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ORIGINAL PAPER

On the barometric formula inside the Earth

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Abstract The assumptions leading to the barometric formula are discussed, with remarks on the influence of temperature, gravitational field, Earth rotation, and non-equilibrium conditions. A generalization of the barometric formula for negative heights, e.g., for pressure inside shafts and deep tunnels, is also presented, and some surprising conclusions obtained. Related historical aspects are also discussed.

Keywords Barometric equation · Non-ideality · Positive and negative heights · Historical and technological aspects

1 Introduction

In a previous article the barometric formula [1] that gives the pressure dependence with height of an isothermal and ideal gas was discussed. Generalizations of the barometric formula for a non-uniform gravitational field and for a vertical temperature gradient were also presented together with a brief historical review. The effect of a gravitational field on fluids and quantum particles as well as a discussion of the van

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der Waals equation of state was carried out in a series of articles. [2–4] Worth citing about a similar type of problem is a quite recent paper by Dubinova [5] that presents an exact barometric equation for a warm isothermal Fermi gas.

In the present paper we will deepen the discussion on the barometric formula with additional remarks on the influence of temperature, gravitational field, Earth rotation, and non-equilibrium conditions. A generalization of the barometric formula for negative heights, i.e., for pressure inside shafts and deep tunnels, is also presented.

In its simplest form, the barometric formula relates the pressure $p(z)$ of an isothermal, ideal gas of molecular mass m at some height z to its pressure $p(0)$ at height $z = 0$, where g_0 is the standard acceleration of gravity, k the Boltzmann constant, m the molecular mass, and T the temperature [1,6],

$$p(z) = p_0 \exp\left(-\frac{mg_0 z}{kT}\right) = p_0 \exp\left(-\frac{z}{H}\right). \tag{1}$$

With a scale height $H = kT/mg_0 = 8.4\,\text{km}$ if $T = 288\,\text{K}$ and $m = 29\,\text{g mol}^{-1}$. The numerator of the quotient in the exponential of Eq. 1 is the potential energy of a molecule, while the denominator, except for a constant numerical factor, is the mean kinetic energy of a molecule. This means that the decreasing pressure with height is the result of a balance between gravity pulling downward and random thermal motion in all directions, upward inclusive.

In spite of its approximate nature, namely owing to the assumption of a static isothermal atmosphere in thermodynamic equilibrium, and also that g is independent of height, i.e., $g(z) = g_0 = 9.8\,\text{m/s}^2$, Eq. 1 applies reasonably well to the lower troposphere, i.e., for altitudes up to 6 km, the error being less than 5%, and also to the stratosphere up to 20 km, where $T = 217\,\text{K}$, that is, $T = -57\,^\circ\text{C}$ [7,8]. Certain assumptions that are necessary for the derivation of the simple barometric formula do not apply to the real atmosphere. Equation 1 hides another simplifying assumption, i.e., that the Earth is flat and that it has no finite vertical extent. To get round this problem is to write the hydrostatic equation in spherical polar coordinates and assume that atmospheric variables depend only on the radial coordinate.

2 Discussion of the assumptions

2.1 The atmosphere is not isothermal

Temperatures in the real atmosphere range widely, between about 15 °C on the surface (average value) to −100 °C in the upper mesosphere, not to mention the thermosphere, a highly rarefied layer whose temperatures can exceed 2,000 °C. Consider the case of uniform gravitational field but with a vertical temperature gradient. Assuming the following linear variation of temperature with height, which is a good approximation for the troposphere ($z < 10\,\text{km}$) [7,8],

$$T(z) = T_0 - \beta z. \tag{2}$$

Here β is a positive constant in K/m (frequently called temperature lapse rate, often given in K/km), we have a result already obtained in [1].

$$p(z) = p_0 \left(1 - \frac{\beta z}{T_0}\right)^{mg_0/k\beta}. \tag{3}$$

This equation represents well the pressure dependence on altitude for the whole troposphere (up to 11 km), with $p_0 = 10^5$ Pa, $T_0 = 288$ K (15 °C) and $\beta = 6.5$ K km^{-1}. If Eq. 1 is used instead, the best empirical fit is obtained with $H = 7.8$ km.

The fall of temperature with altitude in the troposphere is due to the fact that air is warmed mainly from the surface of the planet. This fall is, however, smaller than could be expected, because of convection that occurs up to the tropopause [9].

