Lecture 9: More on Quantum Physics

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Overview

- Teleportation
- Measurement in Computational Basis
- Observables
  - Pauli Z Basis
- Expectation Values
- Uncertainty
  - Commutator
- General Heisenberg's Uncertainty Principle
  - Heisenberg's Uncertainty Principle
- Quantum Tunneling
  - Quantum Annealing
- Two-state vector formalism (TSVF)

Teleportation

- It is possible to teleport a qubit from one location to another using an *ebit*
- The two qubits in an *ebit* behave as one unit, even if the qubits are separated
    - Ebit: state with quantum entanglement
- This nonlocal interaction is not limited by speed of light, not mediated by the distance
- The qubit is transferred from one point to another without *traversing* the physical space

Ebit

- We start with the state $|00\rangle$

$$H_1 \otimes I \cdot |00\rangle = (H_1 \cdot |0\rangle) \otimes |0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle = \frac{|00\rangle + |10\rangle}{\sqrt{2}}$$

- To this state represented by the two qubit we apply CX (=cX) gate

$$CX \cdot \left(\frac{|00\rangle + |10\rangle}{\sqrt{2}}\right) = \frac{CX \cdot |00\rangle + CX \cdot |10\rangle}{\sqrt{2}} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

- $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ two qubits that are *entangled* are also called an ebit

- We *separate* the two qubits of the ebit over a distance on *two places* **A** and **B**

$$\frac{|0_A\rangle|0_B\rangle + |1_A\rangle|1_B\rangle}{\sqrt{2}}.$$

In the first step of the teleportation of the qubit $\alpha \cdot |0_A\rangle + \beta \cdot |1_A\rangle$ from the place A to the place B we interact with the corresponding *ebit*

$$(\alpha \cdot |0_A\rangle + \beta \cdot |1_A\rangle) \otimes \left(\frac{|0_A\rangle|0_B\rangle + |1_A\rangle|1_B\rangle}{\sqrt{2}} \right)$$

$$\frac{\alpha \cdot (|0_A\rangle|0_A\rangle|0_B\rangle + |0_A\rangle|1_A\rangle|1_B\rangle) + \beta \cdot (|1_A\rangle|0_A\rangle|0_B\rangle + |1_A\rangle|1_A\rangle|1_B\rangle)}{\sqrt{2}}.$$

After the interaction there are two qubits on the location A and on the location B. In the second step we apply the M_{CNOT} quantum gate to the

- In the second step we apply the Controlled Not quantum gate *CX* to the first two qubits at the location **A** and on the location **B** we do noting

$$CX|00\rangle = |00\rangle \qquad CX|01\rangle = |01\rangle,$$

$$CX|10\rangle = |11\rangle \qquad CX|11\rangle = |10\rangle.$$

$$(CX \otimes I_1) \cdot \frac{1}{\sqrt{2}} \cdot (\alpha \cdot (|0_A\rangle|0_A\rangle|0_B\rangle + |0_A\rangle|1_A\rangle|1_B\rangle) +$$

$$\beta \cdot (|1_A\rangle|0_A\rangle|0_B\rangle + |1_A\rangle|1_A\rangle|1_B\rangle)) =$$

$$\frac{\alpha \cdot (|0_A\rangle|0_A\rangle|0_B\rangle + |0_A\rangle|1_A\rangle|1_B\rangle) + \beta \cdot (|1_A\rangle|1_A\rangle|0_B\rangle + |1_A\rangle|0_A\rangle|1_B\rangle)}{\sqrt{2}}.$$

This can be rewritten as

$$\frac{\alpha \cdot |0_A\rangle \otimes (|0_A\rangle|0_B\rangle + |1_A\rangle|1_B\rangle) + \beta \cdot |1_A\rangle \otimes (|1_A\rangle|0_B\rangle + |0_A\rangle|1_B\rangle)}{\sqrt{2}}.$$

- In the third step we apply the H_1 quantum gate to the first qubit at the location **A** and on the location **B** we do noting

$$(H_1 \otimes I_2) \cdot \frac{1}{\sqrt{2}} \cdot (\alpha \cdot |0_A\rangle \otimes (|0_A\rangle|0_B\rangle + |1_A\rangle|1_B\rangle) +$$

$$\beta \cdot |1_A\rangle \otimes (|1_A\rangle|0_B\rangle + |0_A\rangle|1_B\rangle)) =$$

$$\frac{1}{2} \cdot (\alpha \cdot (|0_A\rangle + 1_A\rangle) \otimes (|0_A\rangle|0_B\rangle + |1_A\rangle|1_B\rangle) +$$

$$+\beta \cdot (|0_A\rangle - 1_A\rangle) \otimes (|1_A\rangle|0_B\rangle + |0_A\rangle|1_B\rangle)) =$$

$$\frac{1}{2} \cdot (\alpha \cdot |0_A\rangle 0_A\rangle|0_B\rangle + \alpha \cdot |0_A|1_A\rangle|1_B\rangle +$$

$$\alpha \cdot |1_A\rangle 0_A\rangle|0_B\rangle + \alpha \cdot |1_A|1_A\rangle|1_B\rangle +$$

$$\beta \cdot |0_A\rangle|1_A\rangle|0_B\rangle + \beta \cdot |0_A\rangle 0_A\rangle|1_B\rangle - \beta \cdot |1_A\rangle 1_A\rangle|0_B\rangle - \beta \cdot |1_A\rangle|0_A\rangle|1_B\rangle)$$

$$\frac{1}{2} \cdot (\alpha \cdot |0_A\rangle 0_A\rangle|0_B\rangle + \alpha \cdot |0_A|1_A\rangle|1_B\rangle +$$

$$\alpha \cdot |1_A\rangle 0_A\rangle|0_B\rangle + \alpha \cdot |1_A|1_A\rangle|1_B\rangle +$$

$$\beta \cdot |0_A\rangle|1_A\rangle|0_B\rangle + \beta \cdot |0_A\rangle 0_A\rangle|1_B\rangle - \beta \cdot |1_A\rangle 1_A\rangle|0_B\rangle - \beta \cdot |1_A\rangle|0_A\rangle|1_B\rangle)$$

- After rewriting the equation, we get the following representation

$$\frac{1}{2} \cdot (|0_A\rangle|0_A\rangle \otimes (\alpha \cdot |0_B\rangle + \beta \cdot |1_B\rangle) + |0_A\rangle|1_A\rangle \otimes (\alpha \cdot |1_B\rangle + \beta \cdot |0_B\rangle) +$$

$$|1_A\rangle|0_A\rangle \otimes (\alpha \cdot |0_B\rangle - \beta \cdot |1_B\rangle) + |1_A\rangle|1_A\rangle \otimes (\alpha \cdot |1_B\rangle - \beta \cdot |0_B\rangle)).$$

- In the fourth step a measurement of the first two qubits at the place **A** is done
- There are four possible results; each of them has an equal probability of being measured.

