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Properties of the Mittag-Leffler relaxation function

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The Mittag-Leffler relaxation function, $E_\alpha(-x)$, with $0 \leqslant \alpha \leqslant 1$, which arises in the description of complex relaxation processes, is studied. A relation that gives the relaxation function in terms of two Mittag-Leffler functions with positive arguments is obtained, and from it a new form of the inverse Laplace transform of $E_\alpha(-x)$ is derived and used to obtain a new integral representation of this function, its asymptotic behaviour and a new recurrence relation. It is also shown that the fastest initial decay of $E_\alpha(-x)$ occurs for $\alpha = 1/2$, a result that displays the peculiar nature of the interpolation made by the Mittag-Leffler relaxation function between a pure exponential and a hyperbolic function.

KEY WORDS: Mittag-Leffler function, Laplace transform, relaxation kinetics

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1. Introduction

The Mittag-Leffler function $E_\alpha(z)$, named after its originator, the Swedish mathematician Gösta Mittag-Leffler (1846–1927), is defined by [1–4]

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{1}$$

where z is a complex variable and $\alpha \geqslant 0$ (for $\alpha = 0$ the radius of convergence of equation (1) is finite, and one has by definition $E_0(z) = 1/(1-z)$). The Mittag-Leffler function is a generalization of the exponential function, to which it reduces for $\alpha = 1$, $E_1(z) = \exp(z)$. For $0 < \alpha < 1$ it interpolates between a pure exponential and a hyperbolic function, $E_0(z) = 1/(1-z)$. The precise nature of this interpolation is the subject of the present study. The Mittag-Leffler function obeys the following relations [3]:

$$E_{1/n}(z^{1/n}) = \mathrm{e}^z \left[n - \sum_{k=1}^{n-1} \frac{\Gamma(1-k/n, z)}{\Gamma(1-k/n)} \right] \quad n = 2, 3, \ldots, \tag{2}$$

$$E_{m\alpha}(z^m) = \frac{1}{m}\sum_{k=0}^{m-1} E_\alpha(z\, e^{2\pi ik/m}) \quad m = 2,\, 3, \ldots . \tag{3}$$

It follows that $E_{1/2}(z) = \exp(z^2)\text{erfc}(-z)$ and $E_2(z) = \cosh(\sqrt{z})$. An explicit expression for any rational value of the parameter $\alpha = m/n$ can be obtained from equations (2) and (3).

The generalized Mittag-Leffler function is [3]

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \tag{4}$$

so that $E_{\alpha,1}(z) = E_\alpha(z)$. In the simplest form $\alpha, \beta \geqslant 0$. Algorithms for the computation of the generalized Mittag-Leffler function were recently discussed [5].

There has been much recent interest in the Mittag-Leffler and related functions in connection with the description of relaxation phenomena in complex physical and biophysical systems [6–18], namely within the framework of fractional (non-integer) kinetic equations. In this work, and having in view the applications of this function to relaxation phenomena, the discussion will be generally restricted to $E_\alpha(-x)$ that corresponds to a relaxation function when x is a non-negative real variable (usually standing for the time) and $0 \leqslant \alpha \leqslant 1$.

2. Basic relation

Using equations (1) and (4), $E_\alpha(-x)$ can be written in terms of two Mittag-Leffler functions with positive arguments,

$$E_\alpha(-x) = E_{2\alpha}\left(x^2\right) - x\, E_{2\alpha,1+\alpha}\left(x^2\right). \tag{5}$$

A particular case of equation (3) follows immediately from equation (5),

$$E_{2\alpha}\left(x^2\right) = \frac{E_\alpha(x) + E_\alpha(-x)}{2}. \tag{6}$$

It also follows from equation (5) that

$$E_\alpha(-i\omega) = E_{2\alpha}\left(-\omega^2\right) - \mathrm{i}\omega\, E_{2\alpha,1+\alpha}\left(-\omega^2\right), \tag{7}$$

a result that will be used in the next section.