On the other hand, there is a temperature rise in the stratosphere ($\beta = -1.0$ K km^{-1} from $z = 20$ to 32 km [8]). This increase is associated with ozone, which concentrates in a layer *ca.* 20 km thick, and centered at an altitude of *ca.* 30 km. This layer strongly absorbs ultraviolet light from the Sun, and subsequently releases the corresponding energy as heat [9]. A flat temperature minimum at −57 °C is observed between 11 and 20 km [7,8], corresponding to a compromise between the cooling and heating profiles.

2.2 The acceleration of gravity is not a constant

This constant, in fact, varies with latitude, longitude, and elevation. These variations, however, are quite small and lead to only a small error. In general case for an isothermal atmosphere,

$$\frac{p(z)}{p_0} = \exp\left(-\int_0^z \frac{mg(u)}{kT} du\right). \tag{4}$$

Acceleration g depends on altitude z. According to the law of gravitation, and noting that the mass of the atmosphere is very small compared to Earth's mass,

$$mg(z) = G\frac{mM}{(R+z)^2}, \tag{5}$$

where M is the mass of the Earth and R is its radius, while at the Earth's surface

$$g_0 = g(R) = G\frac{M}{R^2} \tag{6}$$

hence

$$g(z) = g_0 \frac{1}{(1+z/R)^2}. \tag{7}$$

Inserting Eq. 7 into Eq. 4 we have

$$\frac{p(z)}{p_0} = \exp\left(-\frac{mg_0}{kT_0}\int_0^z \frac{du}{(1+u/R)^2}\right). \tag{8}$$

Integration of this equation gives

$$\begin{aligned}\frac{p(z)}{p_0} &= \exp\left[-\frac{mg_0 R}{kT_0}\left(1-\frac{1}{1+z/R}\right)\right]\\ &= \exp\left(-\frac{mg_0 z}{kT_0}\frac{1}{1+z/R}\right) = \exp\left(-\frac{1}{1+z/R}\frac{z}{H}\right).\end{aligned} \tag{9}$$

Calculations show that a noticeable difference between data obtained from Eqs. 1 and 9 shows up for $z > 0.01R \approx 64\,\text{km}(\approx 8\%, T_0 = 288\,\text{K})$, i.e. well above the stratosphere.

Notice that it follows from Eq. 9 that pressure and molecular concentration do not go to zero as the altitude goes to infinity, which is not physically possible. This aspect, already discussed in [1] for the general 3D case, shows that a static and isothermal atmosphere is intrinsically unstable.

2.3 The Earth spins

Suppose that the Earth atmosphere is rotating with the same angular velocity ω as the Earth (for the lower atmospheric layers this is appropriate). Due to rotation, the weight of gas is different at the pole and at the equator. At the pole, acceleration of gravity is described by Eq. 6. At the equator, the weight is decreased by the centrifugal force $m\omega^2(R+z)$, i.e., the effective acceleration is smaller than acceleration of gravity and obeys the equation

$$g(z) = g_0\frac{1}{(1+z/R)^2} - \omega^2 R\,(1+z/R) = g_0\frac{1}{(1+z/R)^2}\left[1-\frac{\omega^2 R}{g_0}(1+z/R)^3\right]. \tag{10}$$

Assuming that $\omega = 7.27 \times 10^{-5}$ rad/s, $g_0 = 9.8\,\text{m/s}^2$, $R = 6.4 \times 10^6$ m, we have $\omega^2 R/g_0 \approx 3.45 \times 10^{-3}$. It is clear from Eq. 10 that the centrifugal force becomes important (change in gravity acceleration $\geq 1\%$) for

$$\frac{\omega^2 R}{g_0}(1+z/R)^3 \geq 0.01 \Rightarrow z/R \geq 0.43, \tag{11}$$

which leads to a large value ($z > 2{,}700$ km). If it is assumed that the atmosphere rotates with the Earth's as a whole, i.e., irrespective of height, the upper limit of the atmosphere at the equator can be obtained from Eq. 10 by setting the acceleration to zero,

$$1 - \frac{\omega^2 R}{g_0}(1 + z/R)^3 = 0 \Rightarrow z/R = 5.6, \tag{12}$$

which leads again to a very large value ($z = 36{,}000$ km). This is of course meaningless, considering that the density of outer space is attained for altitudes lower than 1,000 km. Escape of molecules, atoms and ions from the upper atmosphere occurs in fact by thermal and photochemical mechanisms still in the presence of a significant inward force.