$|\mathbf{00}\rangle$ is measured the state collapses at place B to

$$\alpha \cdot |0\rangle + \beta \cdot |1\rangle$$

at place B no correction is nesseascary, the qubit described by its amplitude distribution was teleported.

$|\mathbf{01}\rangle$ is measured the state collapses at place B to

$$\alpha \cdot |1\rangle + \beta \cdot |0\rangle$$

at place B a correction is necessary to reconstruct the teleported qubit. NOT gate is applied.

$$X \cdot (\alpha \cdot |1\rangle + \beta \cdot |0\rangle) = \alpha \cdot |0\rangle + \beta \cdot |1\rangle$$

$|\mathbf{10}\rangle$ is measured the state collapses at place B to

$$\alpha \cdot |0\rangle - \beta \cdot |1\rangle$$

at place B a correction is necessary to reconstruct the teleported qubit. Z gate is applied.

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Z \cdot (\alpha \cdot |0\rangle - \beta \cdot |1\rangle) = \alpha \cdot |0\rangle + \beta \cdot |1\rangle$$

$|\mathbf{11}\rangle$ is measured the state collapses at place B to

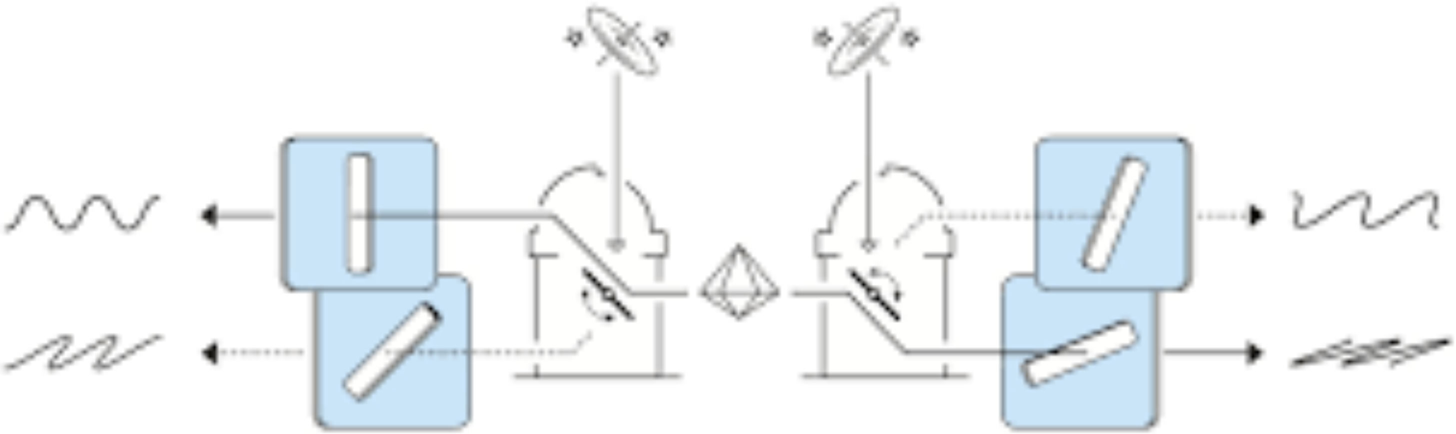
$$\alpha \cdot |1\rangle - \beta \cdot |0\rangle$$

at place B a correction is necessary to reconstruct the teleported qubit. M_{NOT} gate and then the Z gate is applied.

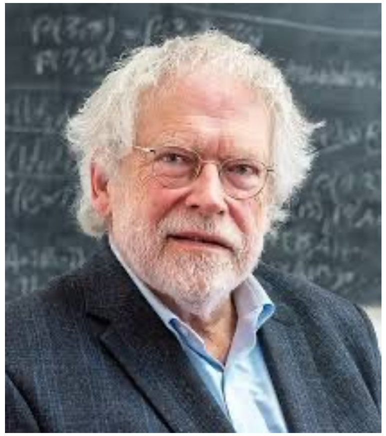
$$Z \cdot X \cdot (\alpha \cdot |1\rangle - \beta \cdot |0\rangle) = \alpha \cdot |0\rangle + \beta \cdot |1\rangle.$$

- For the teleportation of qubits classical **communication is required**
- To indicate how to reconstruct *one* qubit two bits have to be send over a classical channel, since one teleported qubit can take **four different superpositions**

- It follows that an **ebit** cannot be used to send or teleport information without a classical channel



Anton Zeilinger later conducted more tests of Bell inequalities. He created entangled pairs of photons by shining a laser on a special crystal, and used random numbers to shift between measurement settings. One experiment used signals from distant galaxies to control the filters and ensure the signals could not affect each other.



Anton Zeilinger is an Austrian quantum physicist and Nobel laureate in physics of 2022.

Anton Zeilinger's groundbreaking experiment in quantum teleportation stands as a pivotal moment in the history of quantum physics.

In 2017, his Austrian research team achieved a remarkable feat by teleporting quantum information across an astonishing distance of **143** kilometers.

Measurement in Computational Basis

A state vector is just a particular instance of a ket vector

- It is specified by a particular **choice of basis** and refers to **observable** that can have some system properties
- In quantum computation during measurement the "*computational basis"* is used

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

2 dimensional Hilbert space representing 1 qubit

we used the *computational basis* $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, …

$$|0000\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |0001\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |0010\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \cdots, |1111\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

16 dimensional Hilbert space representing 4 qubits

Observables

- Observables are the things that are measured.
  - They are some kind of machines with an input and output.
  - The input is a state vector, and the output represents the result represented by a real number
- Observables can measure the input from different perspectives by rotating the coordinate system
  - Different perspectives can lead to different results.
- In quantum computing the coordinate system is fixed
  - "**Computational basis**"
- Different observables lead to the Heisenberg uncertainty principle and to the quantum tunneling effect
  - Quantum tunneling plays an important role in quantum biology.