3. Inverse Laplace transform

A simple form of the inverse Laplace transform of $E_\alpha(-x)$ can be obtained by the method outlined in [17]. Briefly, the three following equations can be used for the direct inversion of a function $I(x)$ to obtain its inverse $H(k)$,

$$H(k) = \frac{e^{ck}}{\pi} \int_0^\infty [\mathrm{Re}[I(c+i\omega)\,]\cos(k\omega) - \mathrm{Im}[I(c+i\omega)]\sin(k\omega)]d\omega, \quad (8)$$

$$H(k) = \frac{2e^{ck}}{\pi} \int_0^\infty \mathrm{Re}[I(c+i\omega)]\cos(k\omega)d\omega \quad k > 0, \quad (9)$$

$$H(k) = -\frac{2e^{ck}}{\pi} \int_0^\infty \mathrm{Im}[I(c+i\omega)]\sin(k\omega)d\omega \quad k > 0. \quad (10)$$

where c is a real number larger than c_0, c_0 being such that $I(z)$ has some form of singularity on the line $\mathrm{Re}(z) = c_0$ but is analytic in the complex plane to the right of that line, i.e., for $\mathrm{Re}(z) > c_0$.

Using equation (7), application of equation (9) to $E_\alpha(-x)$ with $c = 0$, implying $0 \leqslant \alpha \leqslant 1$, yields a general relation for its inverse Laplace transform $H_\alpha(k)$,

$$H_\alpha(k) = \frac{2}{\pi} \int_0^\infty E_{2\alpha}(-\omega^2)\cos(k\omega)d\omega \quad k > 0, \quad 0 \leqslant \alpha \leqslant 1, \quad (11)$$

hence, for instance

$$H_1(k) = \frac{2}{\pi} \int_0^\infty \cosh(i\omega)\cos(k\omega)d\omega = \frac{2}{\pi} \int_0^\infty \cos(\omega)\cos(k\omega)d\omega = \delta(k-1), \quad (12)$$

$$H_{1/2}(k) = \frac{2}{\pi} \int_0^\infty e^{-\omega^2}\cos(k\omega)d\omega = \frac{1}{\sqrt{\pi}}e^{-k^2/4}, \quad (13)$$

$$H_{1/4}(k) = \frac{2}{\pi} \int_0^\infty e^{\omega^4}\mathrm{erfc}(\omega^2)\cos(k\omega)d\omega, \quad (14)$$

$$H_0(k) = \frac{2}{\pi} \int_0^\infty \frac{\cos(k\omega)}{1+\omega^2}d\omega = e^{-k}. \quad (15)$$

Another integral representation of $H_\alpha(k)$, based on the Lévy one-sided distribution $L_\alpha(k)$ [8], was obtained by Pollard [19] (see also [17,18]),

$$H_\alpha(k) = \frac{1}{\alpha}\, k^{-(1+\frac{1}{\alpha})} L_\alpha\left(k^{-\frac{1}{\alpha}}\right). \quad (16)$$

If equation (8) is used instead of equation (9), and taking into account equation (7),

$$H_\alpha(k) = \frac{1}{\pi}\int_0^\infty \left[E_{2\alpha}(-\omega^2)\cos(k\omega) + \omega E_{2\alpha,1+\alpha}(-\omega^2)\sin(k\omega)\right]\mathrm{d}\omega \quad 0 \leqslant \alpha \leqslant 1. \tag{17}$$

This form is less simple than equation (11), but is (formally) necessary for finding the asymptotic expansion of $E_\alpha(-x)$, as will be done in Section 5.

4. Complete monotonicity

It is known [19,20] that $E_\alpha(-x)$ is completely monotonic for $x \geqslant 0$ if $0 \leqslant \alpha \leqslant 1$, i.e., that

$$(-1)^n \frac{\mathrm{d}^n E_\alpha(-x)}{\mathrm{d}x^n} \geqslant 0, \quad x \geqslant 0, \quad 0 \leqslant \alpha \leqslant 1. \tag{18}$$