Using Eqs. 10 and 4, we obtain for the barometric equation in the case of an isothermal atmosphere,

$$\frac{p(z)}{p_0} = \exp\left(-\frac{mg_0}{kT_0}\int_0^z \frac{1}{(1+u/R)^2}\left(1 - \frac{\omega^2 R}{g_0}(1+u/R)^3\right) du\right). \tag{13}$$

Integration of this equation gives, instead of Eq. 9,

$$\begin{aligned}\frac{p(z)}{p_0} &= \exp\left(-\frac{mg_0 z}{kT_0}\left[\frac{1}{1+z/R} - \frac{\omega^2 R}{g_0}\left(1 + \frac{z}{2R}\right)\right]\right)\\ &= \exp\left(-\left[\frac{1}{1+z/R} - \frac{\omega^2 R}{g_0}\left(1 + \frac{z}{2R}\right)\right]\frac{z}{H}\right).\end{aligned} \tag{14}$$

Calculation shows that a noticeable difference ($\approx 2\%$) between data obtained from Eqs. 14 and 9 occurs for $z > 0.01R \approx 64$ km, i.e. well above the stratosphere.

The major problem in applying the barometric formula to the real atmosphere, however, derives from the fact that the atmosphere is not in equilibrium, as mentioned above at several places.

2.4 The atmosphere is not in equilibrium

The fact that the barometric formula is valid only under conditions of equilibrium (hydrostatic or static atmosphere) raises far more significant issues. This is because the atmosphere as a whole is never in a state of general equilibrium, as it continuously exchanges mass and energy with its surroundings. Only locally can a quasi-equilibrium be defined, the so-called local thermodynamic equilibrium. In fact, a state of equilibrium means that (i) the entropy of the parcel of air is maximized, (ii) no measurable differences in temperature exist, (iii) no measurable changes in pressure take place, and (iv) no measurable changes in density take place. Furthermore, there should be no net evaporation or condensation. In other words, there should be neither weather nor even winds.

Daniel Bernoulli (1700–1782) was the first to quantify the observation that for a fluid in motion there is an associated pressure drop. When the flow is laminar, this drop is proportional to the square of the velocity. This means that a flow of air at any elevation in the atmosphere creates a drop in pressure on both the underlying and the overlying air. It is this pressure drop that leads to the phenomenon generally known as "entrainment." This phenomenon occurs when moving air undergoes a net gain in molecular number density due to the fact that more molecules are entering the moving stream of air than are leaving it. Moreover, the pressure drop due to air flow is cumulative, that is, a westward flow at one elevation does not cancel out an eastward flow at a different elevation. Instead, the two (or more) pressure drops are additive. If these winds persist for any appreciable length of time, differential rates of molecular diffusion will transmit the pressure change to other elevations, and, eventually, to the surface.

The effects of vertical movements of air on pressure are even more dramatic (in this case the Bernoulli equation should be modified). Subsidence can create substantial increases in pressure, and updrafts can create substantial decreases in pressure. These are not predicted by the barometric formula. Since, at some elevation or other, winds are almost always blowing, and updrafts and downdrafts are normal phenomena of atmospheric processes, this means that any attempt to use the barometric formula to predict the pressure at elevation z from the pressure at some other elevation is bound to some degree of error. The full Bernoulli equation, which can take care of the elevation, is strictly valid only to the extent that the fluid is ideal. If viscous forces are present thermal energy will be involved.

2.5 Agreement with observations and altimeters

Finally, we should note that when observations are taken via weather balloons, the values for the *observed* pressures do not correspond to the projected values very closely. There is general agreement, in that pressures do indeed decrease with elevation, but there is little in the way of specific agreement. The actual readings (real pressures) have to be "corrected" for variations in density, temperature, humidity, etc. (corrected pressures) before coming very close to the projected values. This is to be expected, of course. Whenever we take a formula that is valid only for an ideal gas under conditions of equilibrium and apply it to the real atmosphere, we should not be surprised if it does not work perfectly. Nevertheless, pressure seldom departs from the average value (which is well predicted by the barometric equation) by more than a few percent. With respect to ground level atmospheric pressure (whose average value is about 101 kPa or 760 mmHg), the highest recorded value is 109 kPa (814 mmHg), and occurred at Mongolia on 19 December 2001. The lowest recorded value is 87.0 kPa (653 mmHg), and occurred in the Western Pacific during Typhoon Tip on 12 October 1979.