Observables

- In quantum physics observables are linear **self-adjoint** operators represented by a **Hermitian** matrix

$$A^* = A \text{ with } a_{ij} = \overline{a_{ji}}.$$

- For real values, the Hermitian matrix is a symmetrical matrix with

$$A^T = A$$

- Hermitian matrices have real eigenvalues and the corresponding eigenvectors are orthogonal

$$A \cdot |x\rangle = \lambda \cdot |x\rangle$$

with eigenvalue λ and eigenvector |x⟩

Notation

- To indicate that the eigenvalue λ_i corresponds to an eigenvector one writes

$$A \cdot |\lambda_i\rangle = \lambda_i \cdot |\lambda_i\rangle$$

- with λ_i being the eigenvalue and $|\boldsymbol{\lambda}_i\rangle$ being the corresponding eigenvector

- For a state represented by a unit-length vector |**x**⟩ the observable *A* rotates the state into a |**y**⟩

$$|y\rangle = A \cdot |x\rangle$$

- After the measurement, the state is in the eigenstate $|\boldsymbol{\lambda}_i\rangle$
  - it collapses to the **basis state** of *A (Think about KL-Transform, PCA)*
- The distinguishable states are the **orthogonal vectors** that define the **coordinate system** of the measurable operator *A*
- They are the basis states of the coordinate system of the observable and thy are spanned by the **eigenvectors**

- The result of the measurement are the real eigenvalues λ_i of the operator that represent the observable
- The state for which the result of a measurement λ_i corresponds to the eigenvector $|\boldsymbol{\lambda}_i\rangle$
- If the system is in the eigenstate $|\boldsymbol{\lambda}_i\rangle$ then the measurement is λ_i since

$$\lambda_i \cdot |\lambda_i\rangle = A \cdot |\lambda_i\rangle.$$

Born's Rule

- The state vector $|\mathbf{x}\rangle$ of the system and the observable A is measured, the probability of the observable λ_i According to the Born rule is

eigenvalue

eigenvector

$$p(\lambda_i) = \langle x|\lambda_i\rangle\langle\lambda_i|x\rangle = |\langle x|\lambda_i\rangle|^2$$

  - The Born rule is a postulate of quantum mechanics that gives the probability that a measurement of a quantum system will yield a given result
- It is always positive and real and can be identified with probability

- If λ_1 and λ_2 are two unequal eigenvalues of an observable, then the corresponding eigenvectors are orthogonal
- The observable

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

has the eigenvalues

$$\lambda_1 = 1, \quad \lambda_2 = -1$$

and the corresponding eigenvectors are

$$|\lambda_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\lambda_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

- However, if two eigenvalues are equal the corresponding eigenvectors can be orthogonal as well
  - The observable represented by the identity matrix

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

has equal eigenvalues

$$\lambda_1 = 1, \quad \lambda_2 = 1$$

and the corresponding eigenvectors are

$$|\lambda_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\lambda_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Example

Fo the state vector

$$|x\rangle = \frac{1}{\sqrt{2}} \cdot |0\rangle + \frac{1}{\sqrt{2}} \cdot |1\rangle \qquad\qquad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

acting with A gives

$$A \cdot |x\rangle = \frac{1}{\sqrt{2}} \cdot |0\rangle - \frac{1}{\sqrt{2}} \cdot |1\rangle.$$

After measuring the result is either in $\lambda_1 = 1$ leaving the system in the state

$$|\lambda_1\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

or $\lambda_1 = -1$ leaving the system in the state

$$|\lambda_2\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Pauli Z Basis

$$A = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- A measurement in the **Pauli Z basis** is the same as the **computational basis** measurement, projecting the state onto one of the states $|0\rangle$ or $|1\rangle$ (the eigenstates of Pauli Z matrix)

Expectation Values

An average of all possible eigenvalue for the observable A is defined as

$$\langle A \rangle = \sum_{i=1}^{n} \lambda_i \cdot p(\lambda_i).$$

weighted mean of eigenvalues

We can represent a quantum state $|x\rangle$ by a vector of the length one by the orthonormal basis of A

$$|x\rangle = \sum_{i=1}^{n} \alpha_i \cdot |\lambda_i\rangle$$

with

eigenvalue

$$A \cdot |x\rangle = \sum_{i=1}^{n} \alpha_i \cdot A \cdot |\lambda_i\rangle = \sum_{i=1}^{n} \alpha_i \cdot \lambda_i \cdot |\lambda_i\rangle$$

eigenvector

$$A \cdot |\lambda_i\rangle = \lambda_i \cdot |\lambda_i\rangle.$$

With

$$\langle\lambda_i| \cdot A = \lambda_i \cdot \langle\lambda_i|$$

we get

$$\langle x| \cdot A = \sum_{i=1}^{n} \alpha_i^* \cdot A \cdot \langle\lambda_i| = \sum_{i=1}^{n} \alpha_i^* \cdot \lambda_i \cdot \langle\lambda_i|.$$

Since the eigenvectors are orthonormal

$$\langle\lambda_i|\lambda_i\rangle = 1$$

and

$$\alpha_i^* \cdot \alpha_i = p(\lambda_i)$$

$$\langle x|A \cdot x\rangle = \langle x|A|x\rangle = \langle A\rangle = \sum_{i=1}^{n}(\alpha_i^* \cdot \alpha_i) \cdot \lambda_i = \sum_{i=1}^{n} \lambda_i \cdot p(\lambda_i).$$

- We have a rule for computing the averages
- After many measurements the result converge to the average of the eigenvalue

$$\langle A\rangle = \sum_{i=1}^{n} \lambda_i \cdot p(\lambda_i).$$

- We call the value ⟨A⟩ the **expected** value of observable *A* in state **x**
  - Each eigenvalue corresponds to the collapse on the eigenstate (basis state of the observable)

