Lecture 3: AI and Quantum Physics and Quantum Computation

Andreas Wichert

Department of Computer Science and Engineering

Técnico Lisboa

Overview

- Unstructured Search
- Quantum Oracle
- Householder Reflection
- Grover’s Amplification
- Iterative Amplification
- Number of Iterations
- Example
- SAT Problem

Lov Kumar Grover (born 1961) is an Indian-American computer scientist.
Lov Grover received his BTech in Electrical Engineering from IIT Delhi in 1981
He is the originator of the Grover database search algorithm used in quantum computing.

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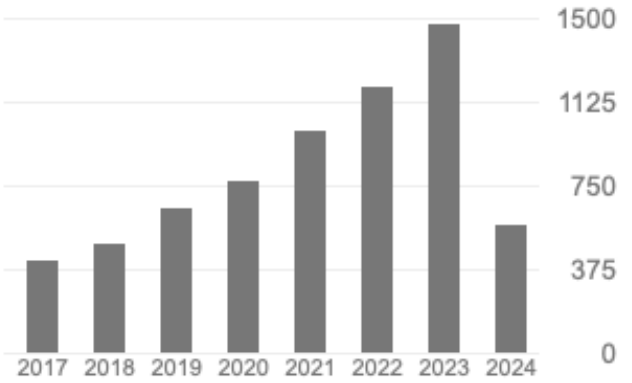


Unstructured Search

- For a function $o_\xi(x)$

$$o_\xi(x) = \begin{cases} 1 & if \quad x = \xi \\ 0 & else \end{cases}$$

- We want to find x for which $o_\xi(x) = 1$ *if* $x = \xi$. The task is equivalent to a decision problem with a binary answer ***1 = yes*** and ***0 = no*** and the instance x.
- Grover's amplification algorithm implements exhaustive search in $O(\sqrt{n})$ steps in n-dimensional Hilbert space
- It follows that using a quantum computer NP − complete problems remain NP − complete

- The search for ξ is based on the three principles of quantum computation:
  - The function $o(x)$ is represented by a quantum Boolean circuit T
  - T can be represented by a unitary operator (matrix)
  - For the function $o(x)$, the solution is encoded by $\boldsymbol{(-1)^{o(x)}}$, the **sign** of the amplitude
- If $o(x)$ is NP – complete, a quantum circuit T with a polynomial number of quantum gates T verifies for a given instance

  x_{ticket} if $o(x_{ticket}) = 1$, $x_{ticket} = \xi$

  or $o(x_{ticket}) = 0$

Solution is encoded by $(-1)^{o(x)}$, the **sign** of the amplitude

- The unitary operator *T* represents the **quantum oracle** function *o(x)* that determines if the configuration is the goal configuration
- The auxiliary qubit *c* is set to one, and the state is represented by m qubits $|0^{\otimes m}\rangle$.
  - First, we set qubits representing the states and the auxiliary qubit in superposition by the Hadamard gate for m + 1 qubits H_{m+1}, and then we execute the unitary operator *T*

$$T \cdot H_{m+1} \cdot |0^{\otimes m}\rangle|1\rangle =$$

$$= \frac{1}{\sqrt{2^{m+1}}} \cdot \sum_{x \in B^m} T \cdot |x\rangle|0\rangle - \frac{1}{\sqrt{2^{m+1}}} \cdot \sum_{x \in B^m} T \cdot |x\rangle|1\rangle$$

$$= \frac{1}{\sqrt{2^{m+1}}} \cdot \sum_{x \in B^m} |x\rangle|o(x) \oplus 0\rangle - \frac{1}{\sqrt{2^{m+1}}} \cdot \sum_{x \in B^m} |x\rangle|o(x) \oplus 1\rangle$$

$$= \frac{1}{\sqrt{2^{m+1}}} \cdot \left(\sum_{x \in B^m} |x\rangle|o(x) \oplus 0\rangle - \sum_{x \in B^m} |x\rangle|o(x) \oplus 1\rangle \right).$$

$$T \cdot H_{m+1} \cdot |0^{\otimes m}\rangle|1\rangle = \frac{1}{\sqrt{2^{m+1}}} \cdot \left(\sum_{x \in B^m} |x\rangle|o(x) \oplus 0\rangle - \sum_{x \in B^m} |x\rangle|o(x) \oplus 1\rangle \right).$$

There are four possible cases with the state $|\xi\rangle$ being the solution:

$$T \cdot |x\rangle|0\rangle = |x\rangle|o(x) \oplus 0\rangle = |x\rangle|0\rangle,$$

$$T \cdot |x\rangle|1\rangle = |x\rangle|o(x) \oplus 1\rangle = |x\rangle|1\rangle,$$

$$T \cdot |\xi\rangle|0\rangle = |\xi\rangle|f(\xi) \oplus 0\rangle = |\xi\rangle|1\rangle,$$

$$T \cdot |\xi\rangle|1\rangle = |\xi\rangle|f(\xi) \oplus 1\rangle = |\xi\rangle|0\rangle.$$

$$
\begin{aligned}
T \cdot H_{m+1} \cdot |0^{\otimes m}\rangle|1\rangle &= \frac{1}{\sqrt{2^{m+1}}} \cdot \left(\sum_{x \neq \xi} |x\rangle|0\rangle + |\xi\rangle|1\rangle - \sum_{x \neq \xi} |x\rangle|1\rangle - |\xi\rangle|0 \right. \\
&\quad \frac{1}{\sqrt{2^{m+1}}} \cdot \left(\sum_{x \neq \xi} |x\rangle \left(|0\rangle - |1\rangle\right) + |\xi\rangle \left(|1\rangle - |0\rangle\right) \right) \\
&\quad = \frac{1}{\sqrt{m}} \sum_{x \in B^m} (-1)^{o(x)} \cdot |x\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right).
\end{aligned}
$$

The value of the function $o(x)$ is encoded by $(-1)^{o(x)}$, the operation is a phase kick-back. We can set the auxiliary qubit $c = \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$ to zero by the Hadamard gate.

Housholder Reflection

- The Householder reflection reflects one vector |x⟩ to its negative and leaves invariant the orthogonal complement of this vectors.
- It is described by the Householder matrix Q_x with ‖|x⟩‖ = 1 representing m qubits with $n = 2^m$ and the projection matrix

$$P = |x\rangle\langle x|$$

$$Q_x = I_m - 2 \cdot P$$

- Suppose P_m is generated by the normalized vector |x⟩ indicating the direction of the bisecting line,

$$|x\rangle = \frac{1}{\sqrt{n}} \cdot |x_1\rangle + \frac{1}{\sqrt{n}} \cdot |x_2\rangle + \cdots + \frac{1}{\sqrt{n}} \cdot |x_n\rangle = \begin{pmatrix} \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \end{pmatrix}$$

then the projection matrix P_m is

$$P_m = |x\rangle\langle x| = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}$$

Householder Reflection

$$P_m = |x\rangle\langle x| = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}$$

The projection matrix P computes for each dimension the mean value

$$\begin{pmatrix} \frac{\sum_{i=1}^{n} x_i}{n} \\ \frac{\sum_{i=1}^{n} x_i}{n} \\ \vdots \\ \frac{\sum_{i=1}^{n} x_i}{n} \end{pmatrix} = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and the Householder reflection

$$Q_x = I_m - 2 \cdot P_m.$$

computes the following mapping,

$$x_i^{new} = x_i^{old} - 2 \cdot \frac{\sum_{i=1}^{n} x_i^{old}}{n}.$$

Q_x is unitary,

$$Q_x \cdot Q_x^* = (I_m - 2 \cdot |x\rangle\langle x|) \cdot (I_m - 2 \cdot |x\rangle\langle x|)^*$$

$$Q_x \cdot Q_x^* = I_m - 2 \cdot |x\rangle\langle x| - 2 \cdot |x\rangle\langle x| + 4 \cdot |x\rangle\langle x| \cdot |x\rangle\langle x|$$

$$Q_x \cdot Q_x^* = I_m - 4 \cdot |x\rangle\langle x| + 4 \cdot \langle x|x\rangle \cdot |x\rangle\langle x| = I_m$$

$\langle x|x\rangle = 1$ because $\||x\rangle\| = 1$. With

$$P = |x\rangle\langle x|.$$

Grover's Amplification

Grover's amplification is based on $-Q_x$. It is a unitary operator with

$$G_m := -Q_x = -I_m + 2 \cdot P_m = 2 \cdot P_m - I_m$$

the mapping is defined as,

$$x_i^{new} = 2 \cdot \frac{\sum_{i=1}^{n} x_i^{old}}{n} - x_i^{old}.$$

Suppose only one amplitude of x_j is negative and the other one are positive. Then the corresponding amplitude grows with

$$x_j^{new} = 2 \cdot \frac{\sum_{i=1}^{n} x_j^{old}}{n} + x_i^{old}$$

$$\begin{pmatrix} 2 \cdot \frac{\sum_{i=1}^{n} x_i}{n} - x_1 \\ 2 \cdot \frac{\sum_{i=1}^{n} x_i}{n} + x_2 \\ \vdots \\ 2 \cdot \frac{\sum_{i=1}^{n} x_i}{n} - x_n \end{pmatrix} = \begin{pmatrix} \frac{2}{n} - 1 & \frac{2}{n} & \cdots & \frac{2}{n} \\ \frac{2}{n} & \frac{2}{n} - 1 & \cdots & \frac{2}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{n} & \frac{2}{n} & \cdots & \frac{2}{n} - 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ -x_2 \\ \vdots \\ x_n \end{pmatrix}$$

The probability of measuring the solution depending on the size n is

$$p(solution) = \left| \frac{3}{\sqrt{n}} - \frac{4}{n \cdot \sqrt{n}} \right|^2$$

and non solution

$$p(non\ solution) = \left| \frac{1}{\sqrt{n}} - \frac{4}{n \cdot \sqrt{n}} \right|^2 .$$

Why?

Projection computes the mean value

$$P_m = |x\rangle\langle x| = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\sum_{i=1}^{n} x_i}{n} \\ \frac{\sum_{i=1}^{n} x_i}{n} \\ \vdots \\ \frac{\sum_{i=1}^{n} x_i}{n} \end{pmatrix} = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

With

$$|y\rangle = \begin{pmatrix} \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \\ -\frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \end{pmatrix} = \frac{(-1)^{f(x)}}{\sqrt{n}} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \frac{(-1)^{f(x)}}{\sqrt{n}} \cdot |\mathbf{1}\rangle$$

the new amplitude distribution is computed by

$$G_m \cdot |y\rangle = (2 \cdot P_m - I_m) \cdot |y\rangle = 2 \cdot P_m \cdot |y\rangle - I_m \cdot |y\rangle.$$

First two times the average amplitude is computed. The amplitude values for non solution are $\frac{1}{\sqrt{n}}$ and for one marked solution $-\frac{1}{\sqrt{n}}$, it follows

$$2 \cdot P_m \cdot |y\rangle = A \cdot |\mathbf{1}\rangle = \frac{2}{n} \cdot \left(n \cdot \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n}} + \left(-\frac{1}{\sqrt{n}} \right) \right) \cdot |\mathbf{1}\rangle$$

$$A \cdot |\mathbf{1}\rangle = \frac{2}{n} \cdot \left((n-1) \cdot \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n}} \right) \cdot |\mathbf{1}\rangle$$

$$A = \frac{2}{\sqrt{n}} \cdot \left(1 - \frac{2}{n} \right) = \frac{2 \cdot n - 4}{n^{\frac{3}{2}}}$$

and the amplitude distribution is

$$G_m \cdot |y\rangle = A \cdot |\mathbf{1}\rangle - I_m \cdot |y\rangle.$$

The amplitude of the state $|\tau\rangle$ indicating the solution in the dimension i is

$$\tau_i = A + \frac{1}{\sqrt{n}} = \frac{3 \cdot n - 4}{n^{\frac{3}{2}}} = \frac{3}{\sqrt{n}} - \frac{4}{n \cdot \sqrt{n}}$$