3 Barometric equation for negative heights

3.1 A historical excursion

Curiosity about the depths of the Earth is as old as man himself. For the primitive men, the cave was both a shelter and a place for magic rites. On the other hand, burial of the dead and a modicum of imagination easily led to the belief on an afterlife spent in the underworld. Volcanoes and other manifestations of an inner heat such has hot springs also suggested the existence of a specific place appropriate for punishment located beneath the surface of the Earth (Tartarus or Hell). These were the prevailing views in ancient Greek and Roman times, as described in Homer's *Iliad* and *Odyssey* (ninth or eighth century BC), and in Virgil's *Aeneid* (first century BC).

On more scientific grounds, Aristotle (fourth century BC) provided evidence for the sphericity of the Earth (in his work *On the Heavens*), giving at the same time but without details an estimate of its radius that is within the correct order of magnitude. Later Eratosthenes (third century BC) computed Earth's radius with much better accuracy. Note that some uncertainty exists on the value in meters of the unit of length used by both authors, the stadium.

In medieval times (end of the thirteenth century) Dante describes in *The Divine Comedy* a journey through a spherical Earth where Hell is located underneath Jerusalem and extends down to the centre, and Purgatory is a mountain in an island located near the antipodes. Paradise lies, of course, above in the heavens. According to Dante's artistic interpretation of the Bible, Lucifer fell from Heaven straight down to the centre of the Earth, being forever prevented from leaving this place (according to the Aristotelian system, objects had a natural tendency to fall down to the center of the Earth).

In *The Dialogues Concerning the Two Chief World Systems* (1632), Galileo Galilei imagined the terrestrial globe pierced by a hole which passed through the center, and examined the motion of a cannon ball dropped in such a hole. He states that the ball "…would have acquired at the center such an impetus from its speed that it would pass beyond the center and be driven upward through as much space as it had fallen (…) the time consumed in this second ascending motion would be equal to its time of descent." It is thus a fortunate situation that Lucifer was not able to resurface. Or did he?

The motion of an object (neglecting drag) dropped in a bottomless shaft was again considered by Hooke in 1679 [10]. The main point under discussion was the effect of Earth's rotation on the trajectory. Hooke obtained the correct result qualitatively: The object should oscillate like a pendulum, describing an ellipse. In fact, an object dropped in a shaft connecting the poles of a homogeneous and spherical Earth behaves as a one-dimensional harmonic oscillator and strictly obeys Hooke's "law", although this is not the present standard pedagogical example. Newton, on the other hand, initially believed that the trajectory should be a spiral, even in the absence of drag [10]. It has to be remarked that later quantitative treatments of the problem are all based on Newtonian dynamics.

The motion of an object (neglecting drag) dropped in a hypothetical tunnel connecting the two poles was also studied by Euler in 1727 [11]. Surprisingly, he obtained the incorrect result that the object would never go further than the centre, and would then return to the point of departure (as reported in [11]).

Investigation of the internal structure of Earth by digging a very deep shaft was considered by Maupertuis in his *Lettre sur le progrès des sciences* (1752) as an important and timely research project. This proposal was ridicularized by Voltaire (as part of a dispute on the paternity and philosophical meaning of the principle of least action).

In his *Letters to a German princess* (1760–1762), Euler mentions this dispute in Letter XLIX (Voltaire, alive and influential, is alluded to as "Monsieur ***") and also states "if a hole were made to the center of the Earth, the density of the air would increase progressively, attaining that of the water first, and then that of gold" (Letter XI).

The increase of air density with depth is a difficulty mentioned but conveniently underestimated by Jules Verne in his celebrated *Journey to the Centre of the Earth* (1864). It is interesting to remark that at about the same time Thomas Andrews was doing his far-reaching work on the condensation of gases.

In 1882 the respected French civil engineer and applied mathematician Édouard Collignon (1831–1897) speculated on the possibility of travel between cities by means of long linear tunnels inside the Earth, in a kind of partial free-fall planetary subway, for which the transit time *in the absence of drag* is 42 minutes, independently of the location of the two cities. An account of his ideas, published on a semi-humorous tone in the scientific periodical *La Nature*, is suggestively entitled "From Paris to Rio de Janeiro in 42 minutes and 11 seconds" [12]. In it, the effect of pressure is discussed, and considered to be an insurmountable problem. Numerical estimates of enormous pressures at several depths (unfortunately provided without computational details) are given, but differ from our own calculations (see below) by several orders of magnitude.