Uncertainty

- For two observable *A* and *B* there is a basis of state-vectors $|\boldsymbol{\lambda_i},\boldsymbol{\mu_j}\rangle$ that are **the same eigenvectors** of both observables with

$$A \cdot |\lambda_i, \mu_j\rangle = \lambda_i \cdot |\lambda_i, \mu_j\rangle$$

$$B \cdot |\lambda_i, \mu_j\rangle = \mu_j \cdot |\lambda_i, \mu_j\rangle$$

and

$$A \cdot B \cdot |\lambda_i, \mu_j\rangle = A \cdot \mu_j \cdot |\lambda_i, \mu_j\rangle = \lambda_i \cdot \mu_j \cdot |\lambda_i, \mu_j\rangle.$$

Since the eigenvalues are real numbers they are commutative,

$$A \cdot B \cdot |\lambda_i, \mu_j\rangle = B \cdot A \cdot |\lambda_i, \mu_j\rangle = \lambda_i \cdot \mu_j \cdot |\lambda_i, \mu_j\rangle = \mu_j \cdot \lambda_i \cdot |\lambda_i, \mu_j\rangle.$$

$$A \cdot B \cdot |\lambda_i, \mu_j\rangle = B \cdot A \cdot |\lambda_i, \mu_j\rangle = \lambda_i \cdot \mu_j \cdot |\lambda_i, \mu_j\rangle = \mu_j \cdot \lambda_i \cdot |\lambda_i, \mu_j\rangle.$$

or

$$A \cdot B \cdot |\lambda_i, \mu_j\rangle - B \cdot A \cdot |\lambda_i, \mu_j\rangle = 0$$

$$(A \cdot B - B \cdot A) \cdot |\lambda_i, \mu_j\rangle = 0$$

$$(A \cdot B - B \cdot A) = 0$$

Commutator

The commutator between two operators (observables) A and B is defined as

$$[A, B] := A \cdot B - B \cdot A$$

with

$$[A, B] = 0 \Longleftrightarrow A \cdot B = B \cdot A.$$

We can write in a shorter way

$$[A, B] \cdot |\lambda_i, \mu_j\rangle = 0.$$

- If two observables commute, then there is a complete *simultaneous eigenvectors* of two observables
- Two observables can be *simultaneously* measured if they commute
  - One partial measurement does not influence other partial measurement
- However usually matrix multiplication is not commutative

$$A \cdot B \neq B \cdot A \Longleftrightarrow [A, B] \neq 0.$$

General Heisenberg's Uncertainty Principle

The expected value of A is the ordinary average

$$\langle x|A|x\rangle = \langle A\rangle = \sum_{i=1}^{n}(\alpha_i^* \cdot \alpha_i)\cdot\lambda_i = \sum_{i=1}^{n}\lambda_i \cdot p(\lambda_i).$$

The uncertainty in A corresponds to the standard deviation by subtracting from it its expectation value

$$\overline{A} = A - \langle A\rangle \cdot I = A - \langle A\rangle$$

with I being the identity matrix (usually one does not write the identity matrix). The probability distribution of $\overline{A}$ is the same as of A, however its average is zero.

$$\overline{A} = A - \langle A \rangle \cdot I$$

average is zero. For the eigenvalues a_i of A the eigenvalue of $\overline{A}$ is

$$\overline{a_i} = a_i - \langle A \rangle,$$

it is shifted so its average is zero. The square of standard deviation of A is defined as

$$(\Delta A)^2 = \sum_{i=1}^{n} \overline{a}_i^2 \cdot p(a_i) = \sum_{i=1}^{n} (a_i - \langle A \rangle)^2 \cdot p(a_i) = \langle x | \overline{A}^2 | x \rangle.$$

If the expectation of A is zero, we do not need to shift and

$$(\Delta A)^2 = \sum_{i=1}^{n} a_i^2 \cdot p(a_i) = \langle x | A^2 | x \rangle.$$

Cauchy-Schwartz Inequality

With

$$\|\mathbf{y}\| = \sqrt{\langle \mathbf{y}|\mathbf{y}\rangle}, \quad \|\mathbf{z}\| = \sqrt{\langle \mathbf{z}|\mathbf{z}\rangle}$$

and

$$\|\mathbf{y} + \mathbf{z}\| = \sqrt{(\langle \mathbf{y}| + \langle \mathbf{z}|) \cdot (\langle \mathbf{y}| + \langle \mathbf{z}|)}$$

and the triangle inequality

$$\|\mathbf{y}\| + \|\mathbf{z}\| \geq \|\mathbf{y} + \mathbf{z}\|$$

we will arrive at the Heisenberg's uncertainty principle. We square the triangle inequality

$$\|\mathbf{y}\|^2 + |\mathbf{z}\|^2 \geq (\langle \mathbf{y}| + \langle \mathbf{z}|) \cdot (\langle \mathbf{y}| + \langle \mathbf{z}|)$$

$$\|\mathbf{y}\|^2 + |\mathbf{z}\|^2 + 2 \cdot \|\mathbf{y}\| \cdot |\mathbf{z}\| \geq \|\mathbf{y}\|^2 + |\langle \mathbf{y}|\mathbf{z}\rangle + \langle \mathbf{z}|\mathbf{y}\rangle| + \|\mathbf{z}\|^2$$

and we arrive at the Cauchy-Schwartz inequality

$$2 \cdot \|\mathbf{y}\| \cdot \|\mathbf{z}\| \geq |\langle \mathbf{y}|\mathbf{z}\rangle + \langle \mathbf{z}|\mathbf{y}\rangle|.$$

With two observables A and B we define

$$|\mathbf{y}\rangle = A \cdot |\mathbf{x}\rangle, \quad |\mathbf{z}\rangle = i \cdot B \cdot |\mathbf{x}\rangle$$

substituting it in Cauchy-Schwartz inequality leads to

$$2 \cdot \sqrt{\langle x|A^2|x\rangle \cdot \langle x|B^2|x\rangle} \geq |\langle x|AB|x\rangle - \langle x|BA|x\rangle|$$

with the minus sign resulting for i. Simplified with the notation of a commutator

$$2 \cdot \sqrt{\langle A^2\rangle \cdot \langle B^2\rangle} \geq |\langle x|[A, B]|x\rangle|$$

General Version of Heisenberg Uncertainty Principle

$$2 \cdot \sqrt{\langle A^2 \rangle \cdot \langle B^2 \rangle} \geq |\langle x|[A, B]|x\rangle|$$

If the expectation of A and B is zero,

If the expectation of A is zero, we do not need to shift and

$$(\Delta A)^2 = \sum_{i=1}^{n} a_i^2 \cdot p(a_i) = \langle x|A^2|x\rangle.$$

$$(\Delta A)^2 = \langle x|A^2|x\rangle = \langle A^2 \rangle, \quad (\Delta B)^2 = \langle B^2 \rangle$$

we write

$$\Delta A \cdot \Delta B \geq \frac{|\langle x|[A, B]|x\rangle|}{2}.$$

uncertainties of the two observables A and B in a state |x⟩

Example

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Different properties represented by different observables

with

$$[A, B] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$[A, B] = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$$

then

$$\Delta(A)\Delta(B) \geq \frac{\left| \langle 0| \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} |0\rangle \right|}{2} = 1$$

$\Delta(A)$ and $\Delta(B)$ must be grater than 0 since their product needs to be greater equal to one.