and the non solution in the dimension j with $j \neq i$

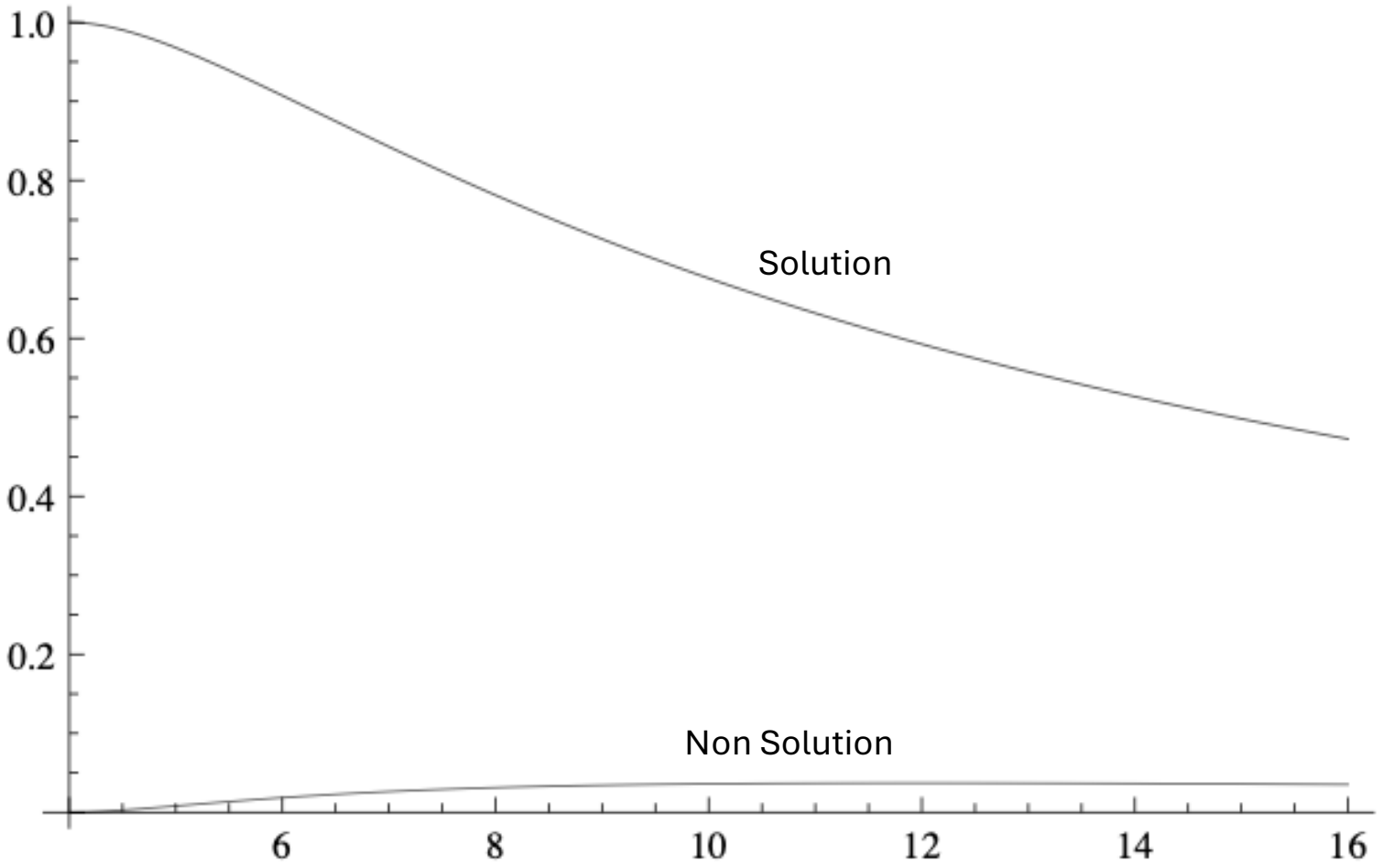
$$\tau_j = A - \frac{1}{\sqrt{n}} = \frac{n - 4}{n^{\frac{3}{2}}} = \frac{1}{\sqrt{n}} - \frac{4}{n \cdot \sqrt{n}}.$$

For $n = 4$ $\tau_i = 1$ and $\tau_j = 0$, for $n = 2^8$ $\tau_i = 0.186523$ and $\tau_j = 0.0615234$.
The probability of measuring the solution depending on the size n is

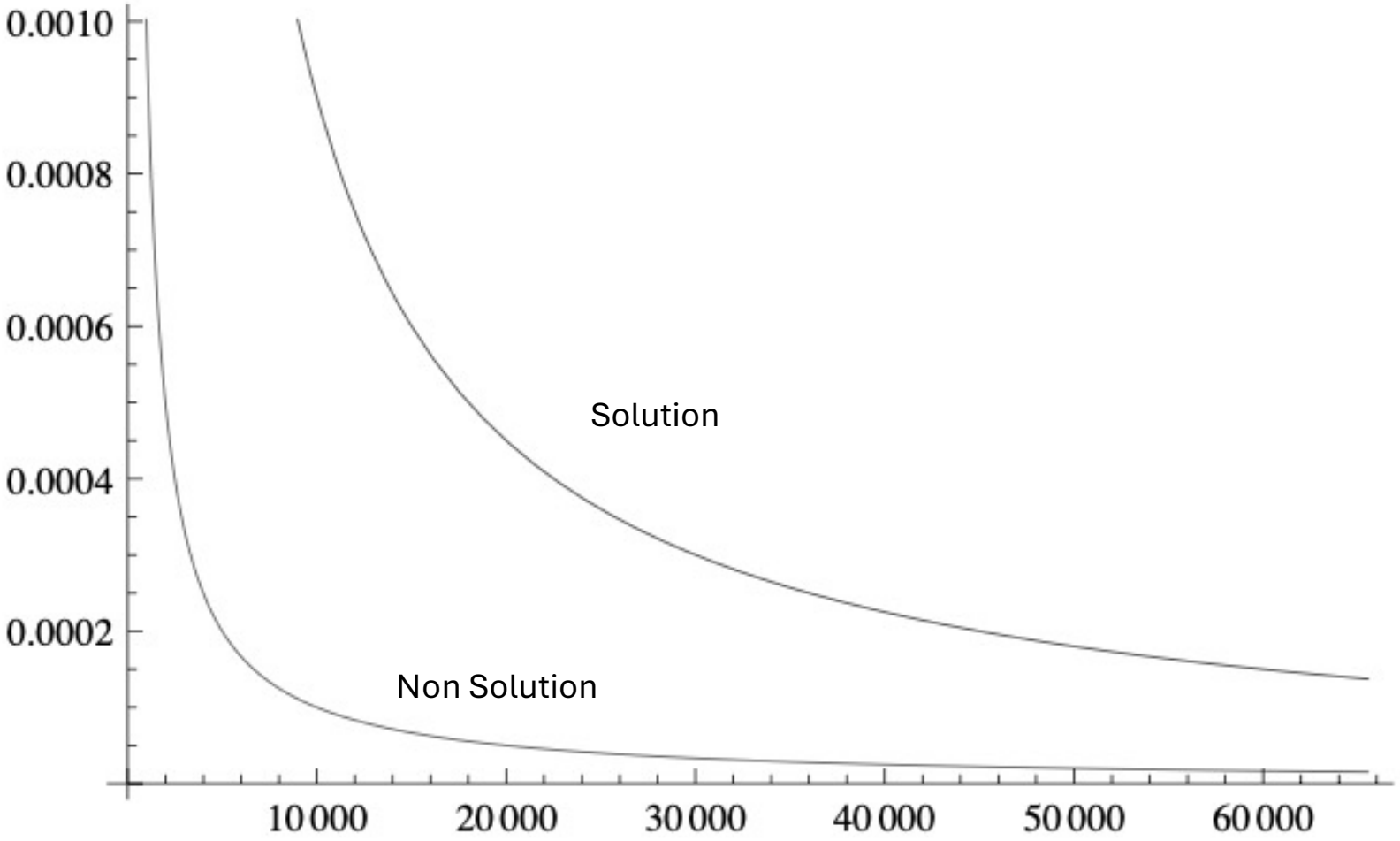
$$p(solution) = \left| \frac{3}{\sqrt{n}} - \frac{4}{n \cdot \sqrt{n}} \right|^2$$

and non solution

$$p(non\ solution) = \left| \frac{1}{\sqrt{n}} - \frac{4}{n \cdot \sqrt{n}} \right|^2.$$



The *probability* of measuring **one** present solution and **one** non solution for $4 \leq n \leq 16$



The *probability* of measuring **one** present solution and **one** non solution for $2^8 \leq n \leq 2^{16}$

Iterative Amplification

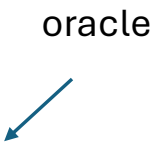
- The probability of seeing one solution should be as close as possible to **1** and the number of iterations should be small
- The number of iterations *r* is the largest integer not greater than t^*

$$r = \lfloor t^* \rfloor = \left\lfloor \frac{\pi}{4} \cdot \sqrt{\frac{2^m}{k}} \right\rfloor$$

- The value of r depends on the relation of $n=2^m$ versus *k*, *k*=number of solutions
  - For $n = 4$ and $k = 1$ we need **only** one rotation
  - For $k=n/4$ we need **only** one rotation to measure **one** solution

Iterative Amplification

With the definition



$$\Gamma_m := (G_m \otimes I_1) \cdot \boldsymbol{T}$$

the resulting state is $|\tau\rangle$. We describe several amplifications by

$$\left(\prod_{t=1}^{r} \Gamma_m\right) \cdot H_{m+1} \cdot |0^{\oplus m}\rangle|1\rangle = \Gamma_m \cdot \Gamma_m \cdot \cdots \cdot \Gamma_m \cdot H_{m+1} \cdot |0^{\oplus m}\rangle|1\rangle =$$

$$= |\tau\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right).$$

How many iterations do we need to preform, or what is the suitable value for r? The probability of seeing one solution should be as close as possible to 1 and r should be as small as possible.

- Suppose that the function o(x) has k solutions with $n \gg k \geq 1$,

$$o_\xi(x) = \begin{cases} 1 & if \quad x = \xi_i \quad i \in \{1, 2, \cdots, k\} \\ 0 & else \end{cases}$$

- The amplitude for the solution at the iteration t will be indicated by α_t non solution by β_t
- For $t = 0$ before the first iteration of Γ_m

$$\alpha_0 = \frac{1}{\sqrt{n}} \qquad \beta_0 = \frac{1}{\sqrt{n}}.$$

- In the first iteration Γ_m two times the average amplitude is computed

$$A = \frac{2}{n} \cdot \left(n \cdot \frac{1}{\sqrt{n}} - \frac{k}{\sqrt{n}} - \frac{k}{\sqrt{n}} \right) = \frac{2}{n} \cdot \left((n-k) \cdot \frac{1}{\sqrt{n}} - \frac{k}{\sqrt{n}} \right)$$

$$A = \frac{2}{n} \cdot \left((n-k) \cdot \beta_0 - k \cdot \alpha_0 \right).$$

The amplitude of the solution is

$$\alpha_1 = A + \frac{1}{\sqrt{n}} = A + \alpha_0 = \frac{2}{n} \cdot ((n-k) \cdot \beta_0 - k \cdot \alpha_0) + \alpha_0,$$

$$\alpha_1 = \frac{1}{n} \cdot (\alpha_0 \cdot (n - 2 \cdot k) + \beta_0 \cdot (2 \cdot n - 2 \cdot k),$$

$$\alpha_1 = \alpha_0 \cdot (1 - \frac{2 \cdot k}{n}) + \beta_0 \cdot (2 - \frac{2 \cdot k}{n}),$$

and the non solution

$$\beta_1 = A - \frac{1}{\sqrt{n}} = A - \beta_0 = \frac{2}{n} \cdot ((n-k) \cdot \beta_0 - k \cdot \alpha_0) - \beta_0,$$

$$\beta_1 = -\alpha_0 \cdot \frac{2 \cdot k}{n} + \beta_0 \cdot (1 - \frac{2 \cdot k}{n}).$$

- We can describe the evolution of the amplitudes in time *t* by two coupled *recurrence* equations.
- They represent a discrete **dynamical system** of two **difference equations**

$$\alpha_{t+1} = \alpha_t \cdot \left(1 - \frac{2 \cdot k}{n}\right) + \beta_t \cdot \left(2 - \frac{2 \cdot k}{n}\right)$$

$$\beta_{t+1} = -\alpha_t \cdot \frac{2 \cdot k}{n} + \beta_t \cdot \left(1 - \frac{2 \cdot k}{n}\right).$$