Several nineteenth century fiction books describe journeys in deep tunnels or shafts. This theme was thus a relatively common topic, which is not surprising in a society where Mechanics (including Kinematics and Dynamics) played a growing role in everyday life, and with new mechanical devices and means of transportation continuously appearing. The fall in a deep hole is a well-known episode of *Alice's Adventures in Wonderland* (1865). In a later fictional work (*Sylvie and Bruno Concluded*, 1893), Lewis Carroll even briefly mentions [Collignon's] underground gravitational train.

A little known fiction story for the youth that involves a planetary tunnel is *Through the Earth* (1898) by the American author C. Fezandié (1865–1959). In it, the construction and operation of an evacuated linear tunnel connecting Australia and the US for commercial purposes is discussed, and the consequences of free fall are fully explored. The action takes place in the 1980s–1990s. The effect of Earth's rotation on the trajectory is compensated by means of electric repulsion between the walls of the tube and the car. This book, conveniently adapted, would provide a good argument for a science fiction movie with a nineteenth century flavour.

During the twentieth century attention continued to be paid to the subject. In 1909, the famous French amateur astronomer and science popularizer C. Flammarion wrote a short article for *The Strand Magazine* entitled "A Hole Through the Earth" [13]. In it, the meager knowledge of the Earth's interior available at the time is discussed, and Maupertuis project again advocated. Flammarion proposed that the shaft be dug in France or Belgium, among several places (but not in England, as erroneously stated in [14]—it is enough to pay attention to the first illustration, where a hand marks the country of the Landes, in the southwest of France), and that the work should be carried out by soldiers (Fig. 1). These words were unintendedly prophetic: A few years later, millions of soldiers would be indeed incessantly digging in France and Belgium, unfortunately not a deep shaft for peaceful purposes, but the shallow and infamous trenches of the First World War from which many would not return.

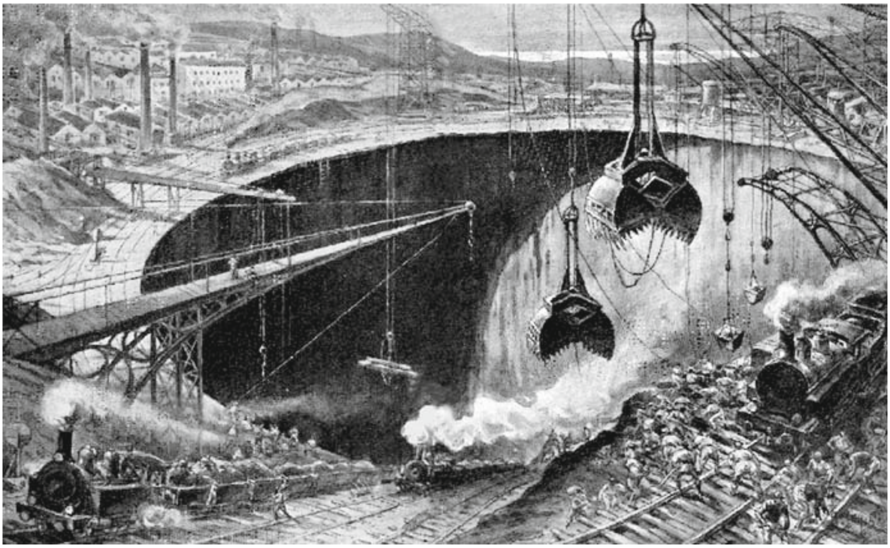

Fig. 1 The orifice of the shaft during construction, showing the factories, railways, enormous cranes, and other machinery, attended by an army of workers [13]

Shortly before this war (1913), another book was published with a section devoted to these matters: *Physics for Entertainment*, by Ya. Perel'man. At the time of his author's death in 1942, during the St. Petersburg (then Leningrad) blockade, the book had already had 13 editions. It underwent many more posthumous editions, and is still in print. In the section devoted to the problem, the period of oscillation and the effect of Earth's rotation are discussed. It is remarked that the period of oscillation is independent of the Earth's radius, and is solely defined by its density. Indeed, $T = \sqrt{\frac{3\pi}{G\rho}}$. A previous (1902) proposal by a Russian author of a 600-km tunnel connecting the two main Russian cities, St. Petersburg and Moscow, is also described.