We remark here that this result follows immediately from

$$(-1)^n \frac{\mathrm{d}^n E_\alpha(-x)}{\mathrm{d}x^n} = \int_0^\infty k^n H_\alpha(k) \mathrm{e}^{-kx}\mathrm{d}k, \tag{19}$$

by noting that $H_\alpha(k) > 0$ for $k \geqslant 0$ if $0 \leqslant \alpha \leqslant 1 (H_\alpha(k)$ is a probability density function), as could be conjectured simply by plotting the function $H_\alpha(k)$ for several values of α. Demonstration that $\mathrm{H}_\alpha(\mathrm{k}) > 0$ for $k \geqslant 0$ is direct for $0 \leqslant \alpha \leqslant 1/2$, using equation (11) and knowing that $E_\alpha(-x)$ $(\alpha \leqslant 1)$ is always positive and decreases monotonically. General demonstrations for $0 \leqslant \alpha \leqslant 1$ were given by Feller [20] and Pollard [19].

5. Behaviour near the origin

Any relaxation function $I(x)$ can be written as

$$I(x) = \exp\left(-\int_0^x k(u)\mathrm{d}u\right), \tag{20}$$

where $k(x)$ is a x-dependent rate coefficient. When the relaxation is exponential, $k(x)$ is obviously constant. For the Mittag-Leffler relaxation function,

$$k(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\ln E_\alpha(-x) = \frac{1}{E_\alpha(-x)}\sum_{n=0}^{\infty}\frac{(n+1)\,(-x)^n}{\Gamma(1+\alpha+\alpha n)}, \tag{21}$$

whose initial value is finite and close to unity,

$$k(0) = \int_0^\infty k\, H_\alpha(k)\mathrm{d}k = \frac{1}{\Gamma(1+\alpha)}. \tag{22}$$

It follows nevertheless from equation (22) that the fastest initial decay occurs for $\alpha = 1/2$, a result hitherto unnoticed, and that shows the peculiar nature of the interpolation between a pure exponential and a hyperbolic function performed by the Mittag-Leffler relaxation function.

6. Asymptotic behaviour

Expansion of equation (17) in a power series gives

$$H_\alpha(k) = \frac{1}{\pi}\sum_{n=0}^{\infty} a_n(\alpha)\, k^n, \quad 0 \leqslant \alpha < 1, \tag{23}$$

with

$$a_0(\alpha) = \int_0^\infty E_{2\alpha}(-\omega^2)\mathrm{d}\omega. \tag{24}$$

The Laplace transform of equation (23) is the asymptotic expansion of $E_\alpha(-x)$,

$$E_\alpha(-x) = \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{a_n(\alpha)}{x^{n+1}}, \quad 0 \leqslant \alpha < 1. \tag{25}$$

Since $a_0(\alpha) \neq 0$ for $0 \leqslant \alpha < 1$, the Mittag-Leffler relaxation function has a hyperbolic (x^{-1}) asymptotic decay for $0 \leqslant \alpha < 1$, and an exponential decay only for $\alpha = 1$. The cross-over between the initial exponential-like decay and the asymptotic hyperbolic decay occurs at a value of x that is the shorter, the smaller the α. It follows from equation (25) that $E_\alpha(-x^2)$ asymptotically decays as x^{-2}, hence equation (24) is clearly convergent for $0 \leqslant \alpha < 1/2$.

7. A recurrence relation

By taking the Laplace transform of equation (11), a recurrence relation is obtained,

$$E_\alpha(-x) = \frac{2x}{\pi}\int_0^\infty \frac{E_{2\alpha}(-\omega^2)}{x^2+\omega^2}\mathrm{d}\omega, \quad 0 \leqslant \alpha \leqslant 1. \tag{26}$$

This relation works in the opposite way of equation (6), and allows the direct calculation of $E_\alpha(-x)$ from $E_{2\alpha}(-x^2)$. In this way, it follows, for instance, that

$$E_1(-x) = \frac{2x}{\pi}\int_0^\infty \frac{\cosh(\mathrm{i}\omega)}{x^2+\omega^2}\mathrm{d}\omega = \mathrm{e}^{-x}, \tag{27}$$

$$E_{1/2}(-x) = \frac{2x}{\pi}\int_0^\infty \frac{\mathrm{e}^{-\omega^2}}{x^2+\omega^2}\mathrm{d}\omega = \mathrm{e}^{x^2}\mathrm{erfc}(x), \tag{28}$$

$$E_{1/4}(-x) = \frac{2x}{\pi}\int_0^{\infty}\frac{e^{\omega^4}\operatorname{erfc}(-\omega^2)}{x^2+\omega^2}d\omega. \tag{29}$$

The asymptotic behaviour of $E_\alpha(-x)$ also follows directly from equation (26) for large $x(\alpha < 1)$.