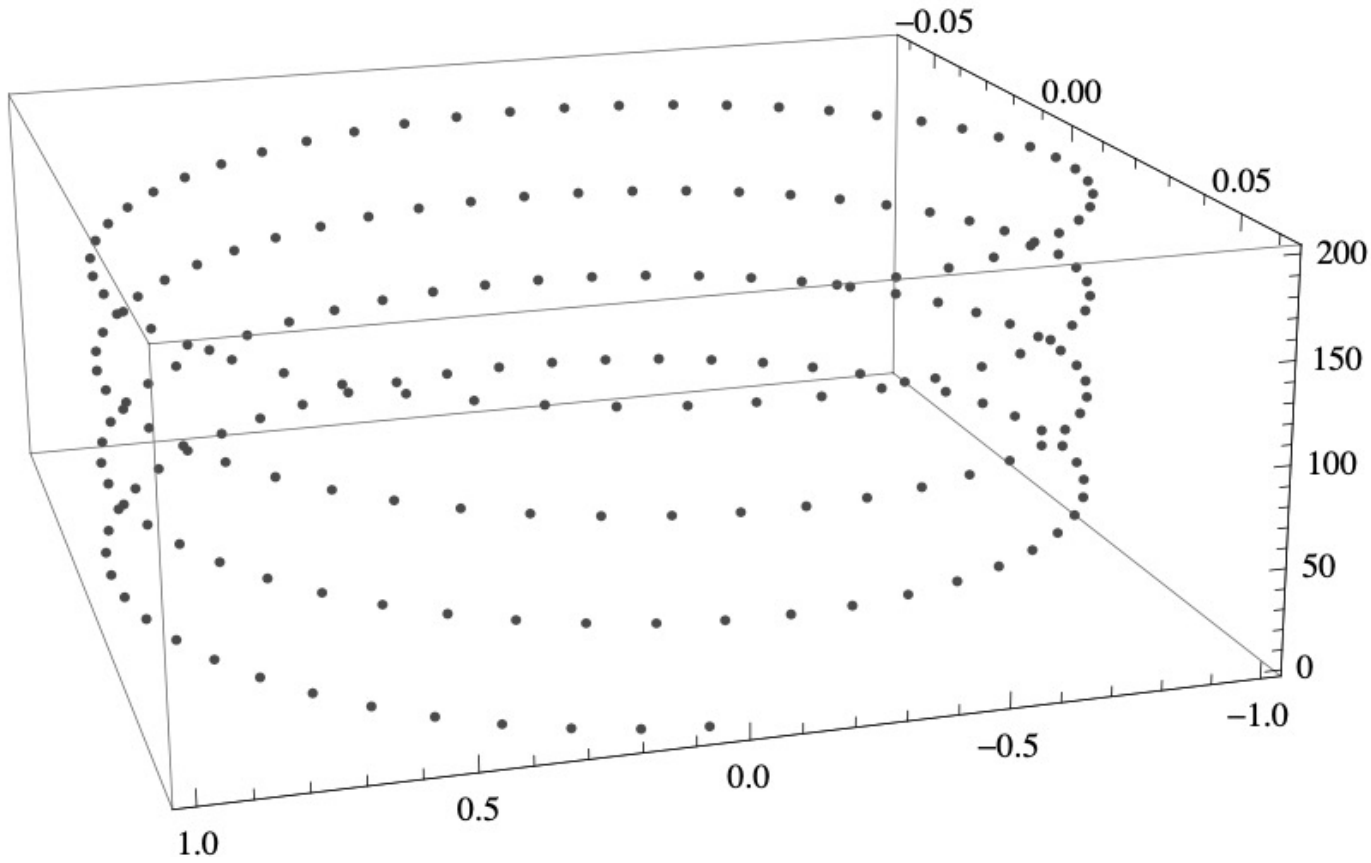


Fig. 10.4 Phase space ofα_t, β_t and t with the boundary condition of $\alpha_0 = \beta_0 = \frac{1}{\sqrt{n}}$. The values are $n = 256$, $k = 1$ and $1 \leq t \leq 200$. The x-axis indicates α_t, the y-axis β_t and the z-axis t.

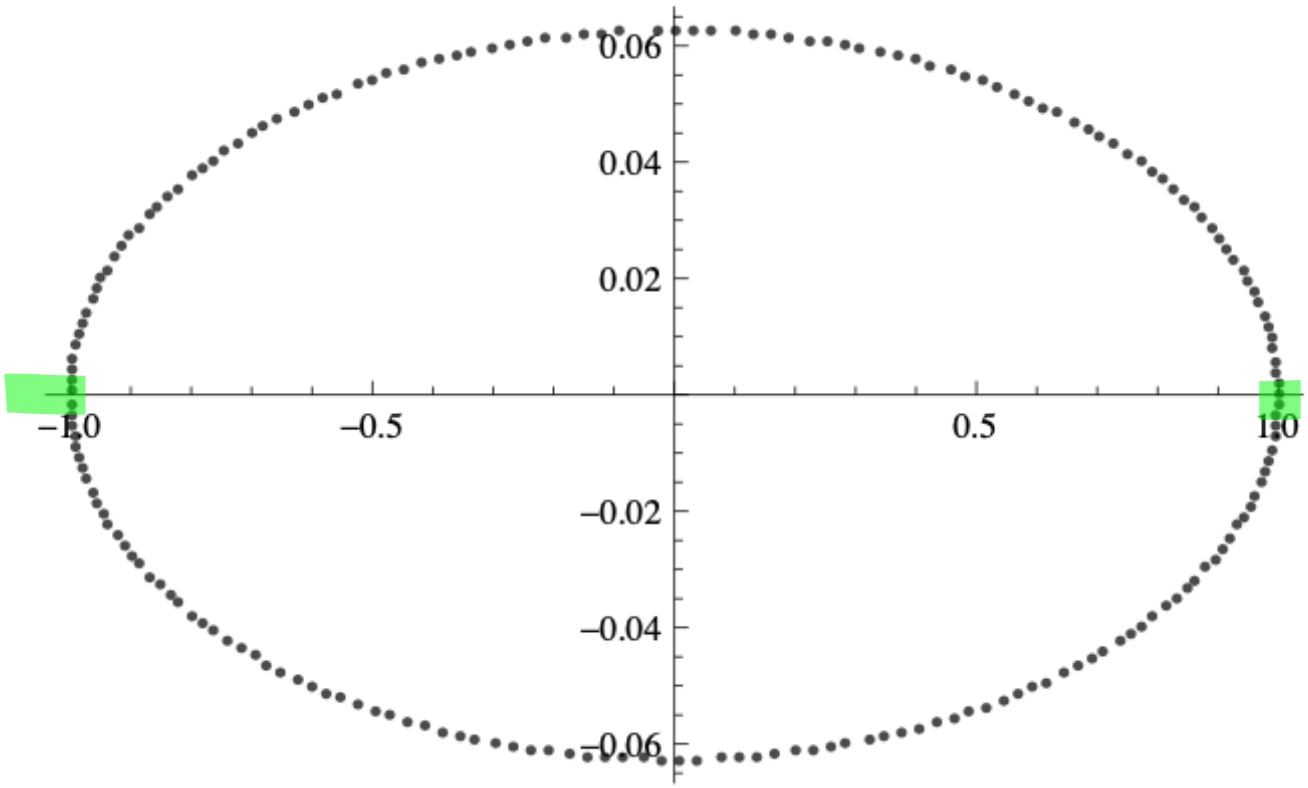


Fig. 10.5 The projected orbit in the two dimensional subspace α_t, β_t with the boundary condition of $\alpha_0 = \beta_0 = \frac{1}{\sqrt{n}}$. The values are $n = 256$, $k = 1$ and $1 \leq t \leq 200$. The x-axis indicates α_t and the y-axis β_t.

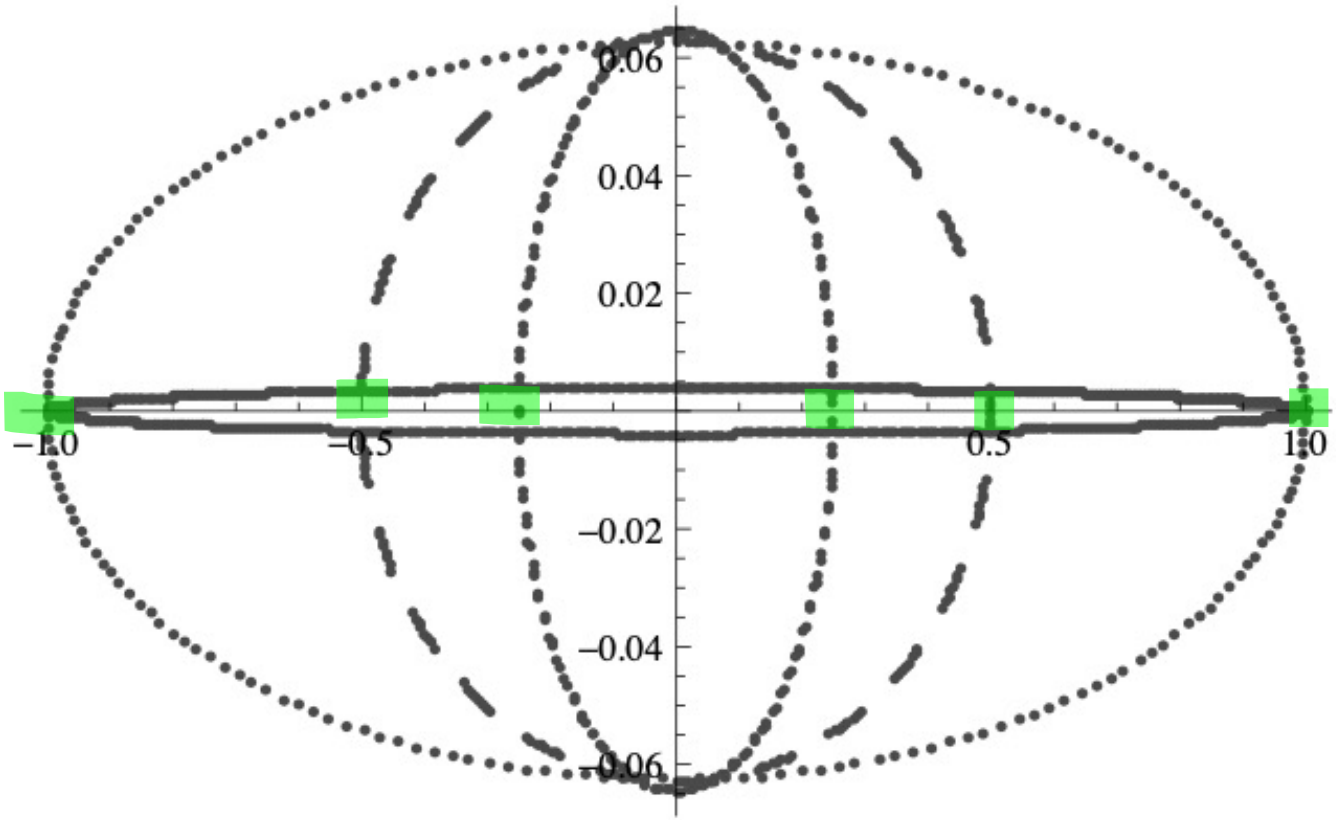


Fig. 10.6 The projected orbit in the two dimensional subspace α_t, β_t with the boundary condition of $\alpha_0 = \beta_0 = \frac{1}{\sqrt{n}}$. The x-axis indicates α_t, the y-axis β_t. The outer ellipse has the values $n = 256$, $k = 1$ as before, the two ellipses with diminishing x-axis radius corresponds to increased k values $k = 4$ and $k = 16$. For all three ellipses with $n = 256$, 200 iterations were done. In the fourth ellipse k is one and $n = 65536 = 2^{16}$, y-axis radius diminish. 1000 iterations were done.

