An Outline for a Kalman Filter and Recursive Parameter Estimation Approach Applied to Stock Market Forecasting

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Abstract:

An outline of a system that models and forecasts stock market processes is described. The method involves a spectral estimation approach to ARMA modelling, forecasting is performed through Kalman filtering, and adaptive parameter estimation performed via the Gauss-Newton algorithm. A system that implements this methodology is currently being coded and will be tested in the near future.

Keywords: Kalman Filter; ARIMA & ARMA Models; Spectral Estimation (SE); Recursive Parameter Estimation (RPE); System Identification.

Introduction:

Stock markets are systems that are dynamic and non-linear in nature and time series processes of these are non-ergodic and non-stationary. Therefore it is necessary to adapt models of these to the time-variant properties of the system through on-line parameter estimation methods.

Efforts in the area have used statistical models and the Box-Jenkins time series approach[1], but the processes were assumed to be ergodic and time-invariant. Other ongoing work uses the Neural Net approach [1, 2, 3] to handle the non-linear nature of stock market processes.

The use of the Kalman filter approach for forecasting and the Recursive Parameter Estimation (RPE) approach have not yet been used for state estimation and for updates of the linearized time-variant models of stock market processes.

This paper describes the outline of an experimental project using a Kalman filter [4][5] and RPE [6] approach to obtain on-line estimates of the state and model and predict future values of composite indices of the stock market using time series data.

The statistical models of the stock market are based on the Box-Jenkins Auto-Regressive Integrated Moving Average, i.e. ARIMA(p,d,q), models of time series [7]. These models are identified by differencing an assumed non-stationary process to obtain a locally Wide Sense Stationary (WSS) and locally ergodic process. The WSS process is initially identified as an Auto-Regressive Moving Average process using modern techniques in Spectral Estimation (SE) [8]. It should be mentioned that a linear modelling approach is applied to processes exhibiting well known non-linear behaviour. However, these processes do exhibit periods of linear behaviour.

Upon obtaining an initial parameter set using SE techniques, the time-variant models and state of the system are then updated and forecasted in real-time via a RPE approach called the Gauss-Newton algorithm [6].

We assume that no knowledge or understanding of the mechanisms that drive stock market processes is known. This is the motivation for using the methods described above for system identification and forecasting.

The forecasting system described above will be implemented in the object oriented programming language C++. 

ARMA & ARIMA Models:

Auto-Regressive Moving Average (ARMA) models are
a class of general linear models that exhibit WSS behaviour. In an ARMA model, an input driving sequence \( \{ u_n \} \) and the output sequence \( \{ x_n \} \) are related by the linear difference equation (1).

\[
x_n = - \sum_{k=1}^{p} a_k x_{n-k} + \sum_{k=0}^{q} b_k u_{n-k}
\]

(1)

Here it is assumed that \( x_n \) and \( u_n \) are scalar and that \( u_n \) is a gaussian white noise process with variance \( \sigma^2 \). We also assume that this input driving noise is not a measurable process and can at best be estimated. This makes the identification of the model different from general system identification where the input to the system is measurable.

The ARMA process is denoted as an ARMA\((p,q)\) process. The \( a_k \) coefficients are termed the autoregressive parameters and the \( b_k \) coefficients are termed the moving average parameters.

Suppose the observed process is \( x \). By applying this operator a sufficient number of times, say \( d \), we may obtain a WSS process. The operator \( \Delta^d \) applied to \( z_n \) for \( n \geq d \) is called the \( d^{th} \) difference of \( z_n \). If the number of times that the time series \( z_n \) has been differenced appropriately, the resulting process \( \{ \Delta^d z_n \} \) will be stationary and it may now be represented by an appropriate ARMA model of order \((p,q)\). Now that \( \Delta^d z_n \) is a stationary process, we will represent it by \( x_n \). We say that the \( z_n \) process is an ARIMA process of order \((p,d,q)\) and is described by equation (4).

\[
\Delta^d z_n = - \sum_{k=1}^{p} a_k x_{n-k} + \sum_{k=0}^{q} b_k u_{n-k}
\]

(4)

Details on ARMA and ARIMA processes can be found in the literature [7, 8].

### Spectral Estimation Methods:

The general problem of Spectral Estimation (SE) is that of determining the spectral content of a stochastic process based on a finite set of observations from that process [8]. The most important result that concerns us about SE is that it relates the Power Spectral Density (PSD) and the autocorrelation function (ACF) to the ARMA parameters. In turn, estimates of the ACF and PSD can be used to give estimates of the ARMA parameters [8].

We must first introduce the PSD. Suppose we have a complex WSS random process \( \{ x_n \} \), then its PSD, denoted \( P_{xx}(f) \), is defined as:

\[
P_{xx}(f) = \sum_{k=-\infty}^{\infty} r_{xx}[k] \exp(\pm j2\pi fk), \quad -0.5 \leq f \leq 0.5.
\]

(3)

The SE approach to identifying ARMA processes relates the parameters of the AR and MA branches to the PSD function of the process.

\[
P_{ARMA}(f) = P_{xx}(f) = \sigma^2 \left| \frac{B(f)}{A(f)} \right|^2
\]

(6)

Here \( A(f) \) denotes \( A(z) \) and \( B(f) \) denotes \( B(z) \) where \( z \) is evaluated along the unit circle, i.e. \( z = \exp(\pm j2\pi f) \) for \(-0.5 \leq f \leq 0.5\) (refer to eq. (2)).