In 1965, an article entitled "High-Speed Tube Transportation" appeared in *Scientific American* [15]. In it, a high-speed subway connecting Boston and Washington, D.C. (630 km) in 90 min was proposed. It was based on pneumatic propulsion and to a small extent on gravity. Note that present-day Maglev (magnetic levitation) trains running on the surface attain similar average speeds. In the following issue of the mentioned journal, a letter to the Editor by M. Gardner appeared [16] drawing attention to the possibility of using only gravity for the thrust, and mentioning (not always accurately) some previous works where this possibility was foreseen. In the next year, a pedagogical article discussed again (not citing [15,16] but mentioning a similar article) the gravitational subway [17]. As in [15], the tunnel connected Boston and Washington, D.C. This apparently revolutionary idea was echoed in the press of the time, but it rapidly surfaced that many other authors, namely Collignon, had already speculated on the subject.

More recent works describing in detail the kinematics of this kind of motion, and even taking into account the rotation of the Earth, but always in the absence of drag, are references [14,18,19]. An entire book [20] was recently devoted to some mathematical and historical aspects of Newtonian mechanics in the Earth–Moon system.

3.2 The barometric equation

Not unexpectedly at this point, suppose a shaft is drilled down to the center of the Earth, some 6,370 km below sea level (in fact, the Earth is not perfectly spherical, and the polar radius, 6,357 km, is slightly smaller than the equatorial one, 6,378 km). Notwithstanding the technical impossibility of this feat, namely owing to the immense pressures and temperatures that exist inside the Earth, and to the physical state of its inner layers, it is interesting to imagine what would be the depth dependence of air pressure within this imaginary shaft. Assuming for simplicity that air temperature and Earth's density are both uniform, Eq. 4 applies, with an acceleration of gravity given by

$$g(z) = \frac{4}{3}\pi G\rho\,(R+z) = g_0\left(1+\frac{z}{R}\right). \tag{15}$$

With $-R < z < 0$. In this way, Eq. 4 becomes

$$p(z) = p_0 \exp\left[-\frac{mg_0}{kT}\left(1+\frac{z}{2R}\right)z\right] = p_0 \exp\left[-\left(1+\frac{z}{2R}\right)\frac{z}{H}\right]. \tag{16}$$

The dependence is similar to Eq. 1, apart from the multiplicative factor (<1) in the exponential's argument that slightly reduces the variation, owing to the decrease of g with depth. The deepest gold mines in South Africa presently attain a depth of 3.9 km, for which one obtains $p = 1.6$ atm, in good agreement with observations.

It may be mentioned that the deepest Earth drilling ever carried out was done in Russia, in the Kola peninsula, near the Norwegian border (Kola Superdeep Borehole project) and attained 12.3 km in 1989. The hole was very narrow, less than 25 cm in diameter. As the temperature at the bottom attained 180 °C further drilling, initially planned to go down to 15 km, was abandoned. In connection to this, a "well to Hell hoax" appeared on the Internet in 1997, and is still in circulation (Wikipedia). According to this "urban legend", the drilling was stopped after Hell was hit and screams of the damned recorded. This extreme case is another reminder that notwithstanding all technical advances our civilization has experienced, scientific culture and scientific reasoning are far from being universal, and remain the privilege of a learned minority.

Equation 16 predicts a pressure of 2 atm for $z = -5.8$ km, and a pressure of 1,000 atm for $z = -58$ km, which is roughly the mirror distance of the stratopause with respect to sea level. For $z = -R$ the calculated pressure is

$$p(-R) = p_0 \exp\left(\frac{R}{2H}\right) \simeq 10^{165} p_0. \tag{17}$$

Now this value is meaningless, as the air ceases to behave as an ideal gas for pressures of a few tens of atmospheres.

Assuming that a van der Waals equation with parameters for nitrogen applies, the pressure dependence with depth is initially that expected from Eq. 16, but after the first 50 km approximately it starts to diverge (see Fig. 3 below). Note that nitrogen is supercritical at room temperature and that condensation cannot occur in the van der Waals picture. Computation of the pressure dependence with depth along the lines described in [2] is now presented: For a van der Waals gas, the barometric equation is [2]

$$\frac{dC}{dz} = -\frac{gMC}{\frac{RT}{(1-bC)^2} - 2aC}, \tag{18}$$

where M is the molar mass, R the gas constant, a and b the van der Waals parameters, and C is the molar concentration. Using Eqs. 15 and 18 can be rewritten as

$$\frac{dC}{dz} = -\frac{\frac{1}{H}\left(1+\frac{z}{R}\right)}{\frac{1}{C(1-bC)^2} - \alpha}, \tag{19}$$

where R is the Earth radius, H is the scale height, and

$$\alpha = \frac{2a}{RT}. \tag{20}$$

Equation 19 can be integrated to give

$$(C - C_0)\left[\frac{b}{(1-bC)(1-bC_0)} - \alpha\right] + ln\frac{C(1-bC_0)}{C_0(1-bC)} = -\left(1+\frac{z}{2R}\right)\frac{z}{H}. \tag{21}$$