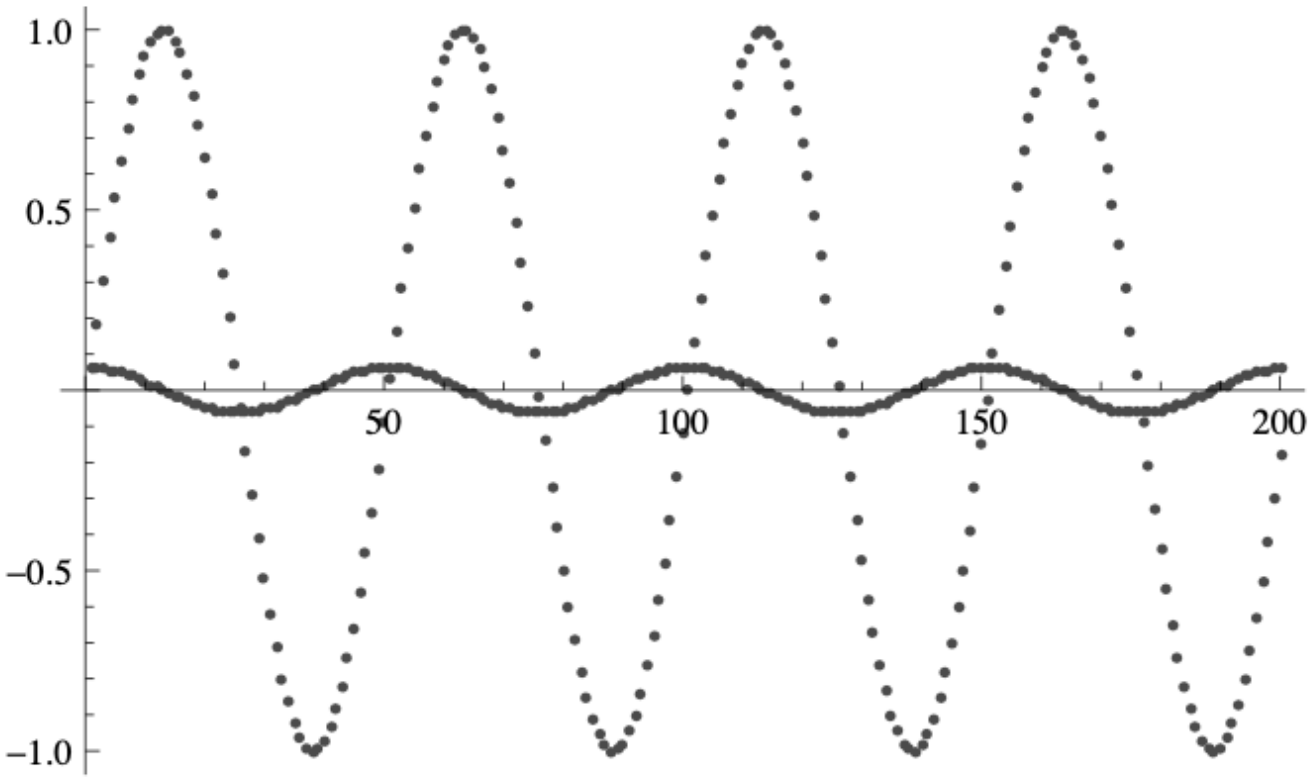
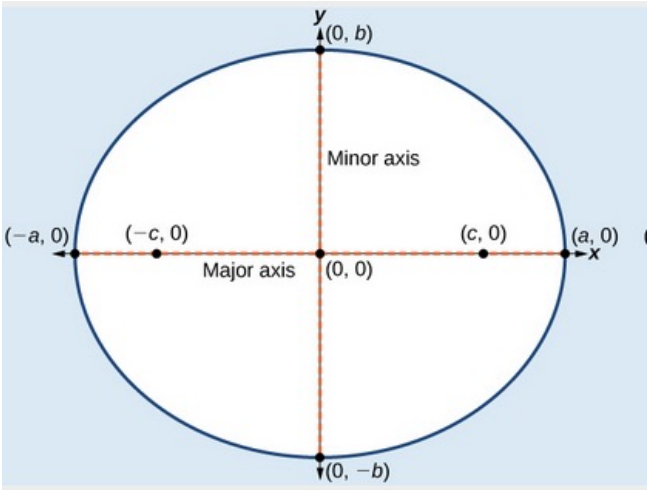


Fig. 10.7 The projected orbit in the two dimensional subspace of amplitude and time t represents a periodic function described by α_t (the dotted curve) and β_t (the continuous curve). The x-axis indicates t and the y-axis the amplitude. The values are $n = 256$, $k = 1$ and $1 \leq t \leq 200$.

- Such a linear and periodic system is usually described by sine and cosine equations
- The ellipse in a Cartesian system is represented by the equation



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- Because α_t and βt are real, following equation representing an ellipse is given by

$$k \cdot \alpha_t^2 + \ (n-k) \cdot \beta_t^2 = 1.$$

Using the the Pythagorean identity

$$\sin^2\theta + \cos^2\theta = 1$$

we can rewrite the equation representing the ellipse as

$$k \cdot \left(\sqrt{\frac{1}{k}} \cdot \sin\theta_t \right)^2 + (n-k) \cdot \left(\sqrt{\frac{1}{n-k}} \cdot \cos\theta_t \right)^2 = 1$$

$$\alpha_{t+1} = \alpha_t \cdot \left(1 - \frac{2 \cdot k}{n}\right) + \beta_t \cdot \left(2 - \frac{2 \cdot k}{n}\right)$$

$$\beta_{t+1} = -\alpha_t \cdot \frac{2 \cdot k}{n} + \beta_t \cdot \left(1 - \frac{2 \cdot k}{n}\right).$$

it follows that

$$\alpha_t = \sqrt{\frac{1}{k}} \cdot \sin\theta_t \quad \textit{and} \quad \beta_t = \sqrt{\frac{1}{n-k}} \cdot \cos\theta_t$$

and we can rewrite the the two coupled recurrence equations as

$$\sqrt{\frac{1}{k}} \cdot \sin\theta_{t+1} = \sqrt{\frac{1}{k}} \cdot \sin\theta_t \cdot \left(1 - \frac{2 \cdot k}{n}\right) + \sqrt{\frac{1}{n-k}} \cdot \cos\theta_t \cdot \left(2 - \frac{2 \cdot k}{n}\right)$$

$$\sqrt{\frac{1}{n-k}} \cdot \cos\theta_{t+1} = -\sqrt{\frac{1}{k}} \cdot \sin\theta_t \cdot \frac{2 \cdot k}{n} + \sqrt{\frac{1}{n-k}} \cdot \cos\theta_t \cdot \left(1 - \frac{2 \cdot k}{n}\right)$$

simplified as

$$\sin\theta_{t+1} = \sin\theta_t \cdot \left(1 - \frac{2\cdot k}{n}\right) + \cos\theta_t \cdot \frac{2\cdot\sqrt{k}\cdot\sqrt{n-k}}{n}$$

$$\cos\theta_{t+1} = -\sin\theta_t \cdot \frac{2\cdot\sqrt{k}\cdot\sqrt{n-k}}{n} + \cos\theta_t \cdot \left(1 - \frac{2\cdot k}{n}\right).$$

Trigonometric simplification

Because

$$-1 \leq \left(1 - \frac{2 \cdot k}{n}\right) \leq 1$$

we can represent it as

$$\cos \omega = 1 - \frac{2 \cdot k}{n}.$$

Because of the Pythagorean identity

$$\left(1 - \frac{2 \cdot k}{n}\right)^2 + \left(\frac{2 \cdot \sqrt{k} \cdot \sqrt{n-k}}{n}\right)^2 = 1$$

it follows that

$$\sin \omega = \frac{2 \cdot \sqrt{k} \cdot \sqrt{n-k}}{n}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

- We can rewrite again the the two coupled recurrence equations as

$$\sin\theta_{t+1} = \sin\theta_t \cdot \cos\omega + \cos\theta_t \cdot \sin\omega$$

$$\cos\theta_{t+1} = -\sin\theta_t \cdot \sin\omega + \cos\theta_t \cdot \cos\omega$$

- Because of the trigonometric identities, addition and subtraction theorem, the two coupled recurrence equations with the boundary condition θ_0 are

$$\sin\theta_{t+1} = \sin(\theta_t + \omega) = \sin(\theta_0 + t\cdot\omega + \omega)$$

$$\cos\theta_{t+1} = \cos(\theta_t + \omega) = \cos(\theta_0 + t\cdot\omega + \omega)$$

$$\sin\theta_{t+1} = \sin(\theta_t + \omega) = \sin(\theta_0 + t \cdot \omega + \omega)$$

$$\cos\theta_{t+1} = \cos(\theta_t + \omega) = \cos(\theta_0 + t \cdot \omega + \omega)$$

we can rewrite the recurrence equations into two simple equations

$$\alpha_t = \sqrt{\frac{1}{k}} \cdot \sin\theta_t \longrightarrow \alpha_t = \frac{1}{\sqrt{k}} \cdot \sin(\theta_0 + t \cdot \omega)$$

$$\beta_t = \sqrt{\frac{1}{n-k}} \cdot \cos\theta_t \longrightarrow \beta_t = \frac{1}{\sqrt{n-k}} \cdot \cos(\theta_0 + t \cdot \omega).$$

- With the boundary conditions

$$\alpha_0 = \beta_0 = \frac{1}{\sqrt{n}}$$

$$\alpha_0 = \sqrt{\frac{1}{k}} \cdot \sin\theta_0$$

$$\beta_0 = \frac{1}{\sqrt{n-k}} \cdot \cos\theta_0$$

it follows that:

$$\sin^2\theta_0 = \frac{k}{n} \longrightarrow \theta_0 = \sin^{-1}\left(\sqrt{\frac{k}{n}}\right)$$

$$\cos^2\theta_0 = \frac{n-k}{n} = 1 - \frac{k}{n}$$

and because of the double angle formula

$$\cos(2 \cdot x) = 2 \cdot \cos^2 x - 1 = 1 - 2 \cdot \sin^2 x$$

and

$$\sin(2 \cdot x) = 2 \cdot \sin x \cdot \cos x$$

it follows that

$$\cos \omega = 1 - \frac{2 \cdot k}{n} = 1 - 2 \cdot \sin^2 \theta_0 = \cos(2 \cdot \theta_0).$$

$$\sin \omega = \frac{2 \cdot \sqrt{k} \cdot \sqrt{n-k}}{n} = 2 \sin \theta_0 \cdot \cos_0 \theta_0 = \sin(2 \cdot \theta_0).$$

$$\alpha_t = \frac{1}{\sqrt{k}} \cdot \sin(\theta_0 + t \cdot \omega)$$

$$\beta_t = \frac{1}{\sqrt{n-k}} \cdot \cos(\theta_0 + t \cdot \omega).$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Using the trigonometric identities

$$\alpha_t = \frac{1}{\sqrt{k}} \cdot (\sin\theta_0 \cdot \cos(t \cdot \omega) + \cos\theta_0 \cdot \sin(t \cdot \omega))$$

$$\beta_t = \frac{1}{\sqrt{n-k}} \cdot (-\sin\theta_0 \cdot \sin(t \cdot \omega) + \cos\theta_0 \cdot \cos(t \cdot \omega))$$

Solution:

...with boundary conditions: $\alpha_0 = \beta_0 = \frac{1}{\sqrt{n}}$

$$\alpha_t = \frac{1}{\sqrt{k}} \cdot \sin(\theta_0 + t \cdot 2 \cdot \theta_0) = \frac{1}{\sqrt{k}} \cdot \sin(\theta_0 \cdot (2 \cdot t + 1))$$

$$\beta_t = \frac{1}{\sqrt{n-k}} \cdot \cos(\theta_0 + t \cdot 2 \cdot \theta_0) = \frac{1}{\sqrt{n-k}} \cdot \cos(\theta_0 \cdot (2 \cdot t + 1))$$

with

$$\theta_0 = \sin^{-1}\left(\sqrt{\frac{k}{n}}\right)$$

Number of Iterations

$$\alpha_t = \frac{1}{\sqrt{k}} \cdot \sin(\theta_0 \cdot (2 \cdot t + 1))$$

- The probability of seeing one solution should be as close as possible to 1 and the number of iterations *r* should be as small as possible.
  - Because there are *k* solution, the probability of measuring a state that represents a solution is

$$k \cdot \alpha_t^2 = \sin^2(\theta_0 \cdot (2 \cdot t + 1)) = 1$$

$$\theta_0 = \sin^{-1}\left(\sqrt{\frac{k}{n}}\right)$$

$$\theta_0 \cdot (2 \cdot t + 1) = \frac{\pi}{2}$$

after t^* iterations the probability of measuring a solution is nearly one

$$t^* := t = \frac{\pi}{4 \cdot \theta_0} - \frac{1}{2} = \frac{\pi}{4} \cdot \sqrt{\frac{n}{k}} - \frac{1}{2} = \frac{\pi}{4} \cdot \sqrt{\frac{2^m}{k}} - \frac{1}{2}$$

$$t^* = 0.785398 \cdot \sqrt{\frac{2^m}{k}} - 0.5.$$

Number of Iterations

- The number of iterations r is the largest integer not greater than t^*,

$$r = \lfloor t^* \rfloor = \left\lfloor \frac{\pi}{4} \cdot \sqrt{\frac{2^m}{k}} \right\rfloor$$

The value of r depends on the relation of n versus k. For $n = 4$ and $k = 1$ we need only one rotation, we need as well only one rotation for

$$\frac{n}{4} = k$$

to find **one** of the k solutions. For 16 qubits and one solution, $k = 1$, $n = 65536 = 2^{16}$, $t^* = 200.562$. In this case we need two hundred rotations. The probability of measuring a state that represents a solution is nearly one.