The relationship between the ACF in (3) and the ARMA parameters in (6) is shown in [8].

Many SE methods exist to identify AR, MA and ARMA processes. The methods deemed most important to this study are the Durbin method for MA processes, the Burg and Levinson methods for AR processes, and the Modified Yule-Walker Equations and Recursive MLE methods for ARMA processes. These
methods and others are described in [8, 9]. The Aikake Information Criterion (AIC) [8, page 297] is used to select the order of the model.

**State Space Approach for ARMA Model:**

In order for Kalman Filter predictions of the identified process to begin, we must find a state space model for the ARMA process. Many models were found but these models proved to be difficult to implement. One involved having to estimate a q-dimensional state vector of noise, thus the dimension of the state vector was unnecessarily large. Another would require the derivation of a Kalman filter solution that could handle time correlated input noise; this adds complexity to the implementation of the filter.

Models should be as simple as is, to be compatible with dynamics contained in the data and consistent with data accuracy [10]. Fortunately minimal state space realizations (controllable and observable) [11] exist and one was found in [10] and its demonstration is found in [12, page 237].

This model simplifies the Kalman filter problem somewhat in that only the process and measurement noise are correlated. The minimal realization of an ARMA state space model, of order \((p, q)\), with \(q \leq p\) is given in the following equations:

\[
x_{k+1} = \begin{bmatrix}
-a_{1,k} & 1 & \cdots & 0 & 0 \\
-a_{2,k} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-a_{p-1,k} & \cdots & 0 & 1 & 0 \\
-a_{p,k} & \cdots & \cdots & \cdots & 0 \\
\end{bmatrix} x_k \\
+ \begin{bmatrix}
b_{1,k} - a_{1,k} \\
b_{2,k} - a_{2,k} \\
\vdots \\
b_{p,k} - a_{p,k} \\
\end{bmatrix} v_k \\
\]

\[
x_k = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\end{bmatrix} x_k + v_k 
\]  

We will denote the transition matrix as \(F_k\), the noise coupling matrix as \(G_k\) and the observation matrix as \(H_k\). We will note here that in this model, not only is the noise correlated but it is also equal. Substituting variables for the matrices, the state space model is rewritten as:

\[
x_{k+1} = F_k x_k + G_k v_k \\
z_k = H_k x_k + v_k 
\]

**Kalman Filter Solution:**

Since the state space model of the system has correlated process and measurement noise the popular conventional Kalman filter algorithm will be suboptimal. Fortunately, the solution to this problem is relatively easy to obtain.

We note here in Kalman filter terminology that the noise covariance matrices \(Q, R\), and \(S\) are equal. We first define the Kalman gain \(K_k\), the innovation \(\hat{z}_k\) and the variable \(D_k\) as:

\[
K_k = F_k^{-1} H_k D_k \\
\hat{z}_k = z_k - H_k \hat{x}_k \\
D_k = (H_k F_k^{-1} H_k^T + R_k)^{-1} 
\]

With a few manipulations of the algorithm found in [5, Pages 122-123], the algorithm becomes:

\[
x_{k+1}^- = x_k^- + K_k \hat{z}_k \\
x_{k+1}^+ = P_{k+1}^- = K_k H_k P_k^- \\
\hat{z}_{k+1} = F_k x_k^- + G_k S_k D_k \hat{z}_k \\
P_{k+1}^- = F_k P_k^- F_k^T + G_k S_k G_k^T \\
- (F_k K_k S_k G_k^T) - (F_k K_k S_k G_k^T) \\
- G_k S_k D_k S_k G_k^T 
\]

This algorithm is in the recursive form that the \(a posteriori\) state estimate (and error covariance) are functions of the \(a priori\) state estimate (and error covariance) and \(vice-versa\). This can greatly simplify the implementation of higher order models since only one vector is needed for storage at any time. As well, \(Q_k\) and \(S_k\) may be replaced by \(R_k\).

Preliminary simulations on ARMA processes show that ARMA models are ill-conditioned for this algorithm. The Joseph form [5, Page 70] and square root filtering method, similar to those found in [14] are applied to the computation to ensure symmetric positive semi-definite covariance matrices.

**Adaptive Estimation of ARMA Parameters via the Gauss-Newton Algorithm:**

Stock market time series processes are non-ergodic and non-stationary in nature. In on-line forecasting of processes such as these, it is desirable to adapt models of these to the time-variant properties of the system.

The Gauss-Newton algorithm for identifying time-varying parameters for ARMA models is found in [13] and is given as:

\[
\begin{bmatrix}
\hat{a}_{k+1} \\
\hat{b}_{k+1}
\end{bmatrix} =
\begin{bmatrix}
\hat{a}_k \\
\hat{b}_k
\end{bmatrix} - \mu \overline{R}_{k+1}^{-1} n_k \psi_k
\]

(18)

\[ R_{k+1} = R_k + \mu (\psi_k \psi_k' - R_k) \]

(19)

where

\[ n_k = y_k - \begin{bmatrix} \hat{a}_{k+1}' \\
\hat{b}_{k+1}' \end{bmatrix} \begin{bmatrix} Y_k \\
N_k \end{bmatrix} \]

(20)

This finishes the description of the forecasting system.

Remarks

The forecasting system based on the methods described above is currently being coded and will be finished in the near future. It will be tested in different market environments. It is expected to perform well during periods of linear behaviour. However, the markets do exhibit non-linear, chaotic and sometimes catastrophic behaviour. The performance under these conditions is anxiously awaited.

References


