. (20) to (23) are assumed as constraints). This is effective but allows local minima
to be obtained. In practice most solutions found with this optimisation method are good enough, but they strongly depend on initial estimates of the parameters provided. Some may be discarded because they are unfeasible or lead to unstable loops, but in many cases it is possible to find more than one acceptable fractional PID; in others, only well-chosen initial estimates of the parameters allow finding a solution.

**TUNING RULES**

The first Ziegler-Nichols rule for tuning an integer PID assumes the plant to have an S-shaped unit-step response, as that of Fig. 3, where $L$ is an apparent delay and $T$ may be interpreted as a pole. The method cannot be applied if the unit-step response is shaped otherwise. The simplest plant with such a response is

$$G = \frac{K}{1 + sT} e^{-Ls}$$

(24)

The minimisation tuning method presented above was applied to plants given by Eqn. (24) for several values of $L$ and $T$, with $K = 1$. The parameters of fractional PIDs thus obtained vary in a regular manner. Using the least-squares method it is possible to translate that regularity into formulas to find acceptable values of the parameters from $L$ and $T$.

Two comments. Firstly, to implement the minimisation tuning method the last condition was verified numerically, evaluating argument in Eqn. (23) at two frequencies, equal to $\omega_c/1.222$ and $1.122 \omega_c$ (this corresponds to $1/20$ of a decade). It is of course possible to evaluate the argument at other frequencies around $\omega_c$; actually, the larger the interval where the argument is constant (or nearly so) the better, and thus using more than two points might ensure that. However, it was verified that such stronger requirements often prevent a solution from being found.

Secondly, least-square method-adjusted formulas cannot exactly reproduce every change in parameters. This means that fractional PIDs tuned with the rules presented below never behave as those tuned analytically, neither are they so robust. Conditions in Eqns. (19) to (23) will only approximately be verified.

**First set of rules**

A first set of rules is given in Tab. 1. This is to be read as

$$P = -0.0048 + 0.2664L + 0.4982T$$

$$+ 0.0232L^2 - 0.0720T^2 - 0.0348TL$$

(25)

and so on. They may be used if $0.1 \leq T \leq 50 \land L \leq 2$ and were designed for the following specifications:

$$\omega_c = 0.5 \text{ rad/s}$$

(26)

$$\varphi_m = 2/3 \text{ rad} \approx 38^\circ$$

(27)

$$\omega_h = 10 \text{ rad/s}$$

(28)

$$\omega_i = 0.01 \text{ rad/s}$$

(29)

$$H = -10 \text{ dB}$$

(30)

$$N = -20 \text{ dB}$$

(31)

Recall that specifications are only approximately verified.

**Second set of rules**

A second set of rules is given in Tab. 2. These may be applied if $0.1 \leq T \leq 50 \land L \leq 0.5$. Only one set of parameters is needed in this case because the range of values of $L$ these rules cope with is more reduced. They were designed for the following specifications:

$$\omega_c = 0.5 \text{ rad/s}$$

(32)

$$\varphi_m = 1 \text{ rad} \approx 57^\circ$$

(33)

$$\omega_h = 10 \text{ rad/s}$$

(34)

$$\omega_i = 0.01 \text{ rad/s}$$

(35)

$$H = -20 \text{ dB}$$

(36)

$$N = -20 \text{ dB}$$

(37)

Of course, other sets of rules might have been found in the same way for different specifications.

**EXAMPLES**

In what follows these rules are applied to three different plants, and the goodness of the results asserted and compared to those obtained with integer PIDs tuned with the first Ziegler-Nichols rule.

Two comments. Firstly, as stated above, rules usually lead to results poorer than those they were devised to achieve. (The same happens with Ziegler-Nichols rules: they are expected to result in an overshoot around 25%, but it is not hard to find plants with which the overshoot is 100% or even more.) Secondly, Ziegler-Nichols rules make no attempt to reach always the same gain-crossover frequency, or the same phase margin. Actually, these
two performance indicators vary widely as $L$ and $T$ vary. This adds some flexibility to Ziegler-Nichols rules: they can be applied for wide ranges of $L$ and $T$ and still achieve a controller that stabilises the plant. Rules from the previous section always aim at fulfilling the same specifications, and that is why their application range is never so broad as that of Ziegler-Nichols rules.