- It is possible to adapt the iterations in such a way that the probability of finding a solution is exactly one.
  - One changes θ_0 of the difference equations either in the last step or continuously
- The resulting speed up remains quadratic
- The iterative amplification algorithm requires the value of k in order to determine the number of iterations
- We can determine the value of k by the quantum counting algorithm

Lower Bound

- If the operator T is represented by $O(m)$ gates, then we can get a speed up of maximum $O(\sqrt{n})$ steps, in fact $\Omega(\sqrt{n})$ *is* the **lower bound**.
- The proof for lower bound $\Omega(\sqrt{n})$ for T oracle-based search is based on a sequence in a Hilbert space and the Euclidean norm properties of Hilbert space
- The definition of the probabilities as normed squared amplitudes

- **No better algorithm exists!**

It follows that using a quantum computer NP – complete problems remain NP – complete

Unitary Representation

$$P_m = |x\rangle\langle x| = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}$$

$$Q_x = I_m - 2 \cdot P_m.$$

Grover's amplification (for $m \geq 2$) is based on $G_m = -Q_x$. The unitary operator Λ_m reverses the sign of $|0\rangle$

$$\Lambda_m \cdot |0\rangle = -|0\rangle$$

and for $|x\rangle \neq |0\rangle$

$$\Lambda_m \cdot |x\rangle = |x\rangle.$$

Then we can write

$$G_m = -Q_x = -I_m + 2 \cdot P_m = 2 \cdot P_m - I_m = -(H_m \cdot \Lambda_m \cdot H_m) \tag{5.17}$$

with

$$H_m \cdot \Lambda_m \cdot H_m = \begin{pmatrix} 1 - \frac{2}{n} & -\frac{2}{n} & \cdots & -\frac{2}{n} \\ -\frac{2}{n} & 1 - \frac{2}{n} & \cdots & -\frac{2}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{2}{n} & -\frac{2}{n} & \cdots & 1 - \frac{2}{n} \end{pmatrix}. \tag{5.18}$$

- We will demonstrate the example of $m = 3$, means $8 = 2^3$ different states
  - *G=-H3*L3*H3*

```
H3=
 [[ 0.3536  0.3536  0.3536  0.3536  0.3536  0.3536  0.3536  0.3536]
 [ 0.3536 -0.3536  0.3536 -0.3536  0.3536 -0.3536  0.3536 -0.3536]
 [ 0.3536  0.3536 -0.3536 -0.3536  0.3536  0.3536 -0.3536 -0.3536]
 [ 0.3536 -0.3536 -0.3536  0.3536  0.3536 -0.3536 -0.3536  0.3536]
 [ 0.3536  0.3536  0.3536  0.3536 -0.3536 -0.3536 -0.3536 -0.3536]
 [ 0.3536 -0.3536  0.3536 -0.3536 -0.3536  0.3536 -0.3536  0.3536]
 [ 0.3536  0.3536 -0.3536 -0.3536 -0.3536 -0.3536  0.3536  0.3536]
 [ 0.3536 -0.3536 -0.3536  0.3536 -0.3536  0.3536  0.3536 -0.3536]]
```

```
L3=np.matrix([[-1., 0., 0., 0., 0., 0., 0., 0.],
        [0., 1., 0., 0., 0., 0., 0., 0.],
        [0., 0., 1., 0., 0., 0., 0., 0.],
        [0., 0., 0., 1., 0., 0., 0., 0.],
        [0., 0., 0., 0., 1., 0., 0., 0.],
        [0., 0., 0., 0., 0., 1., 0., 0.],
        [0., 0., 0., 0., 0., 0., 1., 0.],
        [0., 0., 0., 0., 0., 0., 0., 1.]])
```

*G=-H3*L3*H3*

m = 3, n=8 = 2^3

$$G_m = \begin{pmatrix} \frac{2}{n}-1 & \frac{2}{n} & \cdots & \frac{2}{n} \\ \frac{2}{n} & \frac{2}{n}-1 & \cdots & \frac{2}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{n} & \frac{2}{n} & \cdots & \frac{2}{n}-1 \end{pmatrix}$$

$$G3 = \begin{pmatrix} -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} \end{pmatrix}$$

In the next step we represent the state vector with equally distributed amplitudes and mark the solution with a minus sign

```
a=1/np.sqrt(8)
x1=np.array([a,a,a,a,-a,a,a,a])
print("x1=\n",x1)
x1=
 [ 0.3536  0.3536  0.3536  0.3536 -0.3536  0.3536  0.3536  0.3536]
```

and perform a step in Grover's amplification as with the resulting amplitudes

```
x1=G.dot(x1)
print("x1=\n",x1)
x1=
 [[0.1768 0.1768 0.1768 0.1768 0.8839 0.1768 0.1768 0.1768]]
```

and the second iteration with the resulting amplitudes

```
x2=np.array([0.1768, 0.1768, 0.1768, 0.1768, -0.8839, 0.1768, 0.1768, 0.1768])
x2=G.dot(x2)
print("x2=\n",x2)
x2=
 [[-0.0884 -0.0884 -0.0884 -0.0884  0.9723 -0.0884 -0.0884 -0.0884]].
```

With two rotations we achieved the maximal amplitude value that corresponds to the probability value $0.945367 = |0.9723|^2$. After the third rotation the amplitudes diminish because of the periodic property of Grover's amplification

Decomposition

With the Grove's amplification we have

$$G_m = -(H_m \cdot \Lambda_m \cdot H_m).$$

How can we decompose Λ_m by quantum gates? We note that for one qubit wi

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

and

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

For three qubits we can define Λ_3 accordingly with

$$X_3 = X \otimes X \otimes X$$

$$H_0 = I \otimes I \otimes H$$

$$\Lambda_3 = X_3 \cdot H_0 \cdot CCX \cdot H_0 \cdot X_3$$

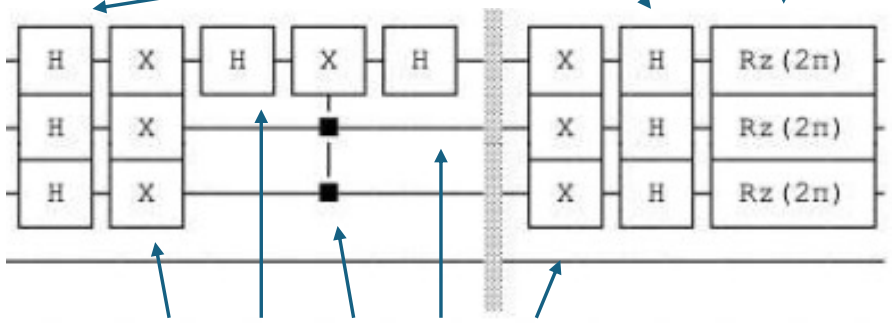
and for the minus sign operation

$$RZ(2 \cdot \pi)_3 = RZ(2 \cdot \pi) \otimes RZ(2 \cdot \pi) \otimes RZ(2 \cdot \pi)$$

it follows

$$G_m = -(H_3 \cdot \Lambda_3 \cdot H_3) = H_3 \cdot \Lambda_3 \cdot H_3 \cdot RZ(2 \cdot \pi)_3$$

"Grover Diffuser"



$$\Lambda_3 = X_3 \cdot H_0 \cdot CCX \cdot H_0 \cdot X_3$$

Example

- In the next step we apply Grover's amplification to a marked state of three qubits
- Our solution corresponds to the Boolean formula $\neg x \wedge y \wedge \neg z$
- for which it evaluates true, which is the case for
  - $x = 0$, $y = 1$ and $z = 0$
- In this case the state determined by the oracle function is
  - $o(010) = 1$ with the solution encoded by $(-1)^{o(x)}$
- The unitary operator T

$$T = (X \otimes I \otimes X \otimes I) \cdot MCX \cdot (X \otimes I \otimes X \otimes I)$$

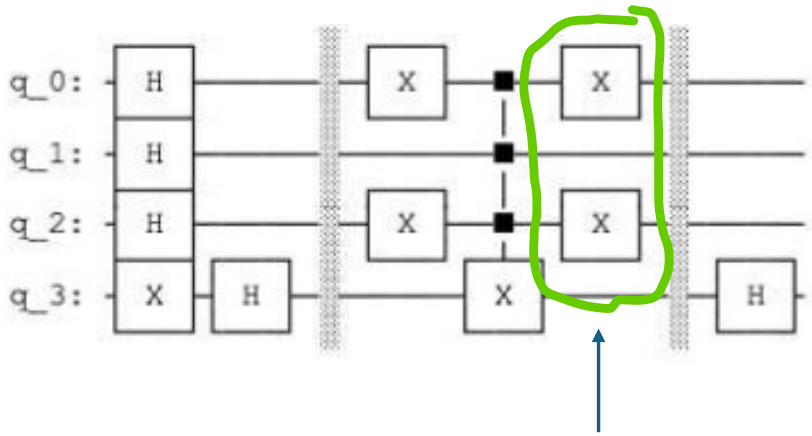
Un-Computation

$$H_0 = I \otimes I \otimes I \otimes H$$

and

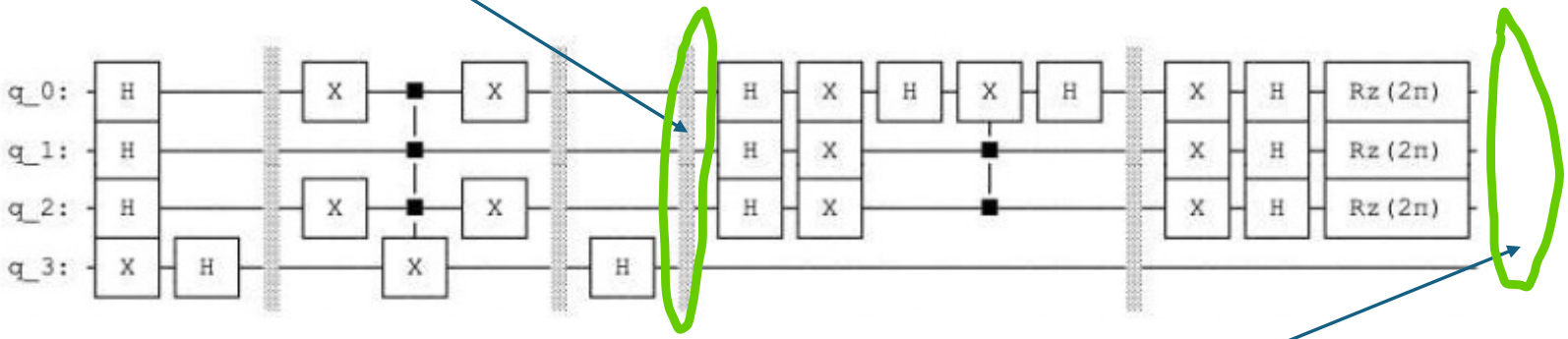
$$H_0 \cdot T \cdot H_4 \cdot |0001\rangle = \frac{1}{\sqrt{8}} \sum_{x \in B^3} (-1)^{o(x)} \cdot |x\rangle \otimes |0\rangle$$

the value of the function $o(x)$ is encoded by $(-1)^{o(x)}$, see as well Figure 5



$$Statevector = \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{1}{\sqrt{8}}\ \frac{1}{\sqrt{8}}\ -\frac{1}{\sqrt{8}}\ \frac{1}{\sqrt{8}}\ \frac{1}{\sqrt{8}}\ \frac{1}{\sqrt{8}}\ \frac{1}{\sqrt{8}}\ \frac{1}{\sqrt{8}}\right]$$

$$Statevector = \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{1}{\sqrt{8}}\ \frac{1}{\sqrt{8}}\ -\frac{1}{\sqrt{8}}\ \frac{1}{\sqrt{8}}\ \frac{1}{\sqrt{8}}\ \frac{1}{\sqrt{8}}\ \frac{1}{\sqrt{8}}\ \frac{1}{\sqrt{8}}\right]$$



$$Statevector = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0.1768\ 0.1768\ 0.8839\ 0.1768\ 0.1768\ 0.1768\ 0.1768\ 0.1768]$$