Heisenberg's Uncertainty Principle

A state-vector that is expressed as an integral rather then a sum is continuous and corresponds to a wave function $\psi(a, b, c, \cdots)$ with the observables $A, B, C, \cdots$ basis vectors $|a, b, c, \cdots\rangle$ and the eigenvalues $a, b, c, \cdots$. The probability of the commuting observables $a, b, c, \cdots$ is

$$p(a, b, c, \cdots) = \psi^*(a, b, c, \cdots)\psi(a, b, c, \cdots).$$

A particle is described by a wave function $\psi(x)$ that indicates its position x and the inverse Fourier transform $\tilde{\psi}(p)$ of its momentum p

$$\psi(x) = \frac{1}{\sqrt{2 \cdot \pi}} \int dpe^{\frac{ipx}{h}} \tilde{\psi}(p)$$

and the Fourier transform

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2 \cdot \pi}} \int dpe^{\frac{-ipx}{h}} \psi(x)$$

with two observables for position X and momentum P. For a position x the function is multiplied with the value x, the particle is located at the point x, it is zero elsewhere

$$X\psi(x) = x\psi(x).$$

The momentum corresponds to the derivative with $-i$ to make the operator Hermitian and the Planck's constant h for the correct relation

$$P\psi(x) = -ih\frac{d\psi(x)}{dx}.$$

For two measurments

$$XP\psi(x) = -ihx\frac{d\psi(x)}{dx}$$

and

$$PX\psi(x) = -ih\frac{d(x\psi(x))}{dx} = -ihx\frac{d\psi(x)}{dx} - ih\psi(x)$$

with

$$[X, P]\psi(x) = XP\psi(x) - PX\psi(x) = ih\psi(x)$$

The general uncertainty principle is given by

$$\Delta X \cdot \Delta P \geq \frac{|\langle x|[X, P]|x\rangle|}{2}$$

$$\Delta X \cdot \Delta P \geq \frac{|ih\langle\Psi|\Psi\rangle|}{2}.$$

- Since ⟨Ψ|Ψ⟩ = 1

$$\Delta X \cdot \Delta P \geq \frac{h}{2}.$$

  - The Heisenberg's uncertainty principle is applied to the momentum and location of moving particles and is represented by where x is the position and p the momentum of a particle and h the Planck constant
- It represents the relation between

$$\Delta(x)\Delta(p) \approx h$$

Quantum Tunneling

The relation

$$\Delta(x)\Delta(p) \approx h$$

is also valid for energy E and time t

$$\Delta(E)\Delta(t) \approx h.$$

- This arrangement contradicts the first law of thermodynamics, which is the conservation of energy, where the sum of the amount of energy of a system *remains constant*

- In quantum physics, there is uncertainty between the energy E and the time *t*

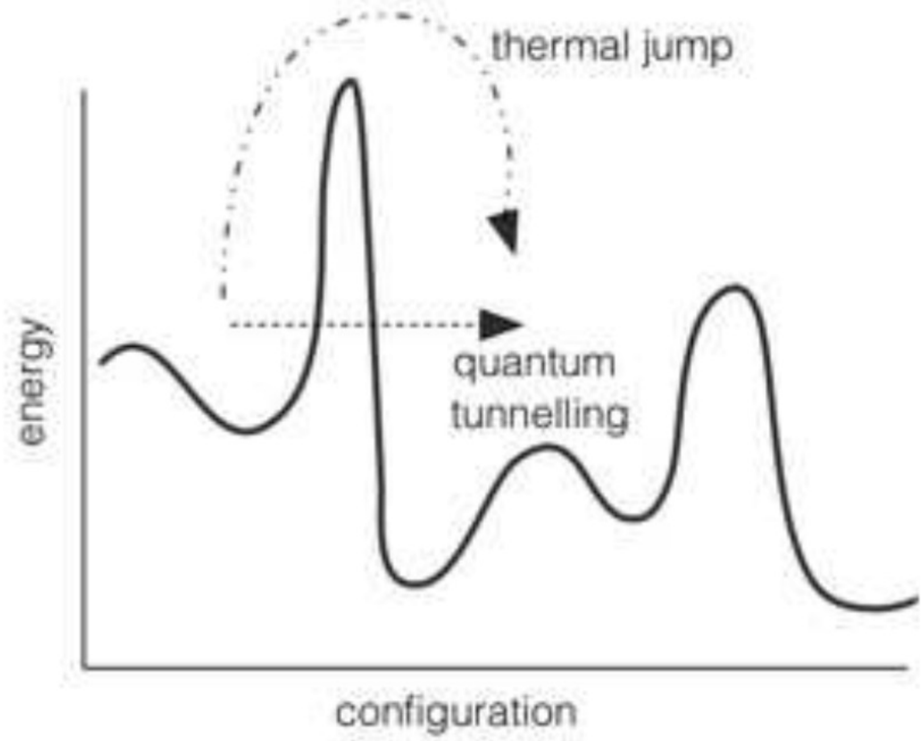
$$\Delta(E) \approx \frac{h}{\Delta(t)}.$$