Bode diagrams presented are exact; all time-responses involving fractional derivatives and integrals were obtained with simulations making use of Oustaloup’s approximation described in the subsection dealing with approximations, with $\omega_0 = 10^{-3} \text{ rad/s}$, $\omega_N = 10^3 \text{ rad/s}$ and $N = 7$.

**First-order plant with delay**

The plant considered was

$$G(s) = \frac{K}{1+s}e^{-0.1s} \quad (38)$$

The nominal value of $K$ is 1. Controllers obtained with the two tuning rules from the previous section and with the first Ziegler-Nichols rule are

$$C_1(s) = \frac{0.4448}{s^{1.4277}} + \frac{0.5158}{s^{1.1230}} + 0.2045s^{1.0202} \quad (39)$$

$$C_2(s) = \frac{1.2507 + 0.3106}{s^{1.1230}} - 0.2589s^{0.1533} \quad (40)$$

$$C_{ZN}(s) = \frac{12.0000}{s} + \frac{0.0002}{s} + 0.6000s \quad (41)$$

(Notice that due to the approximations involved one of the gains is negative; this will not, however, affect results.) Corresponding step responses are given in Figs. 4, 5 and 6. These show what happens for several values of $K$, the plant’s gain, assumed as known with uncertainty. It should be noticed that fractional PIDs can deal with with a clearly broader range of values of $K$. This is likely because the specifications the integer PID tries to achieve are different: that is why responses are all faster, at the cost of greater overshoots. More important is that the overshoot is fairly constant with fractional PIDs, at least for those values closer to 1. This is because fractional PIDs attempt to verify Eqn. (23), which the integer PID does not. Data on these responses is summed up in Tab. 3. (In this and the following tables, the rise time is reckoned according to the 10%–90% rule and the settling time is reckoned according to the ±5% rule.)

Those figures also give the corresponding open-loop Bode diagrams and the gains of sensitivity and closed-loop functions. They show that the desired conditions—given by Eqns. (19) to (23)—are reasonably—though not exactly—followed. Differences are due to the approximations incurred by the least-squares fit.

**Second-order plant**

The plant considered was

$$G(s) = \frac{K}{4.3200s^2 + 19.1801s + 1} \approx \frac{K}{1 + 20s}e^{-0.2s} \quad (42)$$
with a nominal value of $K$ of 1. The approximation stems from the values of $L$ and $T$ obtained from its step-response.

Controllers obtained with the two rules given above and with the first Ziegler-Nichols rule are

$$C_1(s) = 0.0880 + \frac{6.5185}{s^{0.6751}} + 2.5881s^{0.6957}$$  \hspace{1cm} (43)

$$C_2(s) = 6.9928 + \frac{12.4044}{s^{0.0000}} - 4.1063s^{0.7805}$$  \hspace{1cm} (44)

The step-responses obtained (together with open-loop Bode diagrams and sensitivity and closed-loop functions’ gains) are given in Figs. 7, 8 and 9 and in Tab. 4. This time, since there is no delay, the plant is easier to control and a wider variation of $K$ is supported by all controllers. But fractional PIDs still achieve an overshoot that is more constant, in spite of the different structure of the plant.

$$C_{ZN}(s) = 120.0000 + \frac{300.0000}{s} + 12.0000s$$  \hspace{1cm} (45)
### Table 3. DATA ON STEP-RESPONSES OF FIG. 4, FIG. 5 AND FIG 6