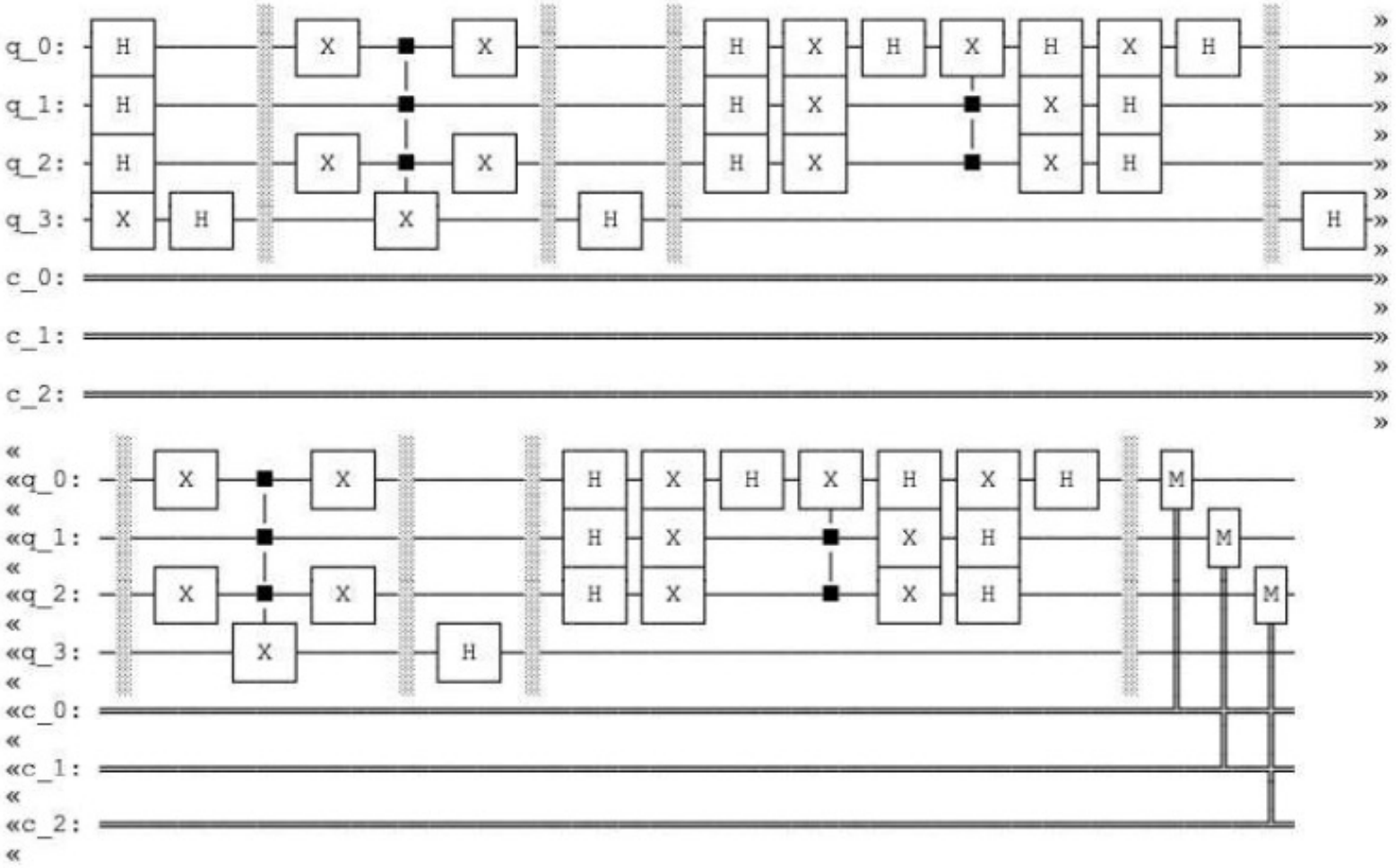
Two Rotations

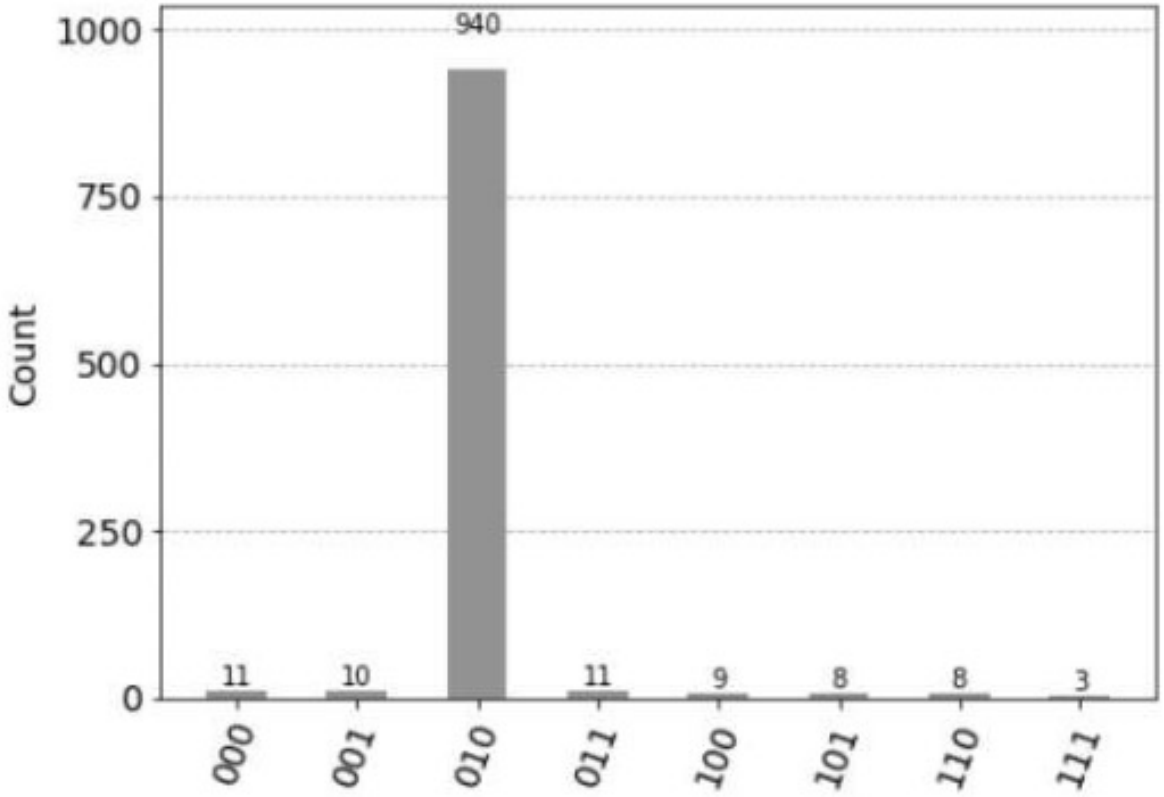
In the next step we will perform two rotations using *qasm simulator*. We do not minus sign operation

$$G_3 = -H_3 \cdot \Lambda_3 \cdot H_3$$

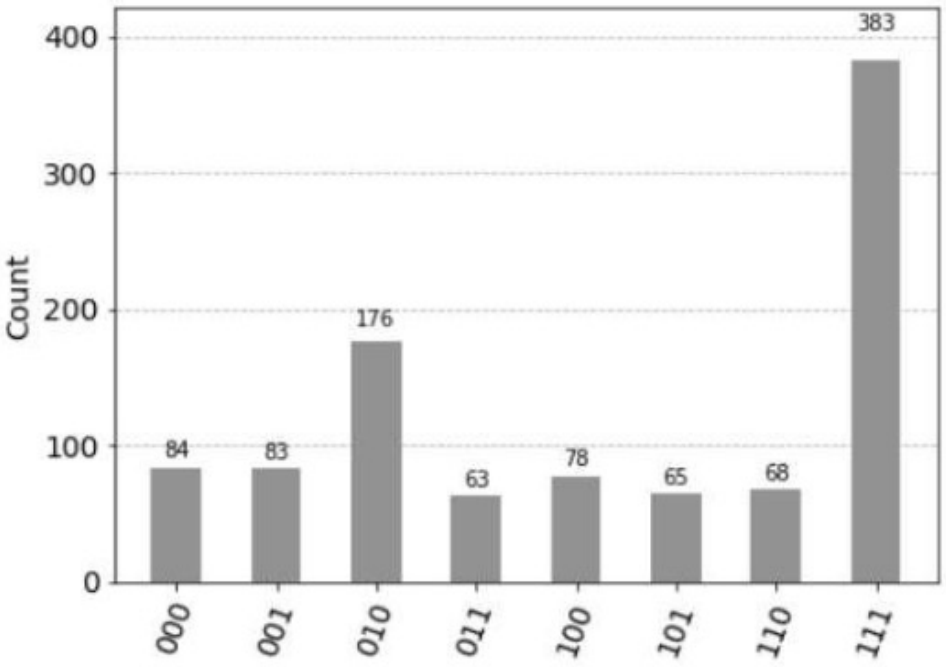
to get the correct result, what changes is the minus sign of over all amplitudes. This has no consequence for the resulting probabilities, so the Grover's amplification can be simplified to

$$G_3 = H_3 \cdot \Lambda_3 \cdot H_3$$


q_0:
q_1:
q_2:
q_3:
c_0:
c_1:
c_2:
H
X
M


1000
750
500
250
0
Count
940
11
10
11
9
8
8
3
000
001
010
011
100
101
110
111

- We may ask what would happen if we do not un-compute and define our oracle simply as $T = MCX \cdot (X \otimes I \otimes X \otimes I).$



- Histogram of Counts after two rotation of Grover's amplification without the un-computation of the oracle
- The resulting histogram of cost does not indicate the correct solution

For m qubits we can define Λ_m accordingly using the MCX gate

```
gate = MCXGate(m)
qc.append(gate, [0, 1, 2,...., m, m+1])
```

with

$$X_m = \underbrace{X \otimes X \otimes \cdots \otimes X}_{M \text{ times}}$$

$$H_0^m = \underbrace{I \otimes I \otimes \cdots \otimes I}_{M-1 \text{ times}} \otimes H$$

$$\Lambda_m = X_m \cdot H_0^m \cdot MCX \cdot H_0^m \cdot X_m$$

Alternatively Λ_m can be implemented efficiently with $f_0(x)$

$$f_0(x) = \begin{cases} 1 & if \quad x = 0 \\ 0 & else \end{cases}$$

as

$$\frac{1}{\sqrt{m}} \sum_{x \in B^m} (-1)^{f_0(x)} \cdot |x\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right).$$

SAT Problem

- The Boolean satisfiability problem (SAT problem) is the problem of determining if there exists an interpretation that satisfies a given Boolean formula, whether the formula evaluates true

The formula satisfiability problem is as follows, is a formula ϕ composed of

- m boolean variables: $x_1, x_2, \cdots, x_m$;
- k boolean connectivities: $\wedge$ (AND), $\vee$ OR $\neg$ (NOT), $\rightarrow$ (implication), $\leftrightarrow$ (if and only if);
- parentheses.

x	y	$x \to y$	$\neg x \vee y$
1	1	1	1
1	0	0	0
0	1	1	1
0	0	1	1

x	y	$x \leftrightarrow y$	$\neg(x \; XOR \; y)$
1	1	1	1
1	0	0	0
0	1	0	0
0	0	1	1

$$\phi = ((x_1 \to x_2) \vee \neg((\neg x_1 \leftrightarrow x_3) \vee x_4)) \wedge \neg x_2$$

has the interpretation $x_1 = 0$, $x_2 = 0$, $x_3 = 1$ and $x_4 = 1$ that satisfies ϕ with

$$\phi = ((0 \to 0) \vee \neg((\neg 0 \leftrightarrow 1) \vee 1)) \wedge \neg 0$$

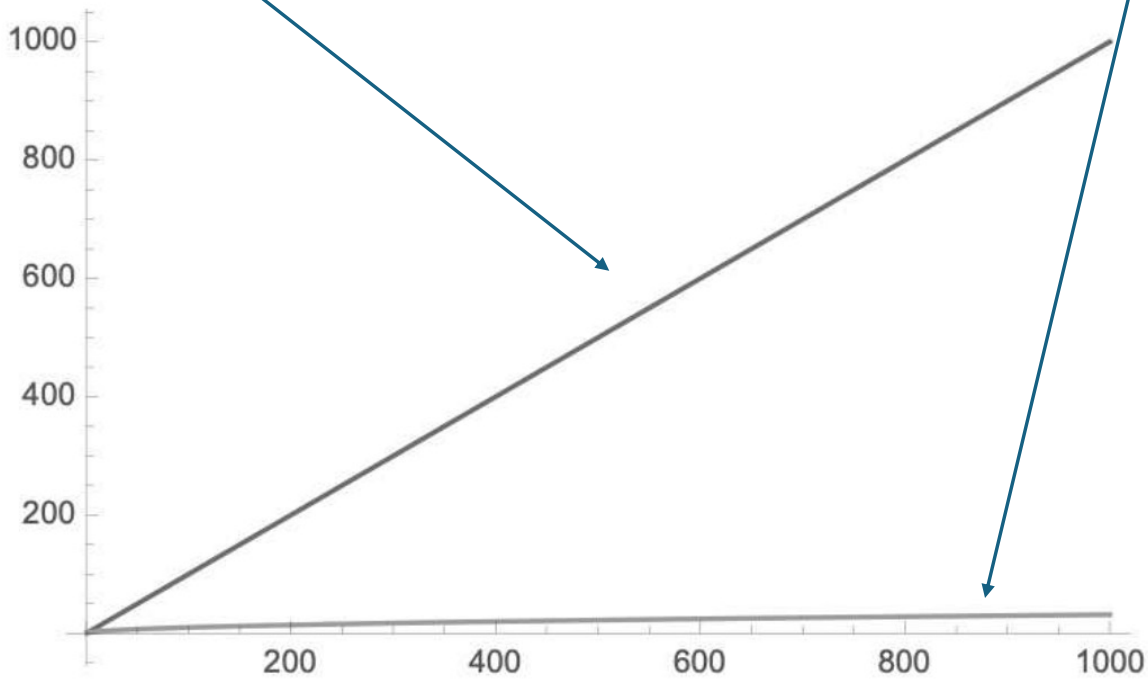
$$\phi = (1 \vee \neg(1 \vee 1)) \wedge 1$$

$$\phi = 1$$

There are 2^m possible assignments in a formula ϕ with m variables, the checking every assignment requires 2^m time on a conventional computer.