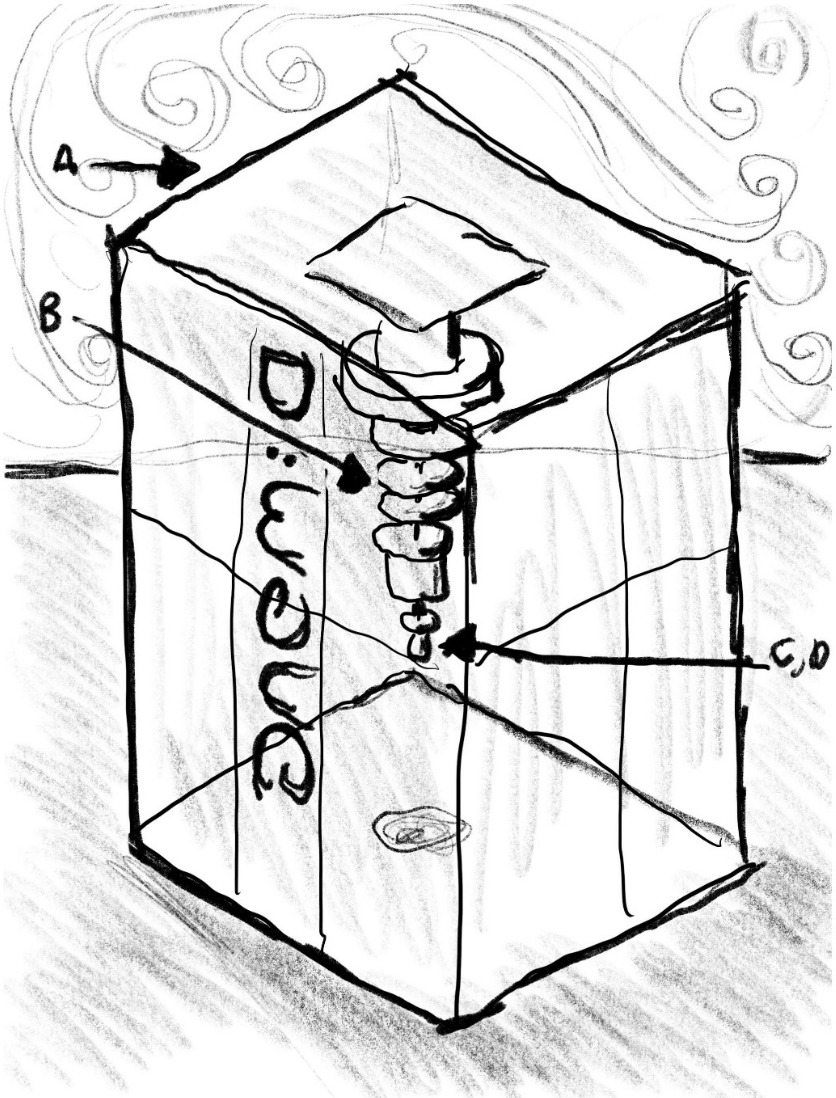
- This uncertainty means that some energy can be borrowed, to overcome some mountain and go out of a minimum as long as we repay it in the time interval

$$\Delta(t) \approx \frac{h}{\Delta(E)}.$$

Quantum Annealing

- Quantum tunneling allows classical energy barriers to be overcome to some extend by the time-energy uncertainty
- A particle cam tunnel through a region in which the potential energy function exceeds the total energy of the particle
- The wave function has some probability of tunneling through a barrier rather than over it





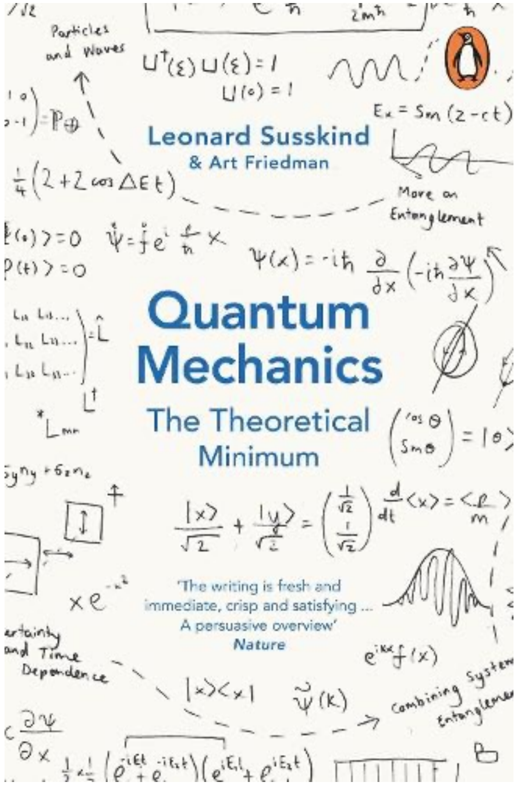
A). Refrigeration system uses liquid helium to cool the D-Wave chip.

B) Gold-plated copper disks draw heat up and away from the chip to keep vibration and other energy from disturbing the quantum state of the processor.

C) A grid of hundreds of tiny niobium loops, which serve as qubits. When cooled, they exhibit a quantum mechanical behavior.

D) Noise shields.

The D-Wave quantum annealer is not a general quantum computer.



Leonard Susskind (1940) is an American theoretical physicist, professor of theoretical physics at Stanford University

Susskind is widely regarded as one of the fathers of string theory

Susskind argues that our universe may be just **one of an enormous number of universes**, each with different physical properties.

Lecture 1 | Modern Physics: Quantum Mechanics (Stanford)

https://www.youtube.com/watch?v=JzhlfbWBuQ8

Standard QM

$$|\Psi\rangle$$

A QM system at a specific time t is described entirely by a quantum state that is defined by measurements conducted before time t.

Two-state vector formalism (TSVF)

- The *two-state vector* is represented by $\langle\Phi| \quad |\Psi\rangle$

- where the state $\langle\Phi|$ evolves backwards from the future and the state $|\Psi\rangle$ evolves forwards from the past.

Satoshi Watanabe (渡辺 慧, Watanabe Satoshi; 26 May 1910 – 15 October 1993) was a theoretical physicis

Past and future measurements, taken together, provide complete information about a quantum system.