<table>
<thead>
<tr>
<th>$K$</th>
<th>rise time</th>
<th>overshoot</th>
<th>settling time</th>
<th>rise time</th>
<th>overshoot</th>
<th>settling time</th>
<th>rise time</th>
<th>overshoot</th>
<th>settling time</th>
</tr>
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<tbody>
<tr>
<td>1/32</td>
<td>22.1 s</td>
<td>26 %</td>
<td>94.6 s</td>
<td>28.1 s</td>
<td>5 %</td>
<td>78.2 s</td>
<td>1.0 s</td>
<td>23 %</td>
<td>3.7 s</td>
</tr>
<tr>
<td>1/16</td>
<td>13.8 s</td>
<td>27 %</td>
<td>59.1 s</td>
<td>15.1 s</td>
<td>5 %</td>
<td>19.2 s</td>
<td>0.6 s</td>
<td>33 %</td>
<td>3.9 s</td>
</tr>
<tr>
<td>1/8</td>
<td>8.9 s</td>
<td>28 %</td>
<td>36.5 s</td>
<td>8.1 s</td>
<td>5 %</td>
<td>10.2 s</td>
<td>0.4 s</td>
<td>40 %</td>
<td>2.8 s</td>
</tr>
<tr>
<td>1/4</td>
<td>5.9 s</td>
<td>30 %</td>
<td>22.6 s</td>
<td>4.4 s</td>
<td>6 %</td>
<td>12.4 s</td>
<td>0.2 s</td>
<td>45 %</td>
<td>1.9 s</td>
</tr>
<tr>
<td>1/2</td>
<td>4.0 s</td>
<td>30 %</td>
<td>19.8 s</td>
<td>2.4 s</td>
<td>8 %</td>
<td>7.7 s</td>
<td>0.1 s</td>
<td>48 %</td>
<td>1.3 s</td>
</tr>
<tr>
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<td>2.6 s</td>
<td>27 %</td>
<td>14.6 s</td>
<td>1.3 s</td>
<td>9 %</td>
<td>4.7 s</td>
<td>0.1 s</td>
<td>74 %</td>
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</tr>
<tr>
<td>2</td>
<td>1.7 s</td>
<td>20 %</td>
<td>7.4 s</td>
<td>0.7 s</td>
<td>8 %</td>
<td>2.8 s</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>4</td>
<td>0.9 s</td>
<td>12 %</td>
<td>5.5 s</td>
<td>0.3 s</td>
<td>8 %</td>
<td>1.4 s</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>8</td>
<td>0.2 s</td>
<td>7 %</td>
<td>3.9 s</td>
<td>—</td>
<td>—</td>
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</tbody>
</table>

![Figure 7](image1.png)

**Figure 7.** (a) STEP RESPONSE OF EQN. (42) CONTROLLED WITH EQN. (43) WHEN $K$ IS 1/32, 1/16, 1/8, 1/4, 1/2, 1 (THICK LINE), 2, 4, 8, 16 AND 32. (b) OPEN-LOOP BODE DIAGRAM WHEN $K$ = 1. (c) SENSITIVITY FUNCTION AND CLOSED-LOOP FUNCTION GAINS WHEN $K$ = 1.

![Figure 8](image2.png)

**Figure 8.** (a) STEP RESPONSE OF EQN. (42) CONTROLLED WITH EQN. (44) WHEN $K$ IS 1/32, 1/16, 1/8, 1/4, 1/2, 1 (THICK LINE), 2, 4, 8, 16 AND 32. (b) OPEN-LOOP BODE DIAGRAM WHEN $K$ = 1. (c) SENSITIVITY FUNCTION AND CLOSED-LOOP FUNCTION GAINS WHEN $K$ = 1.

### Fractional-order plant with delay

The plant considered was

$$ G(s) = \frac{K}{1 + \sqrt{s} e^{-0.5s}} \approx \frac{K}{1 + 1.5s e^{-0.1t}} \quad (46) $$

Controllers obtained with the two rules given above and with a nominal value of $K$ of 1. The approximation is derived from the plant’s step-response at $t = 0.92$ s (the step response at $t = 0.5$ s cannot be used since it has an infinite derivative).
Figure 9. (a) STEP RESPONSE OF EQN. (42) CONTROLLED WITH EQN. (45) WHEN K IS 1/32, 1/16, 1/8, 1/4, 1/2, 1 (THICK LINE), 2, 4, 8, 16 AND 32. (b) OPEN-LOOP BODE DIAGRAM WHEN K = 1. (c) SENSITIVITY FUNCTION AND CLOSED-LOOP FUNCTION GAINS WHEN K = 1.

Table 4. DATA ON STEP-RESPONSES OF FIG. 7, FIG. 8 AND FIG 9.

<table>
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<th>Controller of Eqn. (44)</th>
<th>Controller of Eqn. (45)</th>
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<td></td>
<td>rise time</td>
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<td>settling time</td>
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<td>31.4 s</td>
<td>—</td>
<td>45.5 s</td>
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<td>15.6 s</td>
<td>8 %</td>
<td>38.2 s</td>
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<td>9.1 s</td>
<td>19 %</td>
<td>47.1 s</td>
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<td>28 %</td>
<td>32.1 s</td>
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<td>10.5 s</td>
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<td>7.2 s</td>
</tr>
<tr>
<td>8</td>
<td>0.8 s</td>
<td>23 %</td>
<td>3.3 s</td>
</tr>
<tr>
<td>16</td>
<td>0.5 s</td>
<td>15 %</td>
<td>2.5 s</td>
</tr>
<tr>
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<td>8 %</td>
<td>1.8 s</td>
</tr>
</tbody>
</table>

The step-responses obtained (together with open-loop Bode diagrams and sensitivity and closed-loop functions’ gains) are given in Figs. 10, 11 and 12 and in Tab. 5. The PID performs poorly because it tries to obtain a fast response and thus employs higher gains, but what is relevant here is that fractional PIDs still achieve practically constant overshoots, since, in spite of the different plant structure, the conditions they were expected to verify are still verified to a reasonable degree, as the frequency-response plots show.