- A problem is easy if a conventional computer can determine the instances related to the input for the answer “Yes” in polynomial time
  - Polynomial-time algorithms are said to be fast since they can be executed in an acceptable time on a conventional computer and are called P
- Otherwise, we state that the problem is hard, means the time required grows exponentially and cannot be executed in an acceptable time on a conventional computer, such a problem is called NP
- A problem is called N P – complete if all possible instances must be examined by the conventional computer
  - A conventional computer can only solve a problem by checking all possible instances one after the other one

Despite the fact the saving of Grover's algorithm of $= O(\sqrt{n}) = O(2^{\frac{m}{2}})$ compared to $O(n) = O(2^m)$ is huge, $NP - complete$ problems remain $NP - complete$ on a quantum computer.



Why Huge and another Complexity Class?

saving of $O(2^{\frac{m}{2}})$ compared to $O(2^m)$ is huge, $2^m \neq O(2^{\frac{m}{2}})$ means

$$\Theta(2^{\frac{m}{2}}) \neq \Theta(2^m).$$

For $2^m \neq O(2^{\frac{m}{2}})$ we assume there exist a constant c, that for certain value $m_0 > 0$

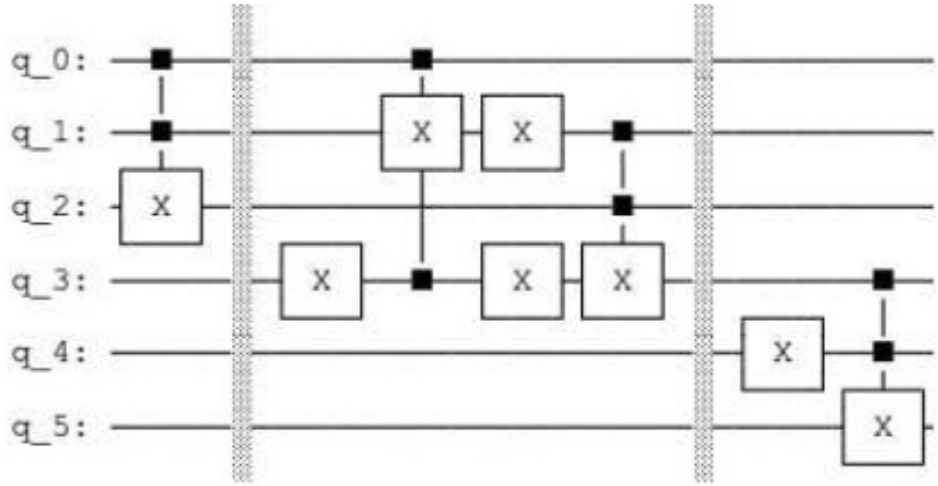
$$0 \leq 2^m \leq c \cdot 2^{\frac{m}{2}} \quad \forall m \geq m_0.$$

However such a constant does not exist because from

$$2^m = 2^{\frac{m}{2}} \cdot 2^{\frac{m}{2}} \leq c \cdot 2^{\frac{m}{2}}$$

follows the simple contradiction

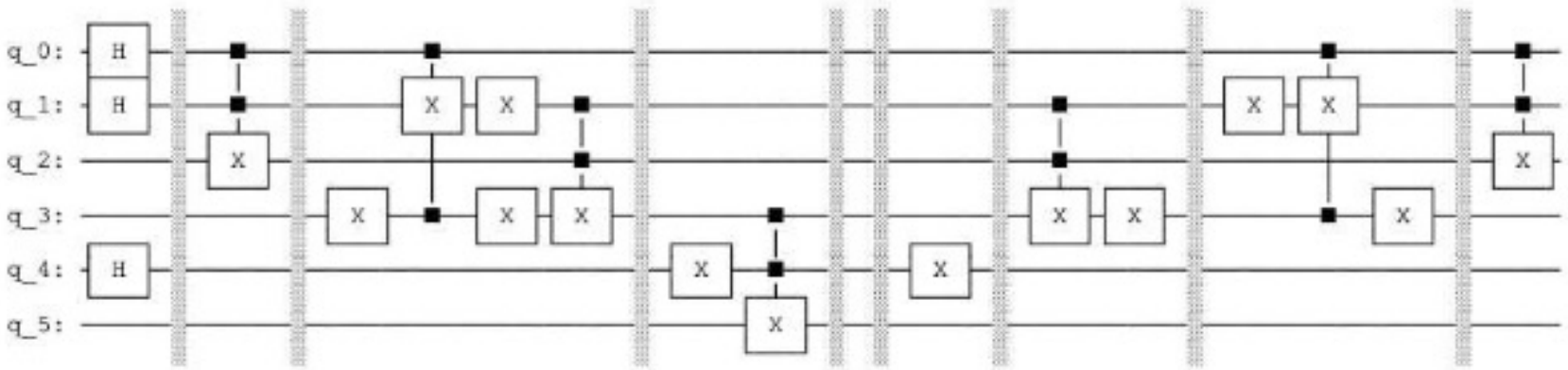
$$2^{\frac{m}{2}} \leq c.$$



In the next step we apply Grover's amplification to a marked state of three qubits. Our solution corresponds to the Boolean formula

$$\phi = (x_1 \leftrightarrow x_2) \wedge (x_1 \wedge x_2) \wedge \neg x_3$$

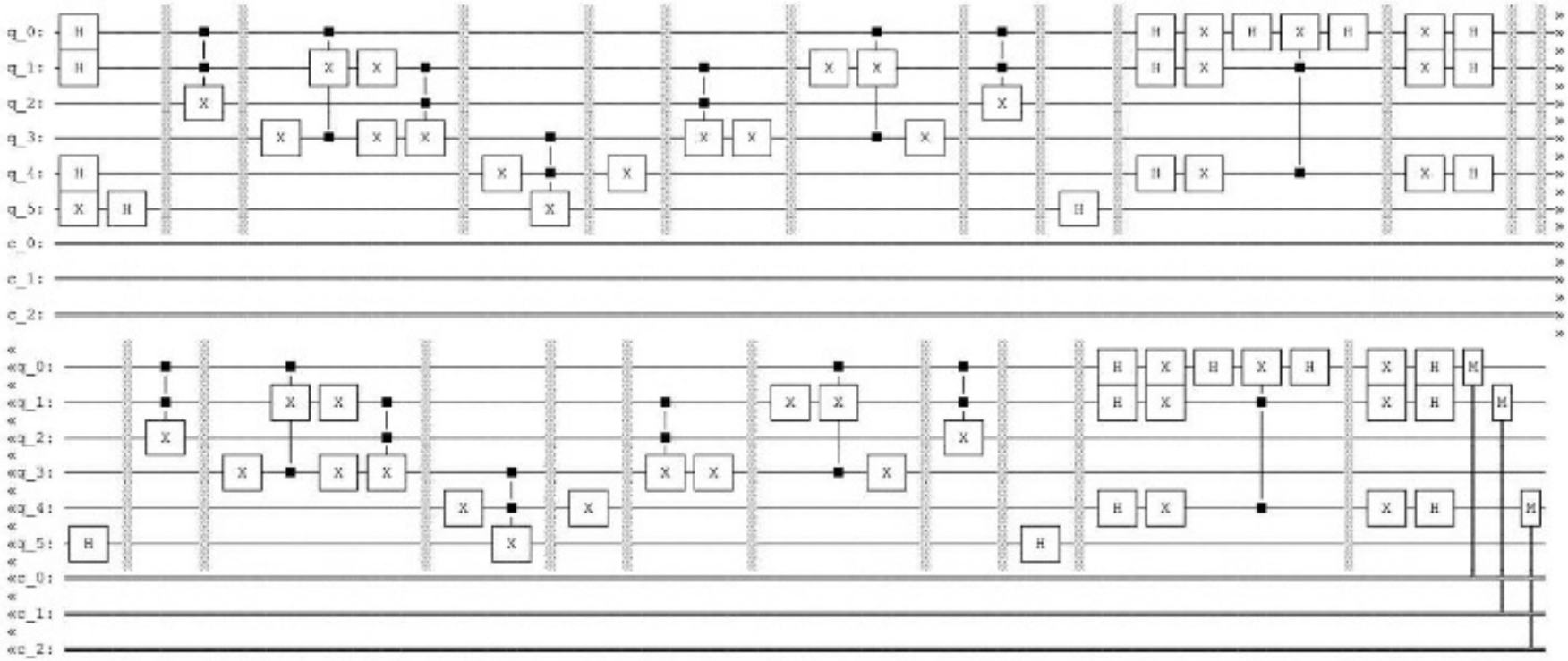
that evaluates true, which is the case for $x_1 = 1$, $x_2 = 1$ and $x_3 = 0$. In this case, the state determined by the oracle function with the solution encoded by $(-1)^{o(x)}$.

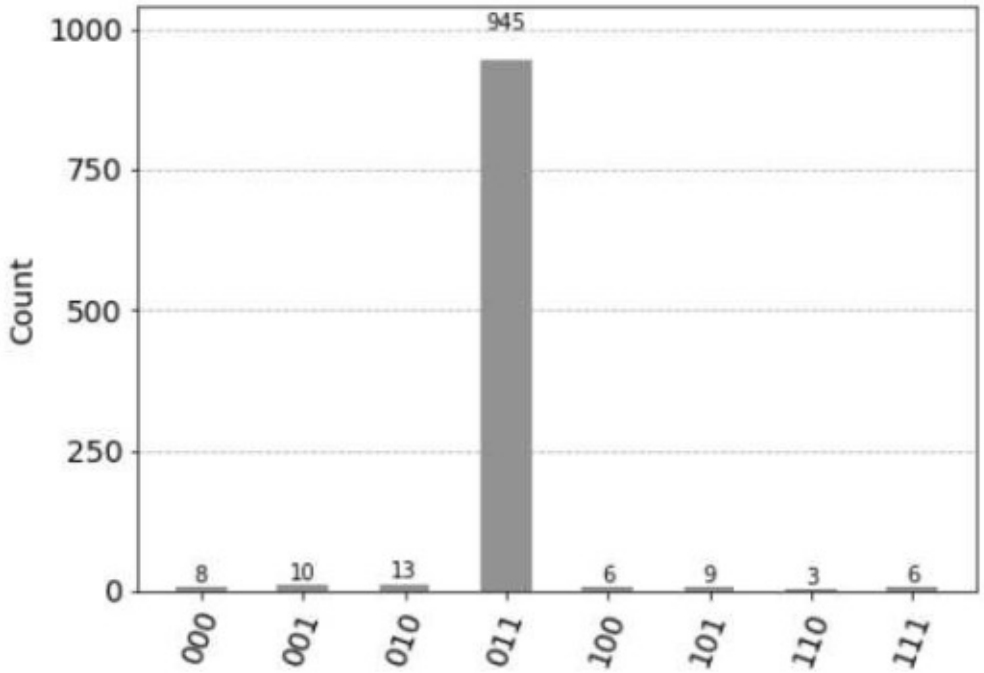


- The circuit representing the Boolean formula $\varphi = (x_1 \leftrightarrow x_2) \wedge (x_1 \wedge x_2) \wedge \neg x_3$.