For $a = 1.408\,\mathrm{dm^6\,bar\,mol^{-2}} = 1.408 \times 10^4\,\mathrm{dm^5\,kg\,s^{-2}\,mol^{-2}}$ and $b = 0.03913\,\mathrm{dm^3\,mol^{-1}}$ [21], one has $\alpha = 0.1180\,\mathrm{dm^3\,mol^{-1}}$. According to the van der Waals equation, the pressure is given by [2]

$$p = \frac{RT}{\frac{1}{C} - b} - aC^2. \tag{22}$$

In Figs. 2–4 are shown the computed concentration and pressure dependences on depth, using Eqs. 21 and 22.

A simple approximate equation that relates pressure and depth for a supercritical fluid can be obtained from Eqs. 21 and 22 and is

$$p = \frac{RT}{b} W\left(\frac{bp_{id}}{RT}\right), \tag{23}$$

where $W(x)$ is the Lambert function, i.e., the function that satisfies $W(x)e^{W(x)} = x$ (this function is implemented in technical computing software e.g. as *ProductLog*[x] in *Mathematica*) and p_{id} is the pressure for an ideal gas, given by Eq. 16. Note that for

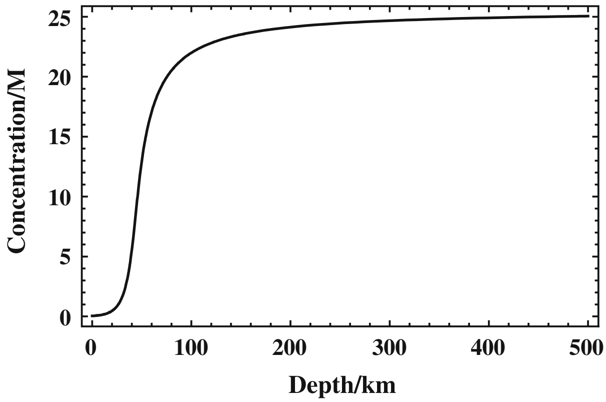

Fig. 2 Air concentration (M) as a function of depth (km) according to Eq. 21

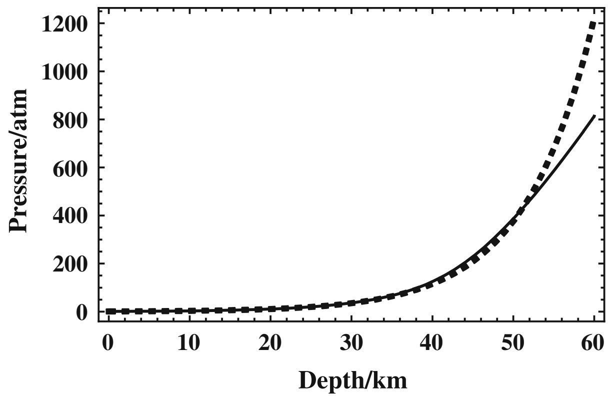

Fig. 3 Pressure (atm) of air as a function of depth (km) for the first 60 km according to Eqs. 21 and 22 (*solid line*). Also shown (*dashed line*) is the result for an ideal gas (Eq. 16)

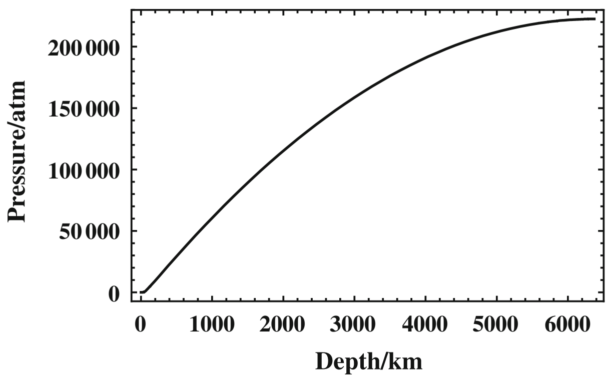

Fig. 4 Pressure (atm) of air as a function of depth (km) according to Eqs. 21 and 22