Comparing these results with those obtained with IMC-tuned fractional PIDs shows that rule-tuned fractional PIDs perform nearly as well as those found with this analytical method. In this case, by letting \( T_F = \frac{1}{2} \), Eqn. (18) leads to

\[
C_{IMC} = \frac{1}{4} + \frac{1}{s} + \frac{1}{s^{3/2}} + \frac{1}{4s} \tag{50}
\]

The step-responses obtained are shown in Fig. 13 and compare well with those of Fig. 10 and Fig. 11.

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In this paper two analytical methods for tuning the parameters of fractional PIDs were reviewed. The optimisation method was then used for developing tuning rules similar to those of the first set of Ziegler-Nichols rules, making use of two parameters (L and T) of the unit-step response of the plant (which should be S-shaped; otherwise rules cannot be applied).

The most obvious difference is that the new rules are clearly more complicated than those of Ziegler-Nichols: they have to be quadratic (approximations of lower order being unsatisfactory). And the broader the application range of the rules is to be, the more complicated they become (the first rule needs two tables of values of L only, needs only one). The usefulness of these rules is that of all sets of rules: they may be applied even if no model of the plant is available, provided a suitable time response is; they may be used as a departing point for fine-tuning (this is relevant if the optimisation tuning method is used, since its results depend...
Table 5. DATA ON STEP-RESPONSES OF FIG. 10, FIG. 11 AND FIG. 12.

<table>
<thead>
<tr>
<th>$K$</th>
<th>Controller of Eqn. (47)</th>
<th>Controller of Eqn. (48)</th>
<th>Controller of Eqn. (49)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>rise time</td>
<td>overshoot</td>
<td>settling time</td>
</tr>
<tr>
<td>1/32</td>
<td>25.1 s</td>
<td>26 %</td>
<td>105.5 s</td>
</tr>
<tr>
<td>1/16</td>
<td>15.8 s</td>
<td>27 %</td>
<td>65.8 s</td>
</tr>
<tr>
<td>1/8</td>
<td>10.3 s</td>
<td>28 %</td>
<td>41.2 s</td>
</tr>
<tr>
<td>1/4</td>
<td>6.9 s</td>
<td>29 %</td>
<td>26.1 s</td>
</tr>
<tr>
<td>1/2</td>
<td>4.4 s</td>
<td>27 %</td>
<td>17.2 s</td>
</tr>
<tr>
<td>1</td>
<td>2.7 s</td>
<td>23 %</td>
<td>11.9 s</td>
</tr>
<tr>
<td>2</td>
<td>1.5 s</td>
<td>17 %</td>
<td>8.6 s</td>
</tr>
</tbody>
</table>

Figure 13. STEP RESPONSE OF EQN. (46) CONTROLLED WITH IMC WHEN $K$ IS 1/32, 1/16, 1/8, 1/4, 1/2, 1 (THICK LINE) AND 2.

significantly from the initial estimate provided); they are easier and faster to apply than analytic methods. Their drawbacks are also those of all sets of rules: their performance is often inferior to the one sought, fine-tuning being often needed; they perform worse than controllers tuned analytically; they cannot be applied to all types of plants, but only to those with a particular sort of time response. These rules compare well with integer PIDs tuned according to the first Ziegler-Nichols rule, even though the comparison is made difficult because Ziegler-Nichols rules achieve different specifications for different values of $T$ and $L$ while rules developed for fractional PIDs attempt to keep always a uniform result. (It is of course likely that carefully tuned integer PIDs perform better than rule-tuned fractional PIDs.)

It is surely possible to improve these tuning rules. Rules similar to the second Ziegler-Nichols rule (making use of a closed-loop response of the plant) are certainly possible. Rules specific for non-minimum phase plants may also be of interest.

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