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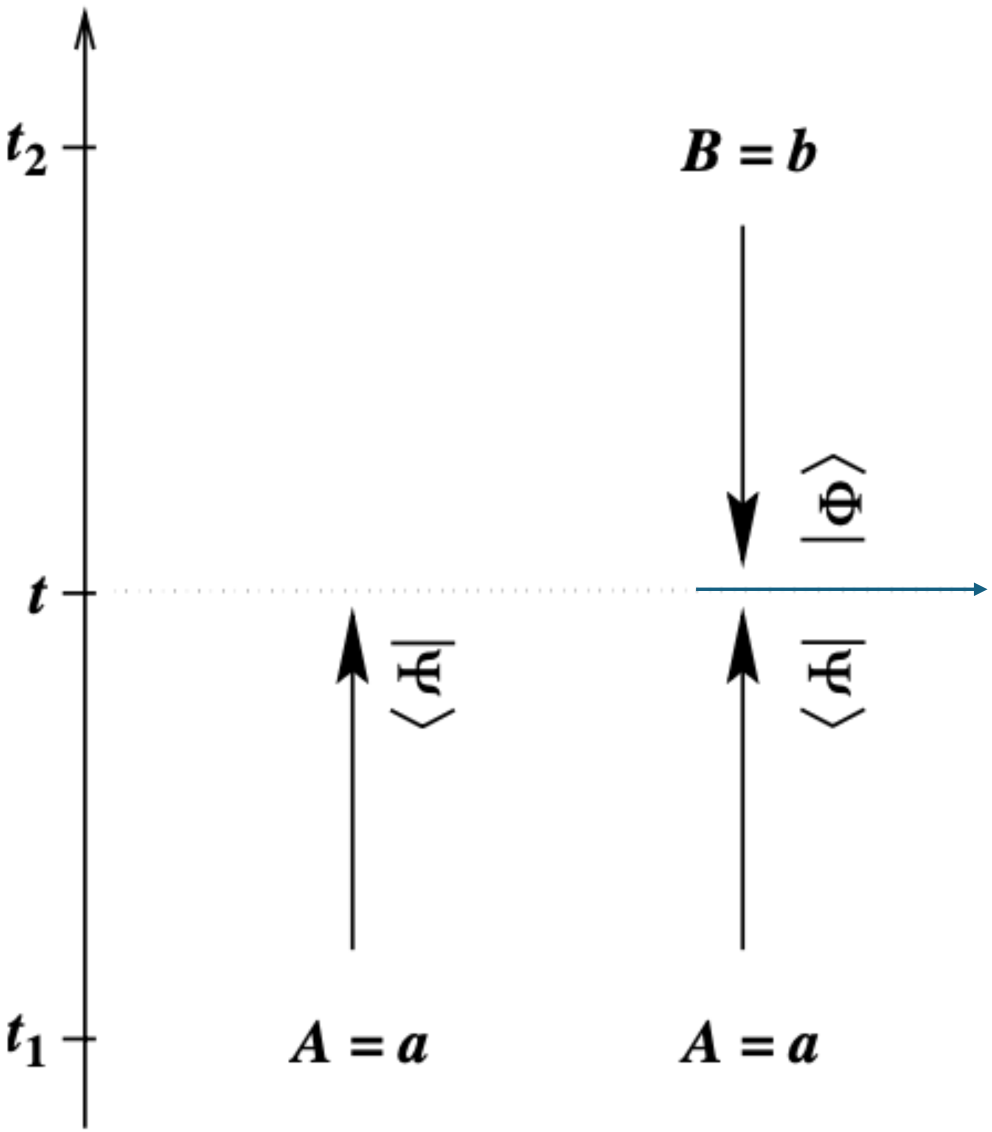
Satoshi Watanabe work rediscovered by Yakir Aharonov, Peter Bergmann and Joel Lebowitz in 1964,



ABL Formula

$$\mathrm{Prob}(c_n) = \frac{|\langle \Phi | \mathbf{P}_{C=c_n} | \Psi \rangle|^2}{\sum_j |\langle \Phi | \mathbf{P}_{C=c_j} | \Psi \rangle|^2}$$

The Pre-selection (The Past):

At an initial time t_1, we prepare the particle so that it has an equal probability of being in any of the three boxes. In quantum terms, this is a superposition state:

$$|\psi_{in}\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle)$$

The Post-selection (The Future):

At a later time t_2, we perform a measurement and "post-select" a very specific final state. We only keep the experimental runs where the particle is found in this exact state:

$$|\psi_{fin}\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle - |3\rangle)$$

$$\langle\Phi|\ |\Psi\rangle = \frac{1}{3}(\langle 1| + \langle 2| - \langle 3|)\ (|1\rangle + |2\rangle + |3\rangle).$$

Suppose we decide to open **Box 1** at time t.

If we calculate the probability of finding the particle in Box 1—given that it *must* end up in our post-selected state at t_2—the mathematics tells us the probability is **100%**.

If we look in Box 1, it will definitely be there.

$$\text{Prob}(\mathbf{P}_1 = 1) = \frac{|\langle\Phi|1\rangle\langle 1|\Psi\rangle|^2}{|\langle\Phi|1\rangle\langle 1|\Psi\rangle|^2 + |\langle\Phi|2\rangle\langle 2|\Psi\rangle + \langle\Phi|3\rangle\langle 3|\Psi\rangle|^2} = \frac{|\frac{1}{3}|^2}{|\frac{1}{3}|^2 + |0|^2} = 1.$$

$$\text{Prob}(\mathbf{P}_2 = 1) = 1.$$

Now, suppose we rewind and run the exact same experiment, but at time t, we decide to open **Box 2** instead.

If we calculate the probability of finding the particle in Box 2—again, given the same initial and final states—the probability is **also 100%**.

The Paradox: How can the particle be guaranteed to be in Box 1 if we look there, but *also* guaranteed to be in Box 2 if we look there instead? Classically, this is impossible. The particle cannot be wholly in two places at once.

Standard quantum mechanics struggles to intuitively explain this because it only looks at the "forward-evolving" state vector from the past ($|\psi_{in}\rangle$).

The Two-State-Vector Formalism (TSVF) Explanation

TSVF, developed by Aharonov, Bergmann, and Lebowitz, argues that a quantum system at any present moment is completely described by **two** state vectors:

1. A state vector evolving **forward** in time from the past pre-selection ($|\psi_{in}\rangle$).
2. A state vector evolving **backward** in time from the future post-selection ($\langle\psi_{fin}|$).

To find the probability of an intermediate measurement, TSVF uses the **ABL rule** (Aharonov-Bergmann-Lebowitz rule). It looks at how the forward and backward vectors overlap (interfere) at the moment of measurement.