8. Integral representations

The starting point is the known Laplace transform of $E_\alpha(-x^\alpha)$, $J_\alpha^\alpha(s)$, which can be obtained in closed form directly from the definition,

$$J_\alpha^\alpha(s) = \int_0^{\infty} E_\alpha(-x^\alpha)\, e^{-sx}dx = \frac{s^{\alpha-1}}{1+s^\alpha}. \tag{30}$$

Application of inversion equation (9) to equation (30) yields

$$E_\alpha(-x^\alpha) = \frac{2}{\pi}\sin(\alpha\pi/2)\int_0^{\infty}\frac{\omega^{\alpha-1}\cos(x\omega)}{1+2\omega^\alpha\cos(\alpha\pi/2)+\omega^{2\alpha}}d\omega, \tag{31}$$

hence a new integral representation for the Mittag-Leffler relaxation function is

$$E_\alpha(-x) = \frac{2}{\pi}\sin(\alpha\pi/2)\int_0^{\infty}\frac{\omega^{\alpha-1}\cos(x^{1/\alpha}\omega)}{1+2\omega^\alpha\cos(\alpha\pi/2)+\omega^{2\alpha}}d\omega. \tag{32}$$

Analogous representations are obtained with equations (8) and (10).

The previously known integral representation for $E_\alpha(-x^\alpha)$,

$$E_\alpha(-x^\alpha) = \frac{\sin(\alpha\pi)}{\pi}\int_0^{\infty}\frac{k^{\alpha-1}}{1+2k^\alpha\cos(\alpha\pi)+k^{2\alpha}}e^{-xk}dk, \tag{33}$$

was obtained from the Bromwich inversion integral [5]. It follows from equation (33) that

$$E_\alpha(-x) = \frac{\sin(\alpha\pi)}{\pi}\int_0^{\infty}\frac{k^{\alpha-1}}{1+2k^\alpha\cos(\alpha\pi)+k^{2\alpha}}e^{-x^{1/\alpha}k}dk. \tag{34}$$

Performing an integration by parts, equation (34) can be rewritten as

$$E_\alpha(-x) = 1 - \frac{1}{2\alpha} + \frac{x^{1/\alpha}}{\pi}\int_0^{\infty}\arctan\left(\frac{k^\alpha+\cos(\alpha\pi)}{\sin(\alpha\pi)}\right)e^{-x^{1/\alpha}k}dk. \tag{35}$$

Equation (34) can be used to compute the numerical coefficient of the leading term of the asymptotic expansion of $E_\alpha(-x)$. Equation (24) becomes

$$a_0(\alpha) = \frac{\alpha}{\pi}\Gamma(\alpha)\sin(2\alpha\pi)\int_0^{\infty}\frac{k^{\alpha-1}}{1+2k^{2\alpha}\cos(2\alpha\pi)+k^{4\alpha}}dk, \quad \alpha < \frac{1}{2}. \tag{36}$$

9. Conclusions

The Mittag-Leffler relaxation function, $E_\alpha(-x)$, with $0 \leqslant \alpha \leqslant 1$, which arises in the description of complex relaxation processes, was studied. From equation (5) that gives the relaxation function in terms of two Mittag-Leffler functions with positive arguments, the inverse Laplace transform of $E_\alpha(-x)$ was obtained, equation (11), and used to derive a new integral representation of this function, equation (32), its asymptotic behaviour, equations (25) and (36), and a new recurrence relation, equation (26). It was also shown that the fastest initial decay of $E_\alpha(-x)$ occurs for $\alpha = 1/2$, a result hitherto unnoticed, and that shows the peculiar nature of the interpolation between a pure exponential and a hyperbolic function performed by the Mittag-Leffler relaxation function.

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