- In quantum computation, it is not possible to reset the information to the pattern representing the initial state
- Instead, we un-compute the output back to the input
  - The input are the qubits 0, 1 and 4 mapped in superposition by Hadamard gates
  - The output is represented in the qubit 5 indicating by the value 1 the presence of the solution

Two Rotations

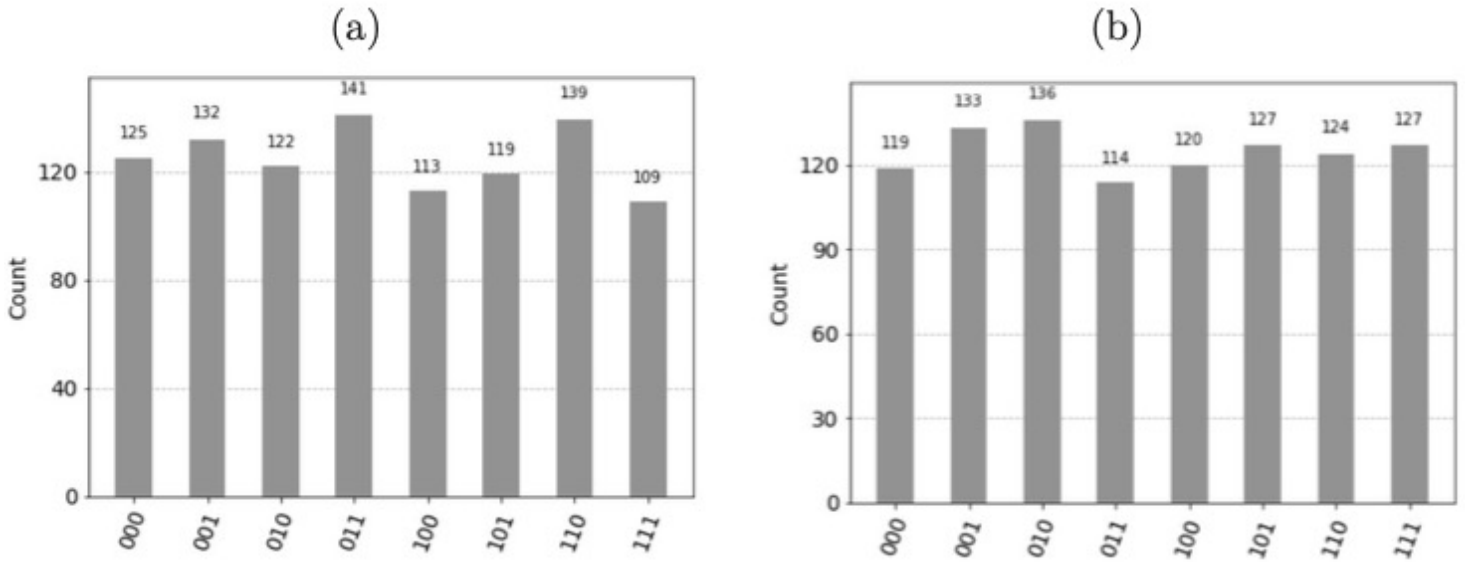




- The qubits 0, 1 and 4 indicate the eight possible instantiations for x_1, $x2$ and x_3 for the $\phi = (x_1 \leftrightarrow x_2) \wedge (x_1 \wedge x_2) \wedge \neg x_3$ SAT problem
- The solution after two rotations for the SAT problem has the highest count with the measured state $|011\rangle$

- For the Boolean formula
  - $\phi=(x_1 \leftrightarrow x_2) \wedge (\neg x_1 \wedge x_2) \wedge \neg x_3$
- no solution exists
  - the formula can never evaluate to the value true.

  - (a) one rotation, (b) two rotations



Grover's Amplitude Amplification

- Grovers Algorithm requires uniform distribution
- Why?
- $n=2^m$

$$|\psi\rangle = \frac{1}{\sqrt{n}} \cdot |x_1\rangle + \frac{1}{\sqrt{n}} \cdot |x_2\rangle + \cdots + \frac{1}{\sqrt{n}} \cdot |x_n\rangle = \begin{pmatrix} \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \end{pmatrix}$$

$$G_{\text{uniform}} = 2 \cdot |\psi\rangle\langle\psi| - I_m$$

$$G_m = H^{\otimes m}(2 \cdot |0^{\otimes m}\rangle\langle 0^{\otimes m}| - I_m)H^{\otimes m}$$

$$G_{\text{uniform}} = H^{\otimes m}(2|0\rangle\langle 0| - I)H^{\otimes m}$$

General Amplitude Amplification

Quantum Amplitude Amplification and Estimation; Gilles Brassard, Peter Hoyer, Michele Mosca, Alain Tapp, 2000, **https://arxiv.org/abs/quant-ph/0005055**

$$G_{\text{uniform}} = H^{\otimes m}(2|0\rangle\langle 0| - I)H^{\otimes m}$$

For non uniform distribution

$$A \neq H^{\otimes m}$$

Gilles Brassard

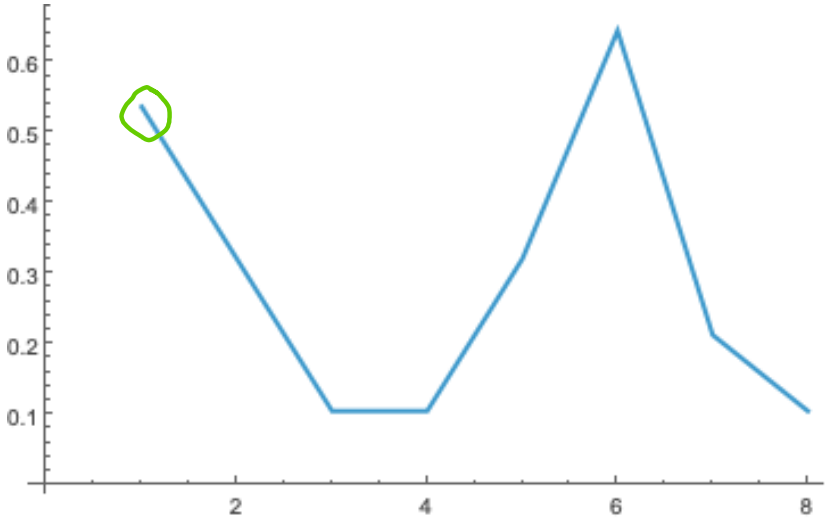
$$A|0\rangle = |\psi\rangle$$

$$D = A(2|0\rangle\langle 0| - I)A^{\dagger}$$

Example for non-uniform distribution

- $|\psi\rangle = \{\{\frac{5}{\sqrt{86}}, \frac{3}{\sqrt{86}}, \frac{1}{\sqrt{86}}, \frac{1}{\sqrt{86}}, \frac{3}{\sqrt{86}}, 3\sqrt{\frac{2}{43}}, \sqrt{\frac{2}{43}}, \frac{1}{\sqrt{86}}\}\}^T$

$$2 \cdot |\psi\rangle\langle\psi| - I_m$$



$$D = \begin{pmatrix} -\frac{18}{43} & \frac{15}{43} & \frac{5}{43} & \frac{5}{43} & \frac{15}{43} & \frac{30}{43} & \frac{10}{43} & \frac{5}{43} \\ \frac{15}{43} & -\frac{34}{43} & \frac{3}{43} & \frac{3}{43} & \frac{9}{43} & \frac{18}{43} & \frac{6}{43} & \frac{3}{43} \\ \frac{5}{43} & \frac{3}{43} & -\frac{42}{43} & \frac{1}{43} & \frac{3}{43} & \frac{6}{43} & \frac{2}{43} & \frac{1}{43} \\ \frac{5}{43} & \frac{3}{43} & \frac{1}{43} & -\frac{42}{43} & \frac{3}{43} & \frac{6}{43} & \frac{2}{43} & \frac{1}{43} \\ \frac{15}{43} & \frac{9}{43} & \frac{3}{43} & \frac{3}{43} & -\frac{34}{43} & \frac{18}{43} & \frac{6}{43} & \frac{3}{43} \\ \frac{30}{43} & \frac{18}{43} & \frac{6}{43} & \frac{6}{43} & \frac{18}{43} & -\frac{7}{43} & \frac{12}{43} & \frac{6}{43} \\ \frac{10}{43} & \frac{6}{43} & \frac{2}{43} & \frac{2}{43} & \frac{6}{43} & \frac{12}{43} & -\frac{39}{43} & \frac{2}{43} \\ \frac{5}{43} & \frac{3}{43} & \frac{1}{43} & \frac{1}{43} & \frac{3}{43} & \frac{6}{43} & \frac{2}{43} & -\frac{42}{43} \end{pmatrix}$$

$$|\psi\rangle = \{-\frac{5}{\sqrt{86}}, \frac{3}{\sqrt{86}}, \frac{1}{\sqrt{86}}, \frac{1}{\sqrt{86}}, \frac{3}{\sqrt{86}}, 3\sqrt{\frac{2}{43}}, \sqrt{\frac{2}{43}}, \frac{1}{\sqrt{86}}\}^T$$

$|\psi\rangle^{\text{one rotation}}$ ={0.990557, -0.0526625, -0.0175542, -0.0175542, -0.0526625, -0.105325, -0.0351083, -0.0175542}^{T}

$$|\psi\rangle = \sqrt{a}\,|\text{good}\rangle + \sqrt{1-a}\,|\text{bad}\rangle$$

$$\sin^2(\theta) = a \quad \Rightarrow \quad \theta = \arcsin(\sqrt{a})$$

$$\text{Success probability after } k \text{ steps} = \sin^2\left((2k+1)\theta\right)$$

Solving:

$$r \approx \frac{\pi}{4\theta} - \frac{1}{2}$$

Substitute $\theta = \arcsin(\sqrt{a})$:

$$r \approx \frac{\pi}{4\,\arcsin(\sqrt{a})} - \frac{1}{2}$$

If $a \ll 1$, then:

$$\arcsin(\sqrt{a}) \approx \sqrt{a}$$

So:

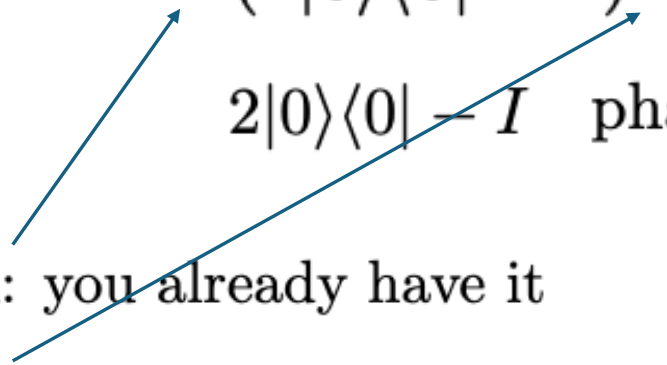
$$r \approx \frac{\pi}{4\sqrt{a}} \approx \frac{1}{\sqrt{a}}$$

How to Implement

- We **have to know** $A|0\rangle = |\psi\rangle$

$$D = A(2|0\rangle\langle 0| - I)A^\dagger$$

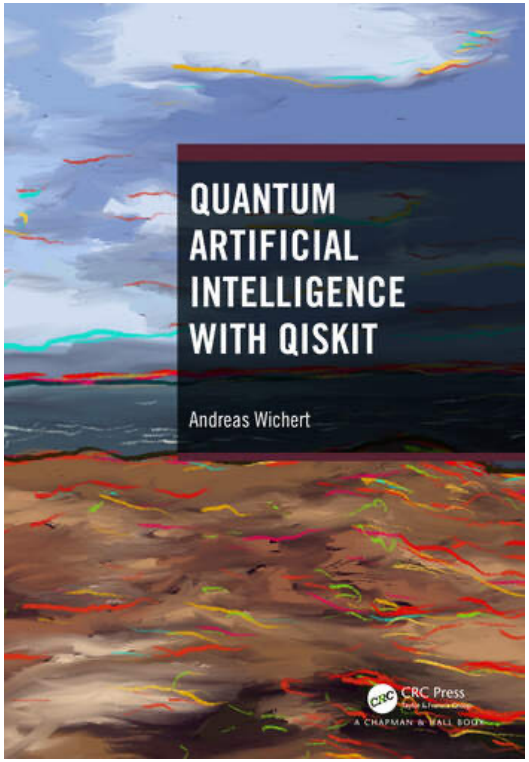
$2|0\rangle\langle 0| - I$ phase flip on $|0\rangle$



A: you already have it

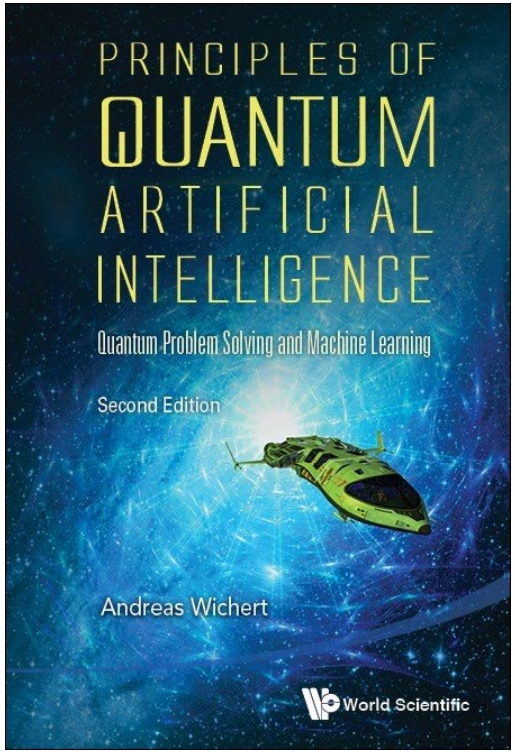
$A^\dagger$: just reverse the circuit

Reflection about your actual state implemented by "uncomputing" with $A^\dagger$



- Chapter 5
- Chapter 6

Quantum Artificial Intelligence with Qiskit, A. Wichert, Chapman and Hall/CRC, 2024



- Chapter 10

Principles of Quantum Artificial Intelligence: Quantum Problem Solving and Machine Learning, 2nd Edition, A. Wichert, World Scientific, 2020