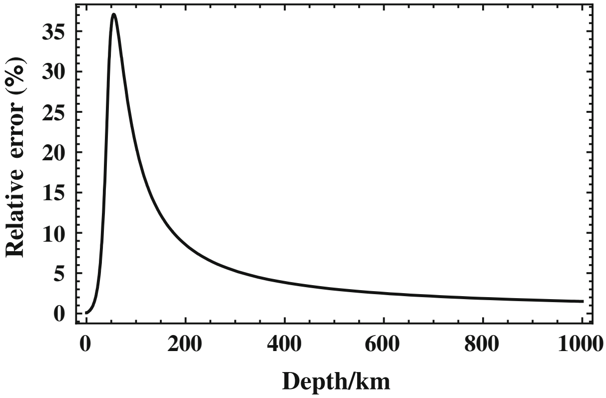

Fig. 5 Relative error (%) corresponding to the use of Eq. 23 as a function of depth (km). The approximate Eq. 23 underestimates the van der Waals result, with a maximum error of 37% at 55 km, but is a good approximation for depths below 20 km and for depths above 250 km

small x one has $W(x) \simeq x$ hence p reduces to p_{id} for small depths. The relative error (in %) is displayed in Fig. 5. The equation is therefore quite useful to complement Eq. 16 in the high depth regime ($-z > 250$ km).

It is seen that the van der Waals equation and the ideal gas equation give similar results down to 50 km. The van der Waals solution for $z = -R$ is $C = 25.5$ M, which is very close to $1/b$. Note that the concentration of liquid nitrogen at 1 atm and 77 K is somewhat higher, 28.9 M, which means that the computed concentration is very likely underestimated owing to the fact that b is a constant obtained from fitting experimental data corresponding to relatively low densities. The pressure corresponding to a concentration of 25.5 M is 223,000 atm. This is again a very rough estimate, as the van der Waals equation is not expected to be accurate for very high pressures. Indeed, it is known that nitrogen at room temperature solidifies at 24,000 bar [22], a phase transition that cannot of course be predicted by the van der Waals equation. This would roughly correspond to a depth of 420 km. Such a depth is still in the mantle part of the Earth, which is known to be solid (but somewhat hot!). It is thus amazing to think that if an *isothermal* shaft with a depth of a few hundred kilometers could be dug in our planet, air would rush into it, and the lower part of the shaft would become filled with *solid* air (a full simulation should account for the initial release of heat on account of the air's decreased gravitational potential energy). In this way, the neglect of air pressure when discussing deep tunnels or shafts reminds us of von Neumann's "dry water" (i.e., liquid water assumed to have zero viscosity in unrealistic hydrodynamical models).

It should be noted that in the giant planets (Jupiter, Saturn), rich in hydrogen, the inner pressure is so high that the core is believed to be in the form of metallic hydrogen. In even more massive bodies (stars) the enormous pressures (and temperatures) render nuclear fusion viable, not to mention the quantum and relativistic effects that exist in neutron stars and black holes.

Curiously, and apart from [12], in none of the mentioned works on the gravitational train is the variation of pressure with depth discussed. At most, it is mentioned that air resistance has to be eliminated by means of vacuum pumps. Nevertheless, both cord links (St. Petersburg–Moscow and Boston–Washington, D.C.) would involve a maximum penetration of about 8 km below the Earth surface. At this depth, the air pressure, estimated with Eqs. 21 and (22), is already 2.8 atm. The effect of gravity on air pressure thus implies that deep tunnels must be kept in a vacuum, not only because of air drag but also on account of the mechanical resistance of the transportation device, as its interior must be kept at or near 1 atm without too much weight. Another major challenge would be the construction of stable tunnel walls. Such a convenient and ecological means of transportation may (literally) never see the daylight, but who knows?

4 Conclusions

The assumptions leading to the barometric formula were discussed, with remarks on the influence of temperature, gravitational field, Earth rotation, and non-equilibrium conditions. A generalization of the barometric formula based on the van der Waals equation for negative heights, e.g., for pressure inside shafts and deep tunnels, was also presented, and some applications discussed, along with relevant historical aspects.

Let us close this study with Thiele's words on Euler [11]: 'The physical word was an occasion to apply mathematics, and if it failed to fit his analysis, it was the physical world, not the mathematics, which was in error", and also with the illuminating Euler's words cited by Thiele: "I did not consider it necessary to confirm my theory by any experiments. For this theory is derived from the most certain principles of mechanics. Hence there can be no doubt, whether it be true or have a place in praxi."

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