The ABL Rule

- The Aharonov-Bergmann-Lebowitz (ABL) rule calculates the probability of an intermediate measurement outcome, given both a fixed initial state (pre-selection) and a fixed final state (post-selection).
- initial state be $|a\rangle$, our final post-selected state be $|b\rangle$, and c_j be the intermediate measurement outcomes we are checking.

$$P(c_j|a,b) = \frac{|\langle b|c_j\rangle\langle c_j|a\rangle|^2}{\sum_i |\langle b|c_i\rangle\langle c_i|a\rangle|^2}$$

- TSVF resolves the paradox by showing that the act of opening a box fundamentally changes the boundary conditions of the system.
- In standard classical logic, statements like "If I had opened Box 2, I would have found the particle" are fixed counterfactuals.
- In TSVF, the "reality" of the present moment is a handshake between the past and the future.
- By choosing to open Box 1 instead of Box 2, you change the intermediate physical reality created by the **interference** of the forward and backward time vectors.

Quantum Cheshire Cat paradox

This paradox was proposed in 2013 and successfully tested using neutrons in 2014

One distinguishing feature of the *Alice*-style Cheshire Cat is the periodic gradual disappearance of its body, leaving only one last visible trace: its iconic grin.

We shoot neutrons through a Mach-Zehnder interferometer, which splits the neutron's path into two routes: the **Left Path** and the **Right Path**. Neutrons also have a property called "**spin**"

- **Pre-selection (The Past):** We fire a neutron in such a way that it is in a spatial superposition of both the Left and Right paths, with its spin pointing in a specific direction.
- **Post-selection (The Future):** At the exit of the interferometer, we only keep the detector clicks for neutrons that exit on a specific path with a specific spin.

Using a technique called **Weak Measurement** (which allows us to measure a quantum system so gently that we don't collapse the wave function), we ask two intermediate questions between the pre- and post-selection:

1. Where is the neutron itself?
2. Where is the neutron's magnetic moment (its "spin")?

The Paradoxical Result

The neutron (*the cat)* is found with certainty traveling down the Left Path

The neutron (*the cat*) is found with certainty traveling down the Right Path.

Weak Measurement

1. **Pre-selection:** Prepare thousands of identical particles (e.g., photons) in the exact same initial state.
2. **Weak Coupling:** Let the particles pass through a measuring device that barely interacts with them. For example, a magnetic field so weak that it shifts the particle's trajectory by an amount smaller than the natural fuzziness (quantum uncertainty) of the particle itself.
3. **Post-selection:** At the end of the experiment, discard any particles that do not match your desired final state.
4. **Averaging:** Look at the intermediate weak measurements *only* for the particles that successfully made it to the post-selection. Average those thousands of tiny, noisy hints together.

Arun Kumar Pati is an Indian physicist notable for his research in quantum information, quantum computation and Foundations of quantum mechanics. He has made pioneering contributions in the area of quantum information

Super Quantum Search Algorithm with Weak Value Amplification and Postselection

Arun Kumar Pati[1,*]

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(Dated: November 5, 2019)

We propose a new model of quantum computation which aims to speed up quantum algorithms assisted by the weak value amplification and ancillay quantum register with the pre- and postelection. Within this model, we show that a quantum computer can solve a data base search of N entries in one step with probability close to one for large N, provided the post-selection on the ancillary quantum register is successful. In this model, to search a database of N entries, the number of qubits grows from n to $2n$, but there is a huge reduction in time complexity. Physically, this can be understood as the effect of weak value amplification that arises due to the pre- and postselection of the ancillary register which interacts with the n qubit register where quantum search is performed. This effectively accelerates the computation and takes the state of quantum computer much ahead in time, compared to what one would obtain without weak value amplification and post-selection.

Super Quantum Search Algorithm with Weak Value Amplification and Postselection

- By using weak values and post-selection, we can find the target item in an unsorted database of $n=2^m$ entries in exactly one single step (O(1) time complexity), regardless of how large the database is.

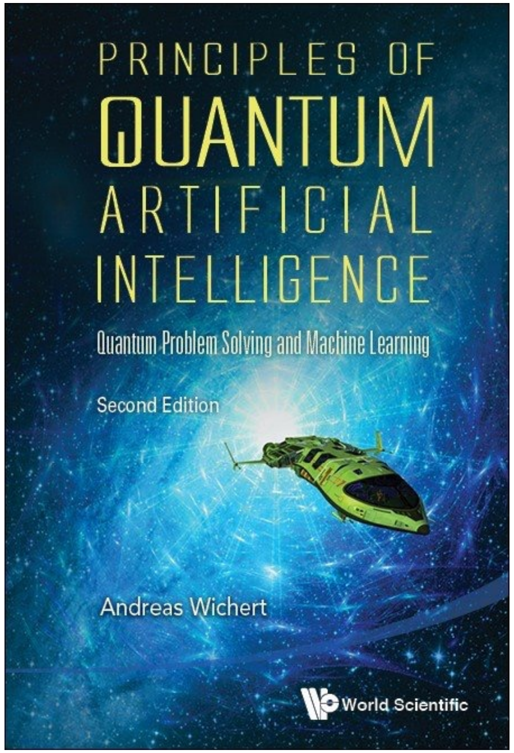
- The Main Register (*m* qubits): This is the actual quantum computer where the search algorithm is performed
- The Ancillary Register (*m* qubits): This is a "helper" system that acts as the catalyst

- The algorithm works by making the Main Register and the Ancillary Register interact with each other very gently (a weak interaction)

  - Pre-selection: Both registers are initialized in specific, known quantum states.
  - The Weak Interaction: The Main Register starts looking for the target item while **weakly interacting** with the Ancillary Register.
  - Post-selection: We perform a strong measurement only on the Ancillary Register. We post-select for one highly specific, incredibly rare final state. *We discard all experimental runs where the Ancillary Register doesn't hit this exact state.*

The catch lies entirely in the post-selection

- For the single-step search to work, the Ancillary Register must land on that specific final state. However, the probability of the ancilla naturally landing in that state is incredibly tiny
- If you run the quantum computer and the ancilla lands on the wrong state, the run is ruined, and you have to start over.
  - So, while the algorithm technically finds the answer in just "one computational step," you will likely have to run that one step millions or billions of times before you successfully pass the post-selection filter.



- Section 8.8.2
- Section 7.9 and 7.10

- *Chapter 16*

Principles of Quantum Artificial Intelligence: Quantum Problem Solving and Machine Learning, 2nd Edition, A. Wichert, World Scientific